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Existence of positive solutions for difference systems coming from a model for burglary

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Abstract: In this paper, we use the Brouwer degree to prove existence results of positive solutions for the following difference systems:

$$D_k\Delta^2(A_{k-1}-A^0_{k-1})-(A_k-A^0_k)+N_kf(k,A_k)=0,\quad k\in[2,n-1]_{\mathbb{Z}},$$
$$\Delta^2N_{k-1}+\Delta[g(k,A_k,\Delta A_{k-1})N_k]-w^2(N_k-1)=0,\quad k\in[2,n-1]_{\mathbb{Z}},$$
$$\Delta A_1=0=\Delta A_{n-1},\quad \Delta N_1=0=\Delta N_{n-1},$$

where the assumptions on w, D_k, A^0_k, f, and g are motivated by some mathematical models for the burglary of houses.

Key words: Neumann problems, Brouwer degree, positive solution, models for house burglary

1. Introduction

Let $\mathbb{Z}$ denote the integer set for $a,b\in\mathbb{Z}$ with $a<b$, $[a,b]_{\mathbb{Z}}:=\{a,a+1,\cdots,b\}$. Let $\mathbb{R}^+=\{x\in\mathbb{R}:x>0\}$, $\mathbb{R}^+_0=\{x\in\mathbb{R}:x\geq 0\}$.

Due to wide applications in many fields such as science, economics, neural networks, ecology, and cybernetics, the theory of nonlinear difference equations has been widely studied since the 1970s; see, for example, [1,9]. At the same time, boundary value problems of difference equations have received much attention from many authors; see [1,2,3,5,9–11,17] and the references therein.

In this paper, we are concerned with the existence of positive solutions of the following difference systems:

$$\begin{aligned}&D_k\Delta^2(A_{k-1}-A^0_{k-1})-(A_k-A^0_k)+N_kf(k,A_k)=0,\quad k\in[2,n-1]_{\mathbb{Z}},\\&\Delta^2N_{k-1}+\Delta[g(k,A_k,\Delta A_{k-1})N_k]-w^2(N_k-1)=0,\quad k\in[2,n-1]_{\mathbb{Z}},\\&\Delta A_1=0=\Delta A_{n-1},\quad \Delta N_1=0=\Delta N_{n-1},\end{aligned}\tag{1}$$

where $w>0$ is a constant, $\mathbf{D}=(D_2,\cdots,D_{n-1})\in\mathbb{R}^{n-2}$ and $D_k>0$, $k\in[2,n-1]_{\mathbb{Z}}$; $\mathbf{A}^0=(A^0_1,\cdots,A^0_n)\in\mathbb{R}^n$ and $A^0_k>0$, $k\in[1,n]_{\mathbb{Z}}$; $f:[2,n-1]_{\mathbb{Z}}\times\mathbb{R}^+_0\to\mathbb{R}$ and $g:[2,n]_{\mathbb{Z}}\times\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

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A solution of (1) is a couple of real vector functions $(\mathbf{A}, \mathbf{N}) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying the system and the boundary conditions. We are interested in positive solutions of this problem, i.e. in solutions $(\mathbf{A}, \mathbf{N})$ such that $A_k > 0$ and $N_k \geq 0$ with $N_k \not\equiv 0$ for all $k \in [1, n]_{\mathbb{Z}}$.

This problem is motivated by the following differential system:

$$\begin{aligned} &D(x)(A - A^0(x))'' - A + A^0(x) + Nf(x, A) = 0, \quad x \in (0, L), \\ &N'' + [g(x, A, A')N]' - w^2(N - 1) = 0, \quad x \in (0, L), \\ &A'(0) = 0 = A'(L), \quad N'(0) = 0 = N'(L), \end{aligned} \tag{2}$$

