Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2016) 40: 784 – 788
© TÜBİTAK
doi:10.3906/mat-1501-34

A remark on singularity of homeomorphisms and Hausdorff dimension

Chun WEI[1,*], **Sheng-you WEN**[2]
[1]Department of Mathematics, South China University of Technology, Guangzhou, P.R. China
[2]Department of Mathematics, Hubei University, Wuhan, P.R. China

Received: 16.01.2015 • **Accepted/Published Online:** 14.10.2015 • **Final Version:** 16.06.2016

Abstract: We prove that there is a homeomorphism of the unit interval onto itself that is so singular that it maps some set E of $\dim_H E = 0$ onto a set F of $\dim_H [0,1] \setminus F = 0$.

Key words: Singularity, homeomorphism, Hausdorff dimension

1. Introduction

Let $E \subset \mathbb{R}^d$, $s > 0$, and $\delta > 0$. A family of sets $\{U_i\}_{i=1}^{\infty}$ is called a δ-covering of the set E if $\cup_{i=1}^{\infty} U_i \supset E$ and $0 < |U_i| \leq \delta$ for all i, where $|U_i|$ denotes the diameter of U_i. The s-dimension Hausdorff measure of the set E is defined by

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E), \tag{1}$$

where

$$\mathcal{H}^s_\delta(E) = \inf\{\sum |U_i|^s : \{U_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-covering of } E\}.$$

There is a unique value of s such that $\mathcal{H}^s(E)$ jumps from ∞ to 0. This value, denoted by $\dim_H E$, is called the Hausdorff dimension of E. Thus,

$$\dim_H E = \sup\{s : \mathcal{H}^s(E) > 0\} = \inf\{s : \mathcal{H}^s(E) < \infty\}. \tag{2}$$

For the properties of the Hausdorff dimension we refer to [1].

A homeomorphism $f : [0,1] \to [0,1]$ is singular if it maps some set of Lebesgue measure zero onto a set of Lebesgue measure 1. Singular homeomorphisms can be used to construct measurable sets that are not Borel. They also act as examples of increasing functions satisfying

$$\int_0^1 f'(x)dx < f(1) - f(0).$$

We may give a finer description for the singularity of a homeomorphism by means of Hausdorff dimension. We say that a homeomorphism is (α, β)-singular if it maps some set E of $\dim_H E \leq \alpha$ onto a set F of

*Correspondence: hbxt1986v@163.com
Supported by the NSFC (No. 11271114) and the Fundamental Research Funds for the Central Universities (No. 2015ZM193).
2010 *AMS Mathematics Subject Classification:* 28A80, 60G30.

$\dim_H[0,1] \setminus F \leq \beta$, where $\alpha, \beta \in [0,1]$. It is known that for any $\alpha, \beta \in (0,1]$ there are (α,β)-singular quasisymmetric homeomorphisms (see [4]). However, quasisymmetric homeomorphisms are not $(0,0)$-singular because they preserve sets of $\dim_H = 0$ by their Hölder-continuity (see [3]). In this note, we shall prove that a general homeomorphism can be $(0,0)$-singular.

Theorem 1 *There is a homeomorphism* $f:[0,1]\to[0,1]$ *and a set* $E \subset [0,1]$ *such that* $\dim_H E = \dim_H[0,1]\setminus f(E) = 0$.

2. Proof of Theorem 1

We start by recalling middle interval Cantor sets. Let $E_0 = [a,b]$ be a closed interval. Let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence of numbers in $(0,1)$. Removing an open interval of length $\lambda_1(b-a)$ from the middle of $[a,b]$, we get a set E_1 consisting of 2 intervals each of length $\frac{1-\lambda_1}{2}(b-a)$. Removing an open interval of length $\lambda_2|I|$ from the middle of every component interval I of E_1, we get a set E_2 consisting of 2^2 intervals each of length $\frac{(1-\lambda_1)(1-\lambda_2)}{2^2}(b-a)$. Proceeding infinitely, we get a nested sequence of compact sets $\{E_i\}_{i=0}^{\infty}$. The set

$$E := \bigcap_{i=0}^{\infty} E_i \tag{3}$$

is called a middle interval Cantor set in $[a,b]$. In this case, we also say that E is a $\{\lambda_i\}_{i=1}^{\infty}$-Cantor set. Obviously, the set E is totally disconnected and has no isolated points. From the definition, the set E_n consists of 2^n disjoint closed intervals each of length

$$\delta_n = \frac{b-a}{2^n}\prod_{i=1}^{n}(1-\lambda_i). \tag{4}$$

The Hausdorff dimension of the $\{\lambda_i\}_{i=1}^{\infty}$-Cantor set E is

$$\dim_H E = \liminf_{n\to\infty} \frac{n\log 2}{-\log \delta_n}. \tag{5}$$

We refer to [2] for a proof of (5). For the $\{\frac{i}{i+1}\}_{i=1}^{\infty}$-Cantor set E in $[a,b]$ it follows from (4) and (5) that

$$\dim_H E = \liminf_{n\to\infty} \frac{n\log 2}{-\log \frac{(b-a)}{(n+1)!2^n}} = 0. \tag{6}$$

We shall use this fact in the following construction.

