Idempotents of the Green algebras of finite dimensional pointed rank one Hopf algebras of nilpotent type

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Abstract: In this paper, we intend to study idempotents of the Green algebra (complexified Green ring) of any finite dimensional pointed rank one Hopf algebra of nilpotent type over the complex number field. We first determine all one dimensional representations of the quotient algebra of the Green algebra modulo its Jacobson radical. This gives rise to all primitive idempotents of the quotient algebra. Then we present explicitly primitive idempotents of the Green algebra by lifting the ones of the quotient algebra. Finally, as an example, we describe all primitive idempotents of the Green algebra of the Taft algebra T_3.

Key words: Hopf algebra, Green algebra, idempotent, Dickson polynomial

1. Introduction

As we all know, the finite dimensional module category of any finite dimensional Hopf algebra is a tensor category, where the tensor product of any two indecomposable modules can always be decomposed as a direct sum of indecomposable modules. However, there is a less obvious way to obtain this decomposition. One method of addressing this problem is to consider the tensor product as the multiplication of the Green ring (or the representation ring), and to study the ring-theoretical properties of the Green ring, due to J. A. Green for the study of the modular representations of a finite group [4]. A lot of work have been done in this direction, see e.g. [1, 3, 12, 18].

According to the Krull–Schmidt theorem, one knows that the Green ring of a Hopf algebra is a free $\mathbb{Z}$-module with the isomorphism classes of indecomposable modules as a basis, see e.g. [2, 11, 16]. It is natural to think about how nilpotent elements, idempotent elements, etc., of the Green ring are expressed as a linear combination of the basis. The problem has been considered in many special cases. For instance, the authors in [12] presented all nilpotent elements of the Green rings of the generalized Taft algebras as a linear combination of a basis. For an acyclic quiver Q, the authors in [9] studied idempotents of the Green ring of Q and gave a general technique for constructing such idempotents and for decomposing the Green ring into a direct product of ideals. For any semisimple almost cocommutative Hopf algebra, all primitive idempotents of its complexified Green ring (Green algebra) were obtained in [19], where the characters of the Green algebra were used to give a linear expression of idempotents.

In [17], we studied the Green ring of any finite dimensional pointed rank one Hopf algebra of nilpotent

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type. It turned out that the Jacobson radical of the Green ring is a principal ideal generated by a special element expressed linearly by a basis and unfortunately the Green ring has only the trivial idempotents. In this paper, we turn to the study of idempotents of the complexified Green ring (Green algebra) over the complex number field $\mathbb{C}$.

The paper is organized as follows. In Section 2 we recall the algebra and coalgebra structure of any finite dimensional pointed rank one Hopf algebra of nilpotent type, and present its Green ring in terms of generators and relations. In section 3, we first determine all one dimensional representations of the quotient algebra of the Green algebra modulo its Jacobson radical. Note that the quotient algebra is commutative semisimple over the complex number field. Each representation of the quotient algebra determines a primitive idempotent of the quotient algebra, and vice versa. This gives rise to all primitive idempotents of the quotient algebra. After that, we obtain the primitive idempotents of the Green algebra by lifting those of the quotient algebra. In section 4, as an example, we completely determine primitive idempotents of the Green algebra of the Taft algebra T_3.

Throughout, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ stand for the sets of natural numbers, integers, and complex numbers, respectively. The symbol $\delta_{i,j}$ is the Kronecker delta. All algebras considered are associative with unity 1 over the complex number field $\mathbb{C}$. The complex primitive l-th root of unity is usually written as $\cos \frac{2\pi}{l} + i \sin \frac{2\pi}{l}$, where $i^2 = -1$. If V is a finite dimensional vector space over $\mathbb{C}$, its dimension as a vector space is denoted $\dim V$. For standard facts about Hopf algebras and related representation theory, we refer the reader to [8, 13].

2. Preliminaries

In this section, we recall some basic facts about finite dimensional pointed rank one Hopf algebras of nilpotent type and present the Green rings of such Hopf algebras in terms of generators and relations. We refer to [10, 17] for more details.

2.1. Hopf algebra structure of H

Throughout, G is a finite group, g is an element in the center of G, χ is a $\mathbb{C}$-linear character of G subject to $\chi^n = 1$, where n is the order of $\chi(g)$. Then l, the order of χ, is divisible by n. Let H be a Hopf algebra constructed through the group (G, χ, g). More explicitly, H is generated as an algebra by y and all h in G such that $\mathbb{C}G$ is a subalgebra of H and

$$y^n = 0, \ yh = \chi(h)hy, \ \text{for } h \in G.$$

H is endowed with a Hopf algebra structure, where the comultiplication $\bigtriangleup$, the counit ε, and the antipode S are given respectively by

$$\bigtriangleup(y) = y \otimes g + 1 \otimes y, \ \varepsilon(y) = 0, \ S(y) = -yg^{-1},$$

$$\bigtriangleup(h) = h \otimes h, \ \varepsilon(h) = 1, \ S(h) = h^{-1},$$

for all $h \in G$.

The Hopf algebra H is indeed a finite dimensional pointed rank one Hopf algebra of nilpotent type with a $\mathbb{C}$-basis $\{y^i h | h \in G, \ 0 \le i \le n-1\}$; see [10, 17]. Thus $\dim H = n|G|$, where $|G|$ is the order of G.

Typical examples of such Hopf algebras include the (generalized) Taft algebras [2, 12, 15], the Radford Hopf algebras [14], the half quantum groups [5], etc. What we need to emphasize is that any finite dimensional pointed rank one Hopf algebra of nilpotent type can always be obtained from this approach [10, Theorem 1].

Observe that if the order of $\chi(g)$ is $n = 1$, then H is nothing but the group algebra $\mathbb{C}G$. To avoid this, we always assume that $n \geq 2$ throughout this paper. In this situation $\chi(g) \neq 1$; this implies that $g \neq 1$ and $\chi \neq \varepsilon$.

2.2. Indecomposable representations of H

Since the Jacobson radical J of H is generated by y and $H/J \cong \mathbb{C}G$, a complete set of nonisomorphic simple $\mathbb{C}G$-modules forms a complete set of nonisomorphic simple H-modules. In the sequel, we fix such a complete set $\{V_i \mid i \in \Omega\}$ consisting of nonisomorphic simple $\mathbb{C}G$-modules (and also simple H-modules). Note that $0 \in \Omega$ as we denote $V_0 = \mathbb{C}$, the trivial H-module.

Let x be a variable. For any $k \in \mathbb{N}$ and $i \in \Omega$, consider $x^k V_i$ as a vector space in the obvious way. Then $x^k V_i$ becomes a $\mathbb{C}G$-module defined by

$$h(x^k v) = \chi^{-k}(h) x^k hv,$$

for any $h \in G$ and $v \in V_i$. For $1 \leq j \leq n$, we define an action of y on the direct sum

$$M(j,i) := V_i \oplus xV_i \oplus \cdots \oplus x^{j-1}V_i$$

as follows:

$$y(x^k v) = \begin{cases} x^{k+1}v, & 0 \leq k \leq j-2, \\ 0, & k = j-1, \end{cases}$$

for any $v \in V_i$. Then $M(j,i)$ becomes an H-module with $\dim M(j,i) = j \dim(V_i)$. It is easy to see that $M(1,i) \cong V_i$. In particular, the set $\{M(j,i) \mid i \in \Omega,\ 1 \leq j \leq n\}$ forms a complete set of finite dimensional indecomposable H-modules up to isomorphism [17, Theorem 2.5(4)].