where $L > 0$; see [8]. In fact, system (2) with $f = \psi(x)A(1 - A)$ (here ψ is a positive continuous function in $[0, L]$) and $g = -\frac{2A'}{A}$ is a one-dimensional version of a problem that arises in the pioneering work of [4,6,14,15] where a very successful model for burglary of houses was obtained by Short et al. See also the related papers [4,6,14,15]. In most of these models, D represents a measure of the degree of spreading of the attractiveness generated by any given burglary event, A^0 the static component of attractiveness, A the attractiveness for a house to be burgled, and N the density of burglars. In addition, $w = \frac{w_2}{w_1}$, where w_2 and w_1 are the mean lifetime of dynamic attractiveness and an active burglar, respectively. Thus, in the discrete case, the restrictions $A_k > 0$ and $N_k \geq 0$ with $N_k \not\equiv 0$ for all $k \in [1, n]_{\mathbb{Z}}$ appear as natural.

However, the discrete analogue of systems (2) has received almost no attention. In this article, we will discuss it in detail. We assume that the following conditions are satisfied:

(H1) $g(2, y, 0) = 0, \quad g(n, y, 0) = 0$ for all $y \in \mathbb{R}^+$.

(H2) There exists $R > osc\mathbf{A}^0$ such that for all $k \in [2, n-1]_{\mathbb{Z}}$,

$$f(k, A_k) \geq 0, \quad 0 \leq A_k \leq R \quad \text{and} \quad f(k, A_k) \leq 0, \quad A_k \geq R.$$

Here $osc\mathbf{A}^0 = \max_{k \in [1,n]_{\mathbb{Z}}} A_k^0 - \min_{k \in [1,n]_{\mathbb{Z}}} A_k^0$.

(H3) Let $\mathbf{A}^0 - \mathbf{D}\Delta^2\mathbf{A}^0$ be a positive function and there exists $R > 0$ such that for all $k \in [2, n-1]_{\mathbb{Z}}$,

$$f(k, A_k) \geq 0, \quad 0 \leq A_k \leq R \quad \text{and} \quad f(k, A_k) \leq 0, \quad A_k \geq R.$$

Our main result for systems (1) is:

Theorem 1 Assume that (H1)–(H2) or (H1) and (H3) hold. Then the systems (1) have at least one positive solution.

The purpose of this paper is to show that analogues of the existence results of solutions for differential problems proved in [8] hold for the corresponding difference systems. However, some basic ideas from differential calculus are not necessarily available in the field of difference equations, such as the intermediate value theorem, the mean value theorem, and Rolle's theorem. Thus, new challenges are faced and innovation is required. The proof is elementary and relies on Brouwer degree theory [7,12].

The paper is organized as follows. In Section 2 we establish important a priori estimates. Section 3 introduces the associated linear operators. Finally, Section 4 contains the proof of the main result and its applications.

We end this section with some notations. Let $n \in \mathbb{N}, n \geq 4$ be fixed and $(x_1, x_2, \cdots, x_{n-1}, x_n)$

$\in \mathbb{R}^n$. Define $(\Delta x_1, \cdots, \Delta x_{n-1}) \in \mathbb{R}^{n-1}$ and $(\Delta^2 x_1, \cdots, \Delta^2 x_{n-2}) \in \mathbb{R}^{n-2}$ by

$$\Delta x_m = x_{m+1} - x_m,\ m \in [1, n-1]_{\mathbb{Z}}$$

and

$$\Delta^2 x_{m-1} = x_{m+1} - 2x_m + x_{m-1},\ m \in [2, n-1]_{\mathbb{Z}}.$$

Let us introduce the vector space

$$V^{n-2} = \{\mathbf{x} \in \mathbb{R}^n : \Delta x_1 = 0 = \Delta x_{n-1}\} \tag{3}$$

endowed with the orientation of $\mathbb{R}^n$. Its elements can be associated to the coordinates $(x_2, \cdots, x_{n-1})$ and correspond to the elements of $\mathbb{R}^n$ of the form

$$(x_2, x_2, \cdots, x_{n-1}, x_{n-1}).$$

We use the norm $\|\mathbf{x}\| := \max_{k\in[2,n-1]_{\mathbb{Z}}} |x_k|$ in V^{n-2}, and $\max_{k\in[1,n-2]_{\mathbb{Z}}} |x_k|$ in $\mathbb{R}^{n-2}$.