For a $\{\lambda_i\}_{i=1}^{\infty}$-Cantor set C in $[a,b]$ we denote by $\mathcal{I}_n(C)$ the set of components of E_n and let $\mathcal{I}(C) = \cup_{n=1}^{\infty}\mathcal{I}_n(C)$. Denote by $\mathcal{G}(C)$ the set of components of $[a,b]\setminus C$. An element in $\mathcal{I}(C)$ will be called a basic interval of C and an element in $\mathcal{G}(C)$ will be called a gap of C.

Now we introduce composite Cantor sets. Let C_1 be a $\{\lambda_i^{(1)}\}_{i=1}^{\infty}$-Cantor set in $[0,1]$. For every gap $J \in \mathcal{G}(C_1)$ we take a $\{\lambda_i^{(2)}\}_{i=1}^{\infty}$-Cantor set C_J in the closure $\overline{J}$ and write

$$C_2 = C_1 \cup \bigcup_{J\in\mathcal{G}(C_1)} C_J,$$

$$\mathcal{G}(C_2) = \bigcup_{J \in \mathcal{G}(C_1)} \mathcal{G}(C_J),$$

$$\mathcal{I}(C_2) = \mathcal{I}(C_1) \cup \bigcup_{J \in \mathcal{G}(C_1)} \mathcal{I}(C_J).$$

Proceeding infinitely, we get three sequences $\{\mathcal{I}(C_k)\}_{k=1}^{\infty}$, $\{\mathcal{G}(C_k)\}_{k=1}^{\infty}$, and $\{C_k\}_{k=1}^{\infty}$. We call the set

$$C := \bigcup_{k=1}^{\infty} C_k \tag{7}$$

a composite Cantor set in $[0,1]$. Clearly, a composite Cantor set consists of a countable number of middle interval Cantor sets. It is not a compact set.

Note that a composite Cantor set C may not be dense in $[0,1]$. The following lemma gives some necessary and sufficient conditions under which C is dense in $[0,1]$.

Lemma 1 *Let C be a composite Cantor set in $[0,1]$. The following statements are equivalent:*

(i) *C is dense in $[0,1]$.*

(ii) *$\frac{1}{2} \in \overline{C}$, where $\overline{C}$ is the closure of C.*

(iii) $\prod_{k=1}^{\infty} \lambda_1^{(k)} = 0$.

Proof. (i) $\Rightarrow$ (ii). This is obvious.

(ii) $\Rightarrow$ (iii). For every $i \geq 1$ let J_i denote the gap of C_i in the midst of $[0,1]$. Then

$$|J_i| = \prod_{j=1}^{i} \lambda_1^{(j)}.$$

Let

$$x_i = \frac{1}{2}(1 - \prod_{j=1}^{i} \lambda_1^{(j)}), \quad i \geq 1.$$

We see that x_i is the left endpoint of the gap J_i. By (ii) and the construction of C, we have $\lim_{i\to\infty} x_i = \frac{1}{2}$, which implies

$$\prod_{k=1}^{\infty} \lambda_1^{(k)} = 0.$$

(iii) $\Rightarrow$ (i). Let $k \geq 1$ and let $J \in \mathcal{G}(C_k)$ be given. Denote by $M(J)$ the middle point of J. We have $M(J) = \inf J + \frac{1}{2}|J|$. Let

$$x_i = \inf J + \frac{1}{2}(1 - \prod_{j=1}^{i} \lambda_1^{(k+j)})|J|, \quad i \geq 1. \tag{8}$$

We see that x_i is the left endpoint of the gap of C_{k+i} in the midst of J. By (iii), we have $\prod_{j=1}^{\infty} \lambda_1^{(k+j)} = 0$, so $\lim_{i\to\infty} x_i = M(J)$. This implies $M(J) \in \overline{C}$. Next we show that $J \subset \overline{C}$. In fact, given $u \in J \setminus C$, we have from the construction of the composite Cantor set C that

$$\mathrm{dist}(u, C) \leq |x_i - M(J)|$$

for all x_i in (8), which gives $u \in \overline{C}$, and thus $J \subset \overline{C}$. Finally, since $J \subset \overline{C}$ for all $J \in \cup_{k=1}^{\infty} \mathcal{G}(C_k)$, it is easy to see that C is dense in $[0,1]$. □

Proof of Theorem 1. Note that if X, Y are two dense composite Cantor sets in $[0,1]$ then we have a unique increasing homeomorphism $f : [0,1] \to [0,1]$ such that $f(X) = Y$. Therefore, to prove Theorem 1, it suffices to construct two dense composite Cantor sets X and Y in $[0,1]$ such that $\dim_H X = \dim_H [0,1] \setminus Y = 0$.