2.3. The Green ring of H

Let $F(H)$ be the free abelian group generated by the isomorphism classes $[M]$ of finite dimensional H-modules M. The abelian group $F(H)$ becomes a ring if we endow $F(H)$ with a multiplication given by the tensor product $[M][N] = [M \otimes N]$. The *Green ring* (or *representation ring*) $r(H)$ of the Hopf algebra H is defined to be the quotient ring of $F(H)$ modulo the relations $[M \oplus N] = [M] + [N]$. The identity of the associative ring $r(H)$ is represented by the trivial H-module $[V_0]$. Note that $r(H)$ has a $\mathbb{Z}$-basis consisting of the isomorphism classes of indecomposable H-modules [17]. For brevity and simplicity we denote by $M[j,i]$ the isomorphism class of indecomposable H-module $M(j,i)$ in $r(H)$. In particular, we set $1 = [V_0]$ and $a = [V_{\chi^{-1}}]$. Then the order of a is l. The following proposition comes from [17, Proposition 4.1].

Proposition 2.1 *The following hold in the Green ring $r(H)$ of H:*

(1) $M[j,i] = [V_i]M[j,0] = M[j,0][V_i]$, *for $i \in \Omega$ and $1 \leq j \leq n$.*

(2) $M[2,0]M[j,0] = M[j,0]M[2,0] = M[j+1,0] + aM[j-1,0]$, *for $2 \leq j \leq n-1$.*

(3) $M[2,0]M[n,0] = M[n,0]M[2,0] = (1+a)M[n,0]$.

(4) $r(H)$ *is a commutative ring generated by $[V_i]$ for $i \in \Omega$ and $M[2,0]$ over $\mathbb{Z}$.*

It can be deduced from Proposition 2.1 (3) that

$$M[s,0]M[n,0] = (1 + a + \cdots + a^{s-1})M[n,0], \text{ for } 1 \leq s \leq n. \tag{2.1}$$

Now we are ready to give the structure of the Green ring $r(H)$ in terms of generators and relations. Let $\mathbb{Z}[y,z]$ be the polynomial ring with variables y and z over $\mathbb{Z}$ and $F_i(y,z)$ the Dickson polynomials (of the second type) defined recursively as follows (see e.g. [7]):

$$F_1(y,z) = 1, \ F_2(y,z) = z, \ F_i(y,z) = zF_{i-1}(y,z) - yF_{i-2}(y,z), \ i \geq 3. \tag{2.2}$$

Then the Green ring $r(H)$ can be described as follows (see [17, Theorem 4.3]).

Theorem 2.2 *The Green ring $r(H)$ is isomorphic to the quotient ring $r(\mathbb{C}G)[z]/I$, where $r(\mathbb{C}G)[z]$ is the polynomial ring with variable z over the Green ring $r(\mathbb{C}G)$ of the group algebra $\mathbb{C}G$, and I is the ideal of $r(\mathbb{C}G)[z]$ generated by $(1 + a - z)F_n(a,z)$.*

The Jacobson radical of $r(H)$ is a principal ideal generated by the element $M[n,0]\theta$, where $\theta = (1-a)(1 + a^n + a^{2n} + \cdots + a^{(\frac{l}{n}-1)n})$. Moreover, the square of $M[n,0]\theta$ is equal to zero by (2.1).

The Green ring $r(H)$ of H can be embedded into the Green ring of a finite dimensional pointed rank one Hopf algebra of nonnilpotent type [18, Section 4]. Note that the latter Green ring has only the trivial idempotents [18, Theorem 6.5]. In view of this, $r(H)$ has only the trivial idempotents as well.

3. Idempotents of the Green algebras

Since the Green ring $r(H)$ has only the trivial idempotents, in this section, we intend to study idempotents of the complexified Green ring $R(H) := \mathbb{C} \otimes_{\mathbb{Z}} r(H)$, called the Green algebra of H. We first determine all primitive idempotents of the quotient algebra $R(H)/J(R(H))$, where $J(R(H))$ is the Jacobson radical of $R(H)$. Then idempotents of the quotient algebra are candidates to be lifted to the idempotents of $R(H)$.

3.1. One dimensional representations of the quotient algebra

Since the Green algebra $R(\mathbb{C}G) := \mathbb{C} \otimes_{\mathbb{Z}} r(\mathbb{C}G)$ is commutative semisimple over the field $\mathbb{C}$, there are $|\Omega|$ simple $R(\mathbb{C}G)$-modules and each of them is of dimension one. Let $\{W_i \mid i \in \Omega\}$ be the set of all nonisomorphic (one dimensional) simple $R(\mathbb{C}G)$-modules and $\{e_i \mid i \in \Omega\}$ the set of all primitive orthogonal idempotents of $R(\mathbb{C}G)$ satisfying $e_iW_j = \delta_{i,j}W_j$, for $i,j \in \Omega$. We denote by $\omega = \cos\frac{2\pi}{l} + i\sin\frac{2\pi}{l}$ a primitive l-th root of unity. Note that the order of a is l. Then the action of a on W_j is a scalar multiple by ω^{t_j}, for some $0 \leq t_j \leq l-1$. Thus,

$$a = \sum_{j \in \Omega} \omega^{t_j} e_j.$$

According to the exponent t_j, one can divide the index set Ω into three parts:

$$\Omega_1 = \{j \mid j \in \Omega, t_j = 0\},$$

$$\Omega_2 = \{j \mid j \in \Omega, t_j \neq 0 \text{ and } \frac{l}{n} \nmid t_j\},$$

$$\Omega_3 = \{j \mid j \in \Omega, t_j \neq 0 \text{ and } \frac{l}{n} \mid t_j\}.$$

Then $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, which is the disjoint union of subsets of Ω.

In order to describe simple modules over $R(H)/J(R(H))$, we need to determine all distinct roots of the equation

$$(1 + \omega^{t_j} - z)F_n(\omega^{t_j}, z) = 0, \tag{3.1}$$

for any $j \in \Omega$.

Lemma 3.1 *Let* $\alpha = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n}$. *Then the distinct roots of the equation (3.1) can be described as follows:*

(1) *If* $j \in \Omega_3$, *then the equation (3.1) has* $n - 1$ *distinct roots:*

$$\alpha_{j,k} = \sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k}), \text{ for } 1 \leq k \leq n - 1.$$

(2) *If* $j \in \Omega_1 \cup \Omega_2$, *then the equation (3.1) has* n *distinct roots:*

$$\alpha_{j,k} = \sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k}), \text{ for } 1 \leq k \leq n - 1 \text{ and } \alpha_{j,n} = \omega^{t_j} + 1.$$