2. The a priori estimates

In order to use Brouwer degree theory to study systems (1), we first introduce the homotopy corresponding to the systems (1) for $\lambda \in [0, 1]$,

$$-D_k\Delta^2(A_{k-1} - A^0_{k-1}) + A_k - A^0_k = \lambda N_k f(k, A_k),\quad k \in [2, n-1]_{\mathbb{Z}},\quad \Delta A_1 = 0 = \Delta A_{n-1}, \tag{4}$$

$$-\Delta^2 N_{k-1} + w^2 N_k = w^2 + \lambda\Delta[g(k, A_k, \Delta A_{k-1})N_k],\quad k \in [2, n-1]_{\mathbb{Z}},\quad \Delta N_1 = 0 = \Delta N_{n-1}. \tag{5}$$

In fact, for $\lambda = 1$, (4)–(5) reduces to (1), and for $\lambda = 0$, (4)–(5) reduces to the nonhomogeneous decoupled linear system

$$-D_k\Delta^2(A_{k-1} - A^0_{k-1}) + A_k - A^0_k = 0,\quad k \in [2, n-1]_{\mathbb{Z}},\quad \Delta A_1 = 0 = \Delta A_{n-1}, \tag{6}$$

$$-\Delta^2 N_{k-1} + w^2 N_k = w^2,\quad k \in [2, n-1]_{\mathbb{Z}},\quad \Delta N_1 = 0 = \Delta N_{n-1}. \tag{7}$$

For convenience, we write, for all $\mathbf{B} \in \mathbb{R}^p$,

$$\min \mathbf{B} := \min_{k\in[1,p]_{\mathbb{Z}}} B_k \quad\text{and}\quad \max \mathbf{B} := \max_{k\in[1,p]_{\mathbb{Z}}} B_k.$$

Lemma 1 Let $(\mathbf{A}, \mathbf{N})$ be any possible solution of (4)–(5) for some $\lambda \in [0, 1]$. Then

$$\sum_{k=2}^{n-1} N_k = n - 2. \tag{8}$$

Proof. Summing the equation of (5) from $k = 2$ to $n-1$, and combining (H1) with Neumann boundary conditions, we have

$$\sum_{k=2}^{n-1} \Delta^2 N_{k-1} = 0$$

and

$$\sum_{k=2}^{n-1}\Delta[g(k,A_k,\Delta A_{k-1})N_k]=\sum_{k=2}^{n-1}[g(k+1,A_{k+1},\Delta A_k)N_{k+1}-g(k,A_k,\Delta A_{k-1})N_k]=0.$$

Thus, $w^2\sum_{k=2}^{n-1}N_k=w^2\sum_{k=2}^{n-1}1=(n-2)w^2$, and hence (8) holds. □

Lemma 2 Assume that (H2) holds. Let $(\mathbf{A},\mathbf{N})$ be any possible positive solution of (4)–(5) for some $\lambda\in[0,1]$. Then for all $k\in[2,n-1]_{\mathbb{Z}}$,

$$0<\underline{A}_{0,1}:=\min\{\min\mathbf{A^0},\ R-\mathrm{osc}\mathbf{A}^0\}\le A_k\le\max\{\max\mathbf{A^0},\ R+\mathrm{osc}\mathbf{A}^0\}=:\overline{A}_{1,1}. \tag{9}$$

Proof. Let $(\mathbf{A},\mathbf{N})$ be a possible positive solution of (4)–(5) for some $\lambda\in[0,1]$. Suppose that there exists $j\in[2,n-1]_{\mathbb{Z}}$ such that $A_j-A_j^0\ge R-\min\mathbf{A}^0$, namely $A_j\ge R-\min\mathbf{A}^0+A_j^0\ge R$. From the assumption (H2), it is deduced that

$$0\ge D_j\Delta^2(A_{j-1}-A_{j-1}^0)=A_j-A_j^0-\lambda N_jf(j,A_j)\ge A_j-A_j^0.$$