The dense composite Cantor set X is constructed as follows: let all middle interval Cantor sets be chosen to be $\{\frac{i}{i+1}\}_{i=1}^{\infty}$-Cantor sets, and then it follows from (6) that $\dim_H X = 0$ because X consists of a countable number of sets of Hausdorff dimension zero. It is also clear that X is dense in $[0,1]$ by Lemma 1.

Finally we construct a dense composite Cantor set Y such that $\dim_H [0,1] \setminus Y = 0$. We will use the following simple fact repeatedly: for any $s, \alpha > 0$ and for any closed interval I there is a $\{\lambda_i\}_{i=1}^{\infty}$-Cantor set C in I with $\lambda_i \in (0, \frac{1}{2})$ such that

$$\sum_{J \in \mathcal{G}(C)} |J|^s \leq \alpha. \tag{9}$$

Let $\{s_i\}_{i=1}^{\infty}$ and $\{\alpha_i\}_{i=1}^{\infty}$ be two fixed sequences of positive numbers such that s_i is decreasing to zero and $\sum \alpha_i = 1$. The desired composite of Cantor sets Y can be inductively constructed as follows:

Choose a $\{\lambda_i^{(1)}\}_{i=1}^{\infty}$-Cantor set C_1 in $[0,1]$ with $\lambda_i^{(1)} \in (0, \frac{1}{2})$ such that

$$\sum_{J \in \mathcal{G}(C_1)} |J|^{s_1} \leq 1.$$

Let $\{J_i\}_{i=1}^{\infty}$ be an enumeration of members of $\mathcal{G}(C_1)$. For every J_i choose a $\{\lambda_i^{(2)}\}_{i=1}^{\infty}$-Cantor set C_{J_i} in the closure $\overline{J_i}$ with $\lambda_i^{(2)} \in (0, \frac{1}{2})$ such that

$$\sum_{J \in \mathcal{G}(C_{I_i})} |J|^{s_2} \leq \alpha_i.$$

Take $C_2 = C_1 \cup \cup_{i=1}^{\infty} C_{J_i}$. Since $\sum \alpha_i = 1$ is assumed, it follows that

$$\sum_{J \in \mathcal{G}(C_2)} |J|^{s_2} = \sum_{i=1}^{\infty} \sum_{J \in \mathcal{G}(C_{J_i})} |J|^{s_2} \leq 1.$$

Proceeding infinitely, we get an increasing sequence $\{C_k\}_{k=1}^{\infty}$ of sets such that

$$\sum_{J \in \mathcal{G}(C_k)} |J|^{s_k} \leq 1 \text{ for all } k \geq 1. \tag{10}$$

Let $Y = \cup_{k=1}^{\infty} C_k$. Then Y is a composite of Cantor sets and Y is dense in $[0,1]$ by Lemma 1. To complete this proof we are going to show $\dim_H [0,1] \setminus Y = 0$. By the construction, $\mathcal{G}(C_k)$ is a covering of $[0,1] \setminus Y$ and every member of $\mathcal{G}(C_k)$ has diameter of at most 2^{-k}. Therefore,

$$\mathcal{H}^{s_i}_{2^{-k}}([0,1]\setminus Y) \le \sum_{J\in\mathcal{G}(C_k)} |J|^{s_i}$$

for all $i,k \ge 1$. Given $i \ge 1$, since s_k has been assumed to be decreasing, it follows from (10) that

$$\mathcal{H}^{s_i}_{2^{-k}}([0,1]\setminus Y) \le \sum_{J\in\mathcal{G}(C_k)} |J|^{s_k} \le 1$$

for any $k > i$, and so $\mathcal{H}^{s_i}([0,1]\setminus Y) \le 1$. Since s_i is also assumed to tend to zero, we get $\dim_H[0,1]\setminus Y = 0$. This completes the proof of Theorem 1. □

Acknowledgment

The authors thank the referee for his or her helpful suggestions.

References

[1] Falconer KJ. The Geometry of Fractal Sets. Cambridge, UK: Cambridge University Press, 1985.

[2] Feng DJ, Rao H, Wu J. The net measure properties for symmetric Cantor sets and their applications. Progress in Natural Sciences 1997; 7: 172-178.

[3] Heinonen J. Lectures on Analysis on Metric Spaces. New York, NY, USA: Springer-Verlag, 2001.

[4] Tukia P. Hausdorff dimension and quasisymmetrical mappings. Math Scand 1989; 65: 152-160.