Proof Let $b_j = \cos(\frac{t_j\pi}{l} + \frac{3\pi}{2}) + i\sin(\frac{t_j\pi}{l} + \frac{3\pi}{2})$. Then $b_j^2 = -\omega^{t_j}$. The relationship between the polynomials $F_k(\omega^{t_j}, z)$ and the Fibonacci polynomials $F_k(-1, z)$ are established by induction on k as follows:

$$F_k(\omega^{t_j}, z) = b_j^{k-1}F_k(-1, b_j^{-1}z), \text{ for } k \geq 1.$$

In particular, $F_n(\omega^{t_j}, z) = b_j^{n-1}F_n(-1, b_j^{-1}z)$. Since the distinct roots of the equation $F_n(-1, z) = 0$ are $2i\cos\frac{k\pi}{n} = i(\alpha^k + \alpha^{-k})$, for $1 \leq k \leq n - 1$, see e.g. [6], it follows that the distinct roots of $F_n(\omega^{t_j}, z) = 0$ are

$$\alpha_{j,k} = 2b_j i \cos\frac{k\pi}{n}$$
$$= (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})(\cos(\frac{t_j\pi}{l} + \frac{3\pi}{2}) + i\sin(\frac{t_j\pi}{l} + \frac{3\pi}{2}))(\alpha^k + \alpha^{-k})$$
$$= (\cos\frac{t_j\pi}{l} + i\sin\frac{t_j\pi}{l})(\alpha^k + \alpha^{-k})$$
$$= \sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k}),$$

for $1 \leq k \leq n - 1$. Here $\sqrt{\omega^{t_j}}$ stands for $\cos\frac{t_j\pi}{l} + i\sin\frac{t_j\pi}{l}$. This implies that the equation (3.1) has roots $\omega^{t_j} + 1$ and $\sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k})$, for $1 \leq k \leq n - 1$. Now $\omega^{t_j} + 1 = \sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k})$ if and only if $\cos\frac{t_j\pi}{l} = \cos\frac{k\pi}{n}$ if and only if $k = s$ and $t_j = \frac{ls}{n}$, for a unique $1 \leq s \leq n - 1$. We obtain the desired results. □

Let $W_{j,k}$ be a simple $R(H)$-module lifted by W_j. That is, $W_{j,k}$ is the same as W_j as an $R(\mathbb{C}G)$-module, while the generator $M[2,0]$ of $R(H)$ that acts on W_j is the scalar multiple by $\alpha_{j,k}$, which is a root of the equation (3.1) by Lemma 3.1. It follows that

$$\{W_{j,k} \mid j \in \Omega_1 \cup \Omega_2, 1 \leq k \leq n\} \cup \{W_{j,k} \mid j \in \Omega_3, 1 \leq k \leq n - 1\}$$

forms a complete set of simple $R(H)$-modules up to isomorphism. Obviously, the set above is also a complete set of simple $R(H)/J(R(H))$-modules since every simple $R(H)$-module is annihilated by the Jacobson radical $J(R(H))$ of $R(H)$. For any $x \in R(H)$, the image of x under the natural map $R(H) \to R(H)/J(R(H))$ is denoted by $\bar{x}$. For any simple $R(H)/J(R(H))$-module $W_{j,k}$, there exists a unique algebra morphism $\Phi_{j,k}$ from $R(H)/J(R(H))$ to $\mathbb{C}$ such that

$$\Phi_{j,k}(\overline{e_i}) = \delta_{i,j}, \ \Phi_{j,k}(\bar{a}) = \omega^{t_j} \text{ and } \Phi_{j,k}(\overline{M[2,0]}) = \alpha_{j,k}. \tag{3.2}$$

Conversely, every algebra morphism from $R(H)/J(R(H))$ to $\mathbb{C}$ is determined in this way since $R(H)/J(R(H))$ is commutative semisimple over $\mathbb{C}$. Hence there is a one to one correspondence between the set of nonisomorphic simple $R(H)/J(R(H))$-modules and the set of distinct algebra morphisms from $R(H)/J(R(H))$ to $\mathbb{C}$.

Lemma 3.2 *For the algebra morphism $\Phi_{j,k}$ defined above, we have the following:*

(1) *If $j \in \Omega_1 \cup \Omega_2 \cup \Omega_3$ and $1 \le k \le n-1$, then*

$$\Phi_{j,k}(\overline{M[s,0]}) = (\sqrt{\omega^{t_j}})^{s-1} \frac{\alpha^{ks} - \alpha^{-ks}}{\alpha^k - \alpha^{-k}},$$

for $1 \le s \le n$. Moreover, $\Phi_{j,k}(\overline{M[n,0]}) = 0$.

(2) *If $j \in \Omega_1 \cup \Omega_2$ and $k = n$, then*

$$\Phi_{j,n}(\overline{M[s,0]}) = \begin{cases} \frac{1-\omega^{st_j}}{1-\omega^{t_j}}, & j \in \Omega_2, \\ s, & j \in \Omega_1, \end{cases}$$

for $1 \le s \le n$.

Proof (1) By induction on s. If $s = 1$, it is trivial since $\overline{M[1,0]}$ is the identity of $R(H)/J(R(H))$. If $s = 2$, then $\Phi_{j,k}(\overline{M[2,0]}) = \alpha_{j,k} = \sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k})$ by (3.2). Suppose it holds for $s \le i$. To prove the case $s = i+1$, we have by Proposition 2.1 (2) that

$$\begin{aligned}
&\Phi_{j,k}(\overline{M[i+1,0]}) \\
&= \Phi_{j,k}(\overline{M[2,0]})\Phi_{j,k}(\overline{M[i,0]}) - \Phi_{j,k}(\bar{a})\Phi_{j,k}(\overline{M[i-1,0]}) \\
&= \sqrt{\omega^{t_j}}(\alpha^k + \alpha^{-k})(\sqrt{\omega^{t_j}})^{i-1}((\alpha^k)^{i-1} + (\alpha^k)^{i-3} + \cdots + (\alpha^k)^{1-i}) \\
&\quad - \omega^{t_j}(\sqrt{\omega^{t_j}})^{i-2}((\alpha^k)^{i-2} + (\alpha^k)^{i-4} + \cdots + (\alpha^k)^{2-i}) \\
&= (\sqrt{\omega^{t_j}})^i((\alpha^k)^i + (\alpha^k)^{i-2} + \cdots + (\alpha^k)^{-i}) \\
&= (\sqrt{\omega^{t_j}})^i \frac{\alpha^{k(i+1)} - \alpha^{-k(i+1)}}{\alpha^k - \alpha^{-k}}.
\end{aligned}$$

Moreover,

$$\Phi_{j,k}(\overline{M[n,0]}) = (\sqrt{\omega^{t_j}})^{n-1} \frac{\alpha^{kn} - \alpha^{-kn}}{\alpha^k - \alpha^{-k}} = 0$$

because $\alpha = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$.

(2) If $j \in \Omega_1 \cup \Omega_2$ and $k = n$, it follows from (3.2) that

$$\Phi_{j,n}(\overline{M[2,0]}) = \alpha_{j,n} = \omega^{t_j} + 1.$$

Now the result follows by induction on s. □

3.2. Two bases of the quotient algebra

Let $E_{j,k}$ belong to $R(H)$ such that $\overline{E_{j,k}}$ is a primitive idempotent of $R(H)/J(R(H))$ and $\Phi_{i,s}(\overline{E_{j,k}}) = \delta_{i,j}\delta_{s,k}$. Then

$$\{\overline{E_{j,k}} \mid j \in \Omega_1 \cup \Omega_2, 1 \le k \le n\} \cup \{\overline{E_{j,k}} \mid j \in \Omega_3, 1 \le k \le n - 1\}$$

forms an orthogonal basis of $R(H)/J(R(H))$. Moreover, there is another basis of $R(H)/J(R(H))$ that we need.