Then for all $k\in[2,n-1]_{\mathbb{Z}}$,

$$A_k-A_k^0\le A_j-A_j^0\le 0,$$

and hence

$$\max\mathbf{A}\le\max\mathbf{A}^0.$$

Similarly, suppose that there exists $j\in[2,n-1]_{\mathbb{Z}}$ such that $A_j-A_j^0\le R-\max\mathbf{A}^0$, namely $A_j\le R-(\max\mathbf{A}^0-A_j^0)\le R$. From the assumption (H2), it deduces that

$$0\le D_j\Delta^2(A_{j-1}-A_{j-1}^0)=A_j-A_j^0-\lambda N_jf(j,A_j)\le A_j-A_j^0.$$

Then for all $k\in[2,n-1]_{\mathbb{Z}}$,

$$0\le A_k-A_k^0\le A_j-A_j^0,$$

and hence

$$\min\mathbf{A}\ge\min\mathbf{A}^0.$$

Consequently, the result follows easily. □

Lemma 3 Assume that (H3) holds. Let $(\mathbf{A},\mathbf{N})$ be any possible positive solution of (4)–(5) for some $\lambda\in[0,1]$. Then for all $k\in[2,n-1]_{\mathbb{Z}}$,

$$0<\underline{A}_{0,2}:=\min\{\min(\mathbf{A}^0-\mathbf{D}\Delta^2\mathbf{A}^0),\ R\}\le A_k\le\max\{\max(\mathbf{A}^0-\mathbf{D}\Delta^2\mathbf{A}^0),\ R\}=:\overline{A}_{1,2}. \tag{10}$$

Proof. Let $(\mathbf{A},\mathbf{N})$ be a possible positive solution of (4)–(5) for some $\lambda\in[0,1]$. Suppose that there exists $j\in[2,n-1]_{\mathbb{Z}}$ such that $A_j\ge R$. Then, by virtue of (H3), we obtain that

$$0\ge D_j\Delta^2A_{j-1}=D_j\Delta^2A_{j-1}^0+A_j-A_j^0-\lambda N_jf(j,A_j)\ge D_j\Delta^2A_{j-1}^0+A_j-A_j^0,$$

and so

$$\max\mathbf{A}=A_j\le A_j^0-D_j\Delta^2A_{j-1}^0\le\max(\mathbf{A}^0-\mathbf{D}\Delta^2\mathbf{A}^0).$$

Similarly, suppose that there exists $j \in [2, n-1]_{\mathbb{Z}}$ such that $A_j \leq R$. Then, using the assumption (H3),

$$0 \leq D_j \Delta^2 A_{j-1} = D_j \Delta^2 A^0_{j-1} + A_j - A^0_j - \lambda N_j f(j, A_j) \leq D_j \Delta^2 A^0_{j-1} + A_j - A^0_j,$$

and so

$$\min \mathbf{A} = A_j \geq A^0_j - D_j \Delta^2 A^0_{j-1} \geq \min (\mathbf{A}^0 - \mathbf{D}\Delta^2 \mathbf{A}^0).$$

Consequently, the result follows easily.

□

From now on, we respectively write $\underline{A}$ and $\overline{A}$ for $\underline{A}_{0,1}$ and $\overline{A}_{1,1}$ or $\underline{A}_{0,2}$ and $\overline{A}_{1,2}$ depending on the assumption made on f. Set $\max |f| := \max_{[2,n-1]_{\mathbb{Z}} \times [\underline{A}, \overline{A}]} |f|$. Moreover, there exists $M_1 > 0$ (depending on $\underline{A}$ and $\overline{A}$) such that $-M_1 \leq \Delta A_k \leq M_1$, $k \in [1, n-1]_{\mathbb{Z}}$ since $\underline{A} \leq A_k \leq \overline{A}$. However, it is also interesting that we give another a priori bound of ΔA_k.