Lemma 3.3 *We have the following:*

(1) *The set $\{e_j M[k,0] \mid j \in \Omega, 1 \le k \le n\}$ forms a basis of $R(H)$.*

(2) *The set $\{e_j M[n,0] \mid j \in \Omega_3\}$ forms a basis of $J(R(H))$.*

(3) *The set $\{\overline{e_j M[k,0]} \mid j \in \Omega_1 \cup \Omega_2, 1 \le k \le n\} \cup \{\overline{e_j M[k,0]} \mid j \in \Omega_3, 1 \le k \le n - 1\}$ forms a basis of $R(H)/J(R(H))$.*

Proof (1) Observe from (2.2) that the k-th Dickson polynomial $F_k(a, z)$ is of degree $k - 1$ with the leading coefficient 1 in the polynomial algebra $R(\mathbb{C}G)[z]$. Let I be the ideal of $R(\mathbb{C}G)[z]$ generated by the element $(1 + a - z)F_n(a, z)$. Then the quotient algebra $R(\mathbb{C}G)[z]/I$ has a $\mathbb{C}$-basis $\overline{e_j F_k(a, z)}$, for $j \in \Omega$ and $1 \le k \le n$. The Green algebra $R(H)$ is isomorphic to $R(\mathbb{C}G)[z]/I$ (see [17, Theorem 4.3]), where the isomorphism is given by

$$R(H) \to R(\mathbb{C}G)[z]/I, \ e_j M[k,0] \mapsto \overline{e_j F_k(a, z)}.$$

We conclude that $e_j M[k,0]$ for $j \in \Omega$ and $1 \le k \le n$ forms a basis of the Green algebra $R(H)$.

(2) If $j \in \Omega_3$, then

$$1 + \omega^{t_j} + \omega^{2t_j} + \cdots + \omega^{(n-1)t_j} = \frac{1 - \omega^{nt_j}}{1 - \omega^{t_j}} = 0.$$

Note that $M[n,0]^2 = (1 + a + \cdots + a^{n-1})M[n,0]$ and $e_j a = \omega^{t_j} e_j$. Then

$$(e_j M[n,0])^2 = e_j M[n,0]^2$$
$$= e_j(1 + a + a^2 + \cdots + a^{n-1})M[n,0]$$
$$= e_j(1 + \omega^{t_j} + \omega^{2t_j} + \cdots + \omega^{(n-1)t_j})M[n,0]$$
$$= 0.$$

This implies that $e_j M[n,0] \in J(R(H))$ for $j \in \Omega_3$. Moreover, it forms a basis of $J(R(H))$ since $e_j M[n,0]$ for $j \in \Omega_3$ is linear independent by Part (1), and the dimension of $J(R(H))$ is equal to the cardinality of Ω_3 by [17, Proposition 5.2].

(3) It follows immediately from Part (1) and Part (2). □

3.3. Basis transformation

In the following, we shall present the primitive orthogonal idempotents $\overline{E_{j,k}}$ as a linear combination of the basis of $R(H)/J(R(H))$ given in Lemma 3.3 (3).

Let $\overline{\Lambda} = \sum_{i,k} \beta_{i,k} \overline{E_{i,k}}$ be an arbitrary element of $R(H)/J(R(H))$ for $\beta_{i,k} \in \mathbb{C}$. The equality $\Phi_{i,k}(\overline{E_{j,s}}) = \delta_{i,j}\delta_{k,s}$ implies that $\Phi_{i,k}(\overline{\Lambda}) = \beta_{i,k}$. It follows that

$$\overline{\Lambda} = \sum_{i,k} \Phi_{i,k}(\overline{\Lambda})\overline{E_{i,k}}.$$

By replacing Λ with $e_j M[s,0]$, we obtain that

$$\overline{e_j M[s,0]} = \sum_{i,k} \Phi_{i,k}(\overline{e_j M[s,0]})\overline{E_{i,k}} = \sum_k \Phi_{j,k}(\overline{M[s,0]})\overline{E_{j,k}}, \qquad (3.3)$$

where the sum $\sum_k$ runs from 1 to n if $j \in \Omega_1 \cup \Omega_2$, and from 1 to $n-1$ if $j \in \Omega_3$.

For any $j \in \Omega_1 \cup \Omega_2 \cup \Omega_3$, we consider the following matrix:

$$\mathbf{A_j} = \begin{pmatrix} \Phi_{j,1}(\overline{M[1,0]}) & \Phi_{j,2}(\overline{M[1,0]}) & \cdots & \Phi_{j,n-1}(\overline{M[1,0]}) \\ \Phi_{j,1}(\overline{M[2,0]}) & \Phi_{j,2}(\overline{M[2,0]}) & \cdots & \Phi_{j,n-1}(\overline{M[2,0]}) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{j,1}(\overline{M[n-1,0]}) & \Phi_{j,2}(\overline{M[n-1,0]}) & \cdots & \Phi_{j,n-1}(\overline{M[n-1,0]}) \end{pmatrix}.$$

By Lemma 3.2, the (s,k)-entry of the matrix $\mathbf{A_j}$ is

$$\Phi_{j,k}(\overline{M[s,0]}) = (\sqrt{\omega^{t_j}})^{s-1} \frac{\alpha^{ks} - \alpha^{-ks}}{\alpha^k - \alpha^{-k}},$$

where $1 \le k \le n-1$ and $\alpha^k - \alpha^{-k} \ne 0$. Let $\mathbf{B}$ be the matrix with (k,s)-entry $\alpha^{ks} - \alpha^{-ks}$ for $1 \le k, s \le n-1$. Let $\mathbf{C_j}$ and $\mathbf{D}$ be two diagonal matrices given respectively by

$$\mathbf{C_j} = \text{diag}(1, \sqrt{\omega^{t_j}}, (\sqrt{\omega^{t_j}})^2, \cdots, (\sqrt{\omega^{t_j}})^{n-2}),$$

and

$$\mathbf{D} = \text{diag}(\alpha - \alpha^{-1}, \alpha^2 - \alpha^{-2}, \cdots, \alpha^{n-1} - \alpha^{-(n-1)}).$$

It is clear that $\mathbf{B}$ is symmetric, $\mathbf{A_j} = \mathbf{C_j}\mathbf{B}\mathbf{D}^{-1}$ and $\mathbf{A_j}$ is invertible with the inverse $\mathbf{A_j^{-1}} = \mathbf{D}\mathbf{B}^{-1}\mathbf{C_j^{-1}}$. Suppose the (k,s)-entry of the matrix $\mathbf{B}^{-1}$ is $\theta_{k,s}$, for $1 \le k, s \le n-1$. Then the (k,s)-entry of the matrix $\mathbf{A_j^{-1}}$ is

$$\beta_{k,s}^{(j)} = (\sqrt{\omega^{t_j}})^{1-s}(\alpha^k - \alpha^{-k})\theta_{k,s},$$

for $1 \le k, s \le n-1$. We are ready to present the primitive orthogonal idempotents $\overline{E_{j,k}}$ as a linear combination of the basis of $R(H)/J(R(H))$ given in Lemma 3.3 (3).

Case 1: $j \in \Omega_3$ and $1 \le k \le n-1$. Then the linear relations (3.3) can be written as follows:

$$\begin{pmatrix} \overline{e_j M[1,0]} \\ \overline{e_j M[2,0]} \\ \vdots \\ \overline{e_j M[n-1,0]} \end{pmatrix} = \mathbf{A_j} \begin{pmatrix} \overline{E_{j,1}} \\ \overline{E_{j,2}} \\ \vdots \\ \overline{E_{j,n-1}} \end{pmatrix}. \qquad (3.4)$$

Observe that the (k, s)-entry of the matrix $\mathbf{A_j^{-1}}$ is $\beta_{k,s}^{(j)}$. It follows from (3.4) that the idempotents $\overline{E_{j,k}}$ could be expressed as a linear combination as follows:

$$\overline{E_{j,k}} = \sum_{s=1}^{n-1} \beta_{k,s}^{(j)} \overline{e_j M[s,0]} = (\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} (\sqrt{\omega^{t_j}})^{1-s} \theta_{k,s} \overline{e_j M[s,0]}, \tag{3.5}$$

for $j \in \Omega_3$ and $1 \le k \le n-1$.

Case 2: $j \in \Omega_1 \cup \Omega_2$ and $1 \le k \le n$. Then the linear relations (3.3) can be written as follows:

$$\begin{pmatrix} \overline{e_j M[1,0]} \\ \overline{e_j M[2,0]} \\ \vdots \\ \overline{e_j M[n,0]} \end{pmatrix} = \begin{pmatrix} \mathbf{A_j} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix} \begin{pmatrix} \overline{E_{j,1}} \\ \overline{E_{j,2}} \\ \vdots \\ \overline{E_{j,n}} \end{pmatrix}, \tag{3.6}$$

where $\begin{pmatrix} \mathbf{A_j} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix}$ is a block matrix with the entries determined by Lemma 3.2. More explicitly, the column vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 + \omega^{t_j} \\ \vdots \\ 1 + \omega^{t_j} + \omega^{2t_j} + \cdots + \omega^{(n-2)t_j} \end{pmatrix},$$

the row vector $\mathbf{0}$ is a zero vector, and the scalar $\delta = 1 + \omega^{t_j} + \omega^{2t_j} + \cdots + \omega^{(n-1)t_j} \ne 0$. Similarly, the matrix $\begin{pmatrix} \mathbf{A_j} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix}$ is invertible with the inverse given by

$$\begin{pmatrix} \mathbf{A_j} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A_j^{-1}} & -\delta^{-1} \mathbf{A_j^{-1}} \mathbf{b} \\ \mathbf{0} & \delta^{-1} \end{pmatrix},$$

where $-\delta^{-1} \mathbf{A_j^{-1}} \mathbf{b}$ is a column vector with the k-th entry

$$-\delta^{-1} \sum_{s=1}^{n-1} (1 + \omega^{t_j} + \cdots + \omega^{(s-1)t_j}) \beta_{k,s}^{(j)},$$

for $1 \le k \le n-1$. Now the idempotents $\overline{E_{j,k}}$ for $j \in \Omega_1 \cup \Omega_2$ and $1 \le k \le n-1$ could be expressed as follows:

$$\overline{E_{j,k}} = \sum_{s=1}^{n-1} \beta_{k,s}^{(j)} \overline{e_j M[s,0]} - \delta^{-1} \sum_{s=1}^{n-1} (1 + \omega^{t_j} + \cdots + \omega^{(s-1)t_j}) \beta_{k,s}^{(j)} \overline{e_j M[n,0]}$$

$$= \sum_{s=1}^{n-1} \beta_{k,s}^{(j)} (\overline{e_j M[s,0]} - \frac{1 + \omega^{t_j} + \cdots + \omega^{(s-1)t_j}}{1 + \omega^{t_j} + \cdots + \omega^{(n-1)t_j}} \overline{e_j M[n,0]}) \tag{3.7}$$

$$= (\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} (\sqrt{\omega^{t_j}})^{1-s} \theta_{k,s} (\overline{e_j M[s,0]} - \frac{1 + \omega^{t_j} + \cdots + \omega^{(s-1)t_j}}{1 + \omega^{t_j} + \cdots + \omega^{(n-1)t_j}} \overline{e_j M[n,0]}).$$

The idempotents $\overline{E_{j,k}}$ for $j \in \Omega_1 \cup \Omega_2$ and $k = n$ could be expressed as follows:

$$\overline{E_{j,n}} = \delta^{-1} \overline{e_j M[n,0]} = \frac{1}{1 + \omega^{t_j} + \cdots + \omega^{(n-1)t_j}} \overline{e_j M[n,0]}. \tag{3.8}$$

3.4. Liftings of idempotents

We have obtained the primitive orthogonal idempotents $\overline{E_{j,k}}$ as a linear combination of a basis of $R(H)/J(R(H))$ as shown in (3.5), (3.7), and (3.8) for each case. In the following, we delete the upper bar in the equations (3.5), (3.7), and (3.8) and obtain the elements $E_{j,k}$ in $R(H)$ as follows:

- $E_{j,k} := (\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} (\sqrt{\omega^{t_j}})^{1-s} \theta_{k,s} e_j M[s,0]$, for $j \in \Omega_3$ and $1 \le k \le n-1$

- $E_{j,k} := (\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} (\sqrt{\omega^{t_j}})^{1-s} \theta_{k,s} e_j (M[s,0] - \frac{1+\omega^{t_j}+\cdots+\omega^{(s-1)t_j}}{1+\omega^{t_j}+\cdots+\omega^{(n-1)t_j}} M[n,0])$, for $j \in \Omega_1 \cup \Omega_2$ and $1 \le k \le n-1$.

- $E_{j,k} := \frac{1}{1+\omega^{t_j}+\cdots+\omega^{(n-1)t_j}} e_j M[n,0]$, for $j \in \Omega_1 \cup \Omega_2$ and $k=n$.

Now the idempotents of $R(H)$ can be described explicitly as follows.

Theorem 3.4 *Let $e_{j,k}$ be the idempotent of $R(H)$ satisfying $\overline{e_{j,k}} = \overline{E_{j,k}}$.*

(1) *If $j \in \Omega_1 \cup \Omega_2$, then $e_{j,k} = E_{j,k}$, for $1 \le k \le n$.*

(2) *If $j \in \Omega_3$, then $e_{j,k} = E_{j,k} + \gamma_{j,k} e_j M[n,0]$, for $1 \le k \le n-1$, where*

$$\gamma_{j,k} = (1 - 2\delta_{k,\frac{nt_j}{l}}) \frac{\alpha^{\frac{nt_j}{l}}(\alpha^k - \alpha^{-k})^2}{\alpha^{\frac{nt_j}{l}} - \alpha^{-\frac{nt_j}{l}}} \sum_{s+t-1 \ge n} \theta_{k,s} \theta_{k,t} (\alpha^{\frac{(s+t)nt_j}{l}} - \alpha^{-\frac{(s+t)nt_j}{l}}).$$

Proof (1) Note that $e_{j,k}$ is the idempotent of $R(H)$ such that $\overline{e_{j,k}} = \overline{E_{j,k}}$. It follows that $e_{j,k} - E_{j,k} \in J(R(H))$. For any $i \ne j$, we obtain that $e_i e_{j,k} \in e_i J(R(H)) \subseteq J(R(H))$ since $e_i(e_{j,k} - E_{j,k}) \in e_i J(R(H))$ and $e_i E_{j,k} = 0$. It follows that $e_i e_{j,k} = 0$ because $e_i e_{j,k}$ is an idempotent as well. Hence $e_{j,k}$ belongs to $e_j R(H)$. By Lemma 3.3 (2), we have that

$$e_{j,k} - E_{j,k} \in e_j R(H) \cap J(R(H)) = e_j J(R(H))$$

$$= \begin{cases} \mathrm{sp}\{e_j M[n,0]\}, & j \in \Omega_3; \\ 0, & j \in \Omega_1 \cup \Omega_2. \end{cases} \tag{3.9}$$

Therefore, Part (1) is proved.

(2) By (3.9), we denote by $e_{j,k} = E_{j,k} + \gamma_{j,k} e_j M[n,0]$ for $j \in \Omega_3$ and $\gamma_{j,k} \in \mathbb{C}$. Note that $(J(R(H)))^2 = 0$. We have

$$E_{j,k} + \gamma_{j,k} e_j M[n,0] = (E_{j,k} + \gamma_{j,k} e_j M[n,0])^2 = E_{j,k}^2 + 2\gamma_{j,k} e_j M[n,0] E_{j,k}.$$

This implies that

$$E_{j,k}^2 - E_{j,k} = \gamma_{j,k}(e_j M[n,0] - 2e_j M[n,0] E_{j,k}).$$

Note that $a = \sum_{j \in \Omega} \omega^{t_j} e_j$ and $e_j a = \omega^{t_j} e_j$. We have

$$e_j M[n,0] E_{j,k}$$

$$= (\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} (\sqrt{\omega^{t_j}})^{1-s} \theta_{k,s} e_j M[n,0] M[s,0]$$

$$= (\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} (\sqrt{\omega^{t_j}})^{1-s} \theta_{k,s} e_j (1 + a + \cdots + a^{s-1}) M[n,0] \tag{3.10}$$

$$= (\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} (\sqrt{\omega^{t_j}})^{1-s} \theta_{k,s} (1 + \omega^{t_j} + \cdots + \omega^{(s-1)t_j}) e_j M[n,0]$$

$$= (\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} (\sqrt{\omega^{t_j}})^{1-s} \theta_{k,s} \frac{\omega^{st_j} - 1}{\omega^{t_j} - 1} e_j M[n,0].$$

Note that $j \in \Omega_3$ implies that $\frac{l}{n} \mid t_j$. Suppose $t_j = \frac{l}{n} p$ for some integer p; then

$$\sqrt{\omega^{t_j}} = \cos \frac{t_j \pi}{l} + i \sin \frac{t_j \pi}{l} = \cos \frac{p\pi}{n} + i \sin \frac{p\pi}{n} = \alpha^p = \alpha^{\frac{nt_j}{l}}.$$

Now (3.10) can be written as

$$e_j M[n,0] E_{j,k} = (\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} \alpha^{\frac{(1-s)nt_j}{l}} \theta_{k,s} \frac{\alpha^{\frac{2snt_j}{l}} - 1}{\alpha^{\frac{2nt_j}{l}} - 1} e_j M[n,0]$$

$$= \frac{\alpha^k - \alpha^{-k}}{\alpha^{\frac{nt_j}{l}} - \alpha^{-\frac{nt_j}{l}}} e_j M[n,0] \sum_{s=1}^{n-1} \theta_{k,s} (\alpha^{\frac{snt_j}{l}} - \alpha^{-\frac{snt_j}{l}})$$

$$= \frac{\alpha^k - \alpha^{-k}}{\alpha^{\frac{nt_j}{l}} - \alpha^{-\frac{nt_j}{l}}} e_j M[n,0] \delta_{k,\frac{nt_j}{l}}$$

$$= \delta_{k,\frac{nt_j}{l}} e_j M[n,0];$$

here $\sum_{s=1}^{n-1} \theta_{k,s} (\alpha^{\frac{snt_j}{l}} - \alpha^{-\frac{snt_j}{l}}) = \delta_{k,\frac{nt_j}{l}}$ since $\mathbf{B}^{-1}\mathbf{B} = \mathbf{E}$, the identity matrix. Hence,

$$E_{j,k}^2 - E_{j,k} = \gamma_{j,k} (1 - 2\delta_{k,\frac{nt_j}{l}}) e_j M[n,0]. \tag{3.11}$$

The rest is to determine the coefficient of the term $e_j M[n,0]$ in $E_{j,k}^2 - E_{j,k}$. Note that $E_{j,k}$ has no term $e_j M[n,0]$. It suffices to consider the coefficient of the term $e_j M[n,0]$ in $E_{j,k}^2$. Note that

$$E_{j,k}^2 = ((\alpha^k - \alpha^{-k}) \sum_{s=1}^{n-1} (\sqrt{\omega^{t_j}})^{1-s} \theta_{k,s} e_j M[s,0])^2$$

$$= (\alpha^k - \alpha^{-k})^2 \sum_{s,t=1}^{n-1} (\sqrt{\omega^{t_j}})^{2-s-t} \theta_{k,s} \theta_{k,t} e_j M[s,0] M[t,0].$$

By [17, Proposition 4.2], the term $e_j M[n,0]$ appears in $e_j M[s,0]M[t,0]$ if and only if $s+t-1 \geq n$. In this case, it is straightforward to check that the term $e_j M[n,0]$ in $e_j M[s,0]M[t,0]$ is

$$\sum_{s+t-1 \geq n} e_j \left(\sum_{q=0}^{s+t-1-n} a^q \right) M[n,0] = \sum_{s+t-1 \geq n} \left(\sum_{q=0}^{s+t-1-n} \omega^{qt_j} \right) e_j M[n,0]$$

$$= \sum_{s+t-1 \geq n} \frac{1 - \omega^{(s+t-n)t_j}}{1 - \omega^{t_j}} e_j M[n,0] = \sum_{s+t-1 \geq n} \frac{1 - \omega^{(s+t)t_j}}{1 - \omega^{t_j}} e_j M[n,0].$$

We conclude that the coefficient of the term $e_j M[n,0]$ in $E_{j,k}^2$ is

$$(\alpha^k - \alpha^{-k})^2 \sum_{s+t-1 \geq n} (\sqrt{\omega^{t_j}})^{2-s-t} \theta_{k,s} \theta_{k,t} \frac{1 - \omega^{(s+t)t_j}}{1 - \omega^{t_j}}$$

$$= \frac{\alpha^{\frac{n t_j}{l}} (\alpha^k - \alpha^{-k})^2}{\alpha^{\frac{n t_j}{l}} - \alpha^{-\frac{n t_j}{l}}} \sum_{s+t-1 \geq n} \theta_{k,s} \theta_{k,t} \left(\alpha^{\frac{(s+t)n t_j}{l}} - \alpha^{-\frac{(s+t)n t_j}{l}} \right)$$

since $\sqrt{\omega^{t_j}} = \alpha^{\frac{n t_j}{l}}$. Comparing the scalars of the equation (3.11), we conclude that

$$\frac{\alpha^{\frac{n t_j}{l}} (\alpha^k - \alpha^{-k})^2}{\alpha^{\frac{n t_j}{l}} - \alpha^{-\frac{n t_j}{l}}} \sum_{s+t-1 \geq n} \theta_{k,s} \theta_{k,t} \left(\alpha^{\frac{(s+t)n t_j}{l}} - \alpha^{-\frac{(s+t)n t_j}{l}} \right) = \gamma_{j,k} \left(1 - 2\delta_{k, \frac{n t_j}{l}} \right).$$

Therefore,

$$\gamma_{j,k} = \left(1 - 2\delta_{k, \frac{n t_j}{l}} \right) \frac{\alpha^{\frac{n t_j}{l}} (\alpha^k - \alpha^{-k})^2}{\alpha^{\frac{n t_j}{l}} - \alpha^{-\frac{n t_j}{l}}} \sum_{s+t-1 \geq n} \theta_{k,s} \theta_{k,t} \left(\alpha^{\frac{(s+t)n t_j}{l}} - \alpha^{-\frac{(s+t)n t_j}{l}} \right).$$

We complete the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

4. An example

As an example, we shall determine all primitive idempotents of the Green algebra of Taft algebra T_3.

4.1. Taft algebra T_3

Let $\alpha = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$ and $\omega = \alpha^2$. Then ω is a primitive 3rd root of unity. The Taft algebra T_3 is generated over the ground field $\mathbb{C}$ by two elements g and y subject to the relations (cf. [2, 15])

$$g^3 = 1, \; y^3 = 0, \; yg = \omega g y.$$

T_3 is a Hopf algebra with comultiplication $\triangle$, counit ε, and the antipode S given respectively by

$$\triangle(y) = y \otimes g + 1 \otimes y, \; \varepsilon(y) = 0, \; S(y) = -yg^{-1},$$

$$\triangle(g) = g \otimes g, \; \varepsilon(g) = 1, \; S(g) = g^{-1}.$$

Note that $\dim T_3 = 9$ and $\{g^i y^j \mid 0 \leq i,j \leq 2\}$ forms a $\mathbb{C}$-basis for T_3. Let G be the cyclic group generated by g and χ a $\mathbb{C}$-linear character of G such that $\chi(g) = \omega$. Then T_3 is the pointed rank one Hopf algebra associated to the group datum (G, χ, g) and $\{M(j,i) \mid i \in \Omega, 1 \leq j \leq 3\}$ forms a complete set of indecomposable T_3-modules up to isomorphism, where $\Omega = \{0,1,2\}$.

The Green ring $r(T_3)$ of T_3 is commutative with a $\mathbb{Z}$-basis $M[j,i]$ for $0 \leq i \leq 2$ and $1 \leq j \leq 3$. Denote by a one of $M[1,i]$ for $i \in \Omega$ such that the character of $M(1,i)$ as a simple $\mathbb{C}G$-module is χ^{-1}. The multiplication formulas of Green ring $r(T_3)$ is stated as follows:

$$a^3 = 1, \ M[j,i] = a^i M[j,0]$$

$$M[2,0]M[2,0] = a + M[3,0],$$

$$M[2,0]M[3,0] = (1+a)M[3,0],$$

$$M[3,0]M[3,0] = (1+a+a^2)M[3,0].$$

By Theorem 2.2, the Green ring $r(T_3)$ is isomorphic to the quotient ring

$$\mathbb{Z}[a,z]/(a^3 - 1, (1 + a - z)F_3(a,z))$$

whose idempotents are trivial. Let $R(T_3)$ be the complexified Green ring. That is, $R(T_3)$ is isomorphic to the algebra $\mathbb{C}[a,z]/(a^3 - 1, (1 + a - z)F_3(a,z))$. In the following, we follow the notations given in Section 3 and determine all primitive idempotents of $R(T_3)$.

4.2. Idempotents of the Green algebra $R(T_3)$

Let $R(\mathbb{C}G)$ be the complexified Green ring of the group algebra $\mathbb{C}G$. Then $R(\mathbb{C}G)$ is isomorphic to $\mathbb{C}[a]/(a^3-1)$, which is a subalgebra of $R(T_3)$. It is obvious that all primitive idempotents of $\mathbb{C}[a]/(a^3 - 1)$ are

$$e_j = \frac{1}{3}(1 + \omega^{-j}a + \omega^{-2j}a^2),$$

for $0 \leq j \leq 2$, see e.g. [16, Equation (2.1)]. It follows that $a = e_0 + \omega e_1 + \omega^2 e_2$. Let W_j for $0 \leq j \leq 2$ be all (one dimensional) simple modules over $R(\mathbb{C}G)$ such that the generator a acts on W_j is a scalar multiple by ω^j (i.e. $t_j = j$ in this case). Then the subsets Ω_1, Ω_2, and Ω_3 of Ω given respectively in Section 3 become $\Omega_1 = \{0\}$, $\Omega_2 = \emptyset$, and $\Omega_3 = \{1,2\}$. Let $W_{i,j}$ be the same as W_j as a simple $R(\mathbb{C}G)$-module while the generator z acts on it as the scalar multiple by $\alpha^i(\alpha^j + \alpha^{-j})$ for $0 \leq i \leq 2$ and $1 \leq j \leq 2$. Moreover, let $W_{0,3}$ be W_0 as an $R(\mathbb{C}G)$-module and z acts on $W_{0,3}$ as the scalar multiple by 2. Then $\{W_{i,j} \mid 0 \leq i \leq 2, 1 \leq j \leq 2\} \cup \{W_{0,3}\}$ forms all simple $R(T_3)$-modules up to isomorphism.

Now the matrices $\mathbf{B}$, $\mathbf{C_j}$ for $0 \leq j \leq 2$ and $\mathbf{D}$ given in Section 3 can be written directly as follows:

$$\mathbf{B} = (\alpha - \alpha^{-1})\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{C_j} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^j \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} \alpha - \alpha^{-1} & 0 \\ 0 & \alpha^2 - \alpha^{-2} \end{pmatrix}.$$

It follows that

$$\mathbf{A_j} = \mathbf{C_j}\mathbf{B}\mathbf{D^{-1}} = \begin{pmatrix} 1 & 1 \\ \alpha^j & -\alpha^j \end{pmatrix},$$

Let $E_{i,j}$ be in $R(T_3)$ such that $\overline{E_{ij}}$ is an idempotent of $R(T_3)/J(R(T_3))$ determined by the simple module $W_{i,j}$, namely, $\overline{E_{i,j}} \cdot W_{k,l} = \delta_{i,k}\delta_{j,l}W_{k,l}$. Then the equations (3.4) and (3.6) become the following:

$$\begin{pmatrix} \overline{e_jM[1,0]} \\ \overline{e_jM[2,0]} \end{pmatrix} = \mathbf{A_j} \begin{pmatrix} \overline{E_{j,1}} \\ \overline{E_{j,2}} \end{pmatrix}, \tag{4.1}$$

for $1 \le j \le 2$, and

$$\begin{pmatrix} \overline{e_0M[1,0]} \\ \overline{e_0M[2,0]} \\ \overline{e_0M[3,0]} \end{pmatrix} = \begin{pmatrix} \mathbf{A_0} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix} \begin{pmatrix} \overline{E_{0,1}} \\ \overline{E_{0,2}} \\ \overline{E_{0,3}} \end{pmatrix}, \tag{4.2}$$

where $\mathbf{b}$ is the column vector $\binom{1}{2}$ and $\delta = 3$. Since $\mathbf{A}_j$ and $\begin{pmatrix} \mathbf{A_0} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix}$ are both invertible with the inverse matrices given respectively by

$$\mathbf{A_j^{-1}} = \frac{1}{2}\begin{pmatrix} 1 & \alpha^{-j} \\ 1 & -\alpha^{-j} \end{pmatrix},$$

for $0 \le j \le 2$, and

$$\begin{pmatrix} \mathbf{A_0} & \mathbf{b} \\ \mathbf{0} & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A_0^{-1}} & -\frac{1}{3}\mathbf{A_0^{-1}b} \\ \mathbf{0} & \frac{1}{3} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 3 & 3 & -3 \\ 3 & -3 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

In view of this, all primitive idempotents of $R(T_3)/J(R(T_3))$ are completely determined by (4.1) and (4.2), and they can be expressed explicitly as follows:

- $\overline{E_{j,1}} = \frac{1}{2}\overline{e_jM[1,0]} + \frac{\alpha^{-j}}{2}\overline{e_jM[2,0]}$, for $1 \le j \le 2$,

- $\overline{E_{j,2}} = \frac{1}{2}\overline{e_jM[1,0]} - \frac{\alpha^{-j}}{2}\overline{e_jM[2,0]}$, for $1 \le j \le 2$,

- $\overline{E_{0,1}} = \frac{1}{2}\overline{e_0M[1,0]} + \frac{1}{2}\overline{e_0M[2,0]} - \frac{1}{2}\overline{e_0M[3,0]}$,

- $\overline{E_{0,2}} = \frac{1}{2}\overline{e_0M[1,0]} - \frac{1}{2}\overline{e_0M[2,0]} + \frac{1}{6}\overline{e_0M[3,0]}$,

- $\overline{E_{0,3}} = \frac{1}{3}\overline{e_0M[3,0]}$.

In the following, we shall lift the idempotents $\overline{E_{i,j}}$ of the quotient algebra $R(T_3)/J(R(T_3))$ to the Green algebra $R(T_3)$. We first delete the upper bar in the above equations and obtain the element $E_{j,k}$ in $R(H)$ as follows:

$$E_{j,1} := e_j(\frac{1}{2}M[1,0] + \frac{\alpha^{-j}}{2}M[2,0]), \text{ for } 1 \le j \le 2,$$

$$E_{j,2} := e_j(\frac{1}{2}M[1,0] - \frac{\alpha^{-j}}{2}M[2,0]), \text{ for } 1 \le j \le 2,$$

$$E_{0,1} := e_0(\frac{1}{2}M[1,0] + \frac{1}{2}M[2,0] - \frac{1}{2}M[3,0]),$$

$$E_{0,2} := e_0(\frac{1}{2}M[1,0] - \frac{1}{2}M[2,0] + \frac{1}{6}M[3,0]),$$

$$E_{0,3} := \frac{1}{3} e_0 M[3,0].$$

We need to compute the scalar $\gamma_{j,k}$ described in Theorem 3.4. Note that the (k,s)-entry of the matrix $\mathbf{B}^{-1}$ is $\theta_{k,s}$. Then

$$\begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{2,1} & \theta_{2,2} \end{pmatrix} = \mathbf{B}^{-1} = \frac{1}{2(\alpha - \alpha^{-1})} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Now by Theorem 3.4, for $1 \le j, k \le 2$, we have

$$\gamma_{j,k} = (1 - 2\delta_{k,j}) \frac{\alpha^j (\alpha^k - \alpha^{-k})^2}{\alpha^j - \alpha^{-j}} \sum_{s+t-1 \ge 3} \theta_{k,s} \theta_{k,t} (\alpha^{(s+t)j} - \alpha^{-(s+t)j})$$

$$= (1 - 2\delta_{k,j}) \frac{\alpha^j (\alpha^k - \alpha^{-k})^2}{\alpha^j - \alpha^{-j}} \theta_{k,2} \theta_{k,2} (\alpha^{4j} - \alpha^{-4j})$$

$$= (1 - 2\delta_{k,j}) \frac{\alpha^j (\alpha^k - \alpha^{-k})^2}{\alpha^j - \alpha^{-j}} \left(\frac{1}{2(\alpha - \alpha^{-1})} \right)^2 (\alpha^{4j} - \alpha^{-4j})$$

$$= \begin{cases} \frac{\alpha}{4}, & (j,k) = (1,1) \\ -\frac{\alpha}{4}, & (j,k) = (1,2) \\ \frac{\alpha^2}{4}, & (j,k) = (2,1) \\ -\frac{\alpha^2}{4}, & (j,k) = (2,2) \end{cases}$$

$$= (-1)^{k-1} \frac{\alpha^j}{4}.$$

It follows from Theorem 3.4 that all primitive idempotents $e_{i,j}$ of $R(T_3)$ can be presented explicitly as follows:

- $e_{j,1} = E_{j,1} + \gamma_{j,1} e_j M[3,0] = e_j (\frac{1}{2} M[1,0] + \frac{\alpha^{-j}}{2} M[2,0] + \frac{\alpha^j}{4} M[3,0])$, for $1 \le j \le 2$,

- $e_{j,2} = E_{j,2} + \gamma_{j,2} e_j M[3,0] = e_j (\frac{1}{2} M[1,0] - \frac{\alpha^{-j}}{2} M[2,0] - \frac{\alpha^j}{4} M[3,0])$, for $1 \le j \le 2$,

- $e_{0,1} = E_{0,1} = e_0 (\frac{1}{2} M[1,0] + \frac{1}{2} M[2,0] - \frac{1}{2} M[3,0])$,

- $e_{0,2} = E_{0,2} = e_0 (\frac{1}{2} M[1,0] - \frac{1}{2} M[2,0] + \frac{1}{6} M[3,0])$,

- $e_{0,3} = E_{0,3} = \frac{1}{3} e_0 M[3,0]$.

For instance, to see that $e_{j,2}^2 = e_{j,2}$, by using the equalities $e_j a = \omega^j e_j = \alpha^{2j} e_j$ and $1 + \alpha^{2j} + \alpha^{4j} = 0$ for

$1 \leq j \leq 2$, we have that

$$
\begin{aligned}
e_{j,2}^2 &= e_j^2 (\frac{1}{2} M[1,0] - \frac{\alpha^{-j}}{2} M[2,0] - \frac{\alpha^j}{4} M[3,0])^2 \\
&= e_j(\frac{1 + \alpha^{-2j} a}{4} M[1,0] - \frac{\alpha^{-j}}{2} M[2,0]) \\
&\quad + e_j((\frac{\alpha^{-2j} - \alpha^j + 1 + a}{4} + \frac{\alpha^{2j}(1 + a + a^2)}{16}) M[3,0]) \\
&= e_j(\frac{1}{2} M[1,0] - \frac{\alpha^{-j}}{2} M[2,0]) \\
&\quad + e_j((\frac{\alpha^{-2j} - \alpha^j + 1 + \alpha^{2j}}{4} + \frac{\alpha^{2j}(1 + \alpha^{2j} + \alpha^{4j})}{16}) M[3,0]) \\
&= e_j(\frac{1}{2} M[1,0] - \frac{\alpha^{-j}}{2} M[2,0] - \frac{\alpha^j}{4} M[3,0]) \\
&= e_{j,2}.
\end{aligned}
$$

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