Lemma 4 Let $(\mathbf{A}, \mathbf{N})$ be any possible positive solution of (4)–(5) for some $\lambda \in [0, 1]$. Then for all $k \in [2, n-1]_{\mathbb{Z}}$,

$$|\Delta A_k| \leq (n-2) \max \Delta^2 \mathbf{A}^0 + \frac{n-2}{\min \mathbf{D}} [A_1 + \max \mathbf{A}^0 + \max |f|] := M_2. \tag{11}$$

Proof. Let $(\mathbf{A}, \mathbf{N})$ be a possible positive solution of (4)–(5) for some $\lambda \in [0, 1]$; it follows from (4) that, for all $k \in [2, n-1]_{\mathbb{Z}}$,

$$|\Delta^2 A_{k-1}| \leq |\Delta^2 A^0_{k-1}| + \frac{1}{D_k} [A_k + A^0_k + N_k |f(k, A_k)|],$$

and hence, using the Neumann boundary conditions, (8) as well as (9) or (10),

$$|\Delta A_k| = |\sum_{j=2}^{k} \Delta^2 A_{k-1}| \leq (n-2) \max \Delta^2 \mathbf{A}^0 + \frac{n-2}{\min \mathbf{D}} [\overline{A} + \max \mathbf{A}^0 + \max |f|].$$

□

Lemma 5 Let $(\mathbf{A}, \mathbf{N})$ be any possible positive solution of (4)–(5) for some $\lambda \in [0, 1]$. Then for all $k \in [2, n-1]_{\mathbb{Z}}$,

$$N_k \leq 1 + 2w^2 (n-2)^2 + (n-2) \max_{[2,n-1]_{\mathbb{Z}} \times [\underline{A}, \overline{A}] \times [-M, M]} |g| := \overline{N}, \tag{12}$$

where $M := \max\{M_1,\ M_2\}$.

Proof. Summing the equation of (5), and using the boundary conditions and (H1), we get, for all $k \in [2, n-1]_{\mathbb{Z}}$,

$$\Delta N_k = w^2 \sum_{j=2}^{k} N_j - w^2 (k-1) - \lambda \sum_{j=2}^{k} \Delta [g(j, A_j, \Delta A_{j-1}) N_j],$$

and by using (8), we have

$$|\Delta N_k| \leq 2w^2 (n-2) + |g(k+1, A_{k+1}, \Delta A_k)| N_{k+1}. \tag{13}$$

From Lemma 1, it is deduced that there exists $j \in [2, n-1]_{\mathbb{Z}}$ such that $N_j \leq 1$. Combining this with (13), for all $k \in [2, n-1]_{\mathbb{Z}}$, we have

$$N_k - 1 \leq N_k - N_j = \sum_{l=j}^{k-1} \Delta N_l \leq \sum_{l=j}^{k-1} |\Delta N_l| \leq 2w^2 (n-2)^2 + (n-2) \max_{[2,n-1]_{\mathbb{Z}} \times [\underline{A}, \overline{A}] \times [-M, M]} |g|,$$

and so (12) holds.

□

3. The associated linear operators

Lemma 6 For each function $\mathbf{h} = (h_2, \cdots, h_{n-1})$, the problem

$$-D_k\Delta^2 A_{k-1} + A_k = h_k, \quad k \in [2, n-1]_{\mathbb{Z}}, \quad \Delta A_1 = 0 = \Delta A_{n-1} \tag{14}$$

has a unique solution. Moreover, if $h_k > 0$ for all $k \in [2, n-1]_{\mathbb{Z}}$, then $A_k > 0, k \in [1, n]_{\mathbb{Z}}$.

Proof. Let us consider the homogeneous problem

$$-D_k\Delta^2 B_{k-1} + B_k = 0, \quad k \in [2, n-1]_{\mathbb{Z}}, \quad \Delta B_1 = 0 = \Delta B_{n-1}. \tag{15}$$

If $\mathbf{B}$ has a positive maximum at some $j \in [2, n-1]_{\mathbb{Z}}$, then

$$0 \geq D_j\Delta^2 B_{j-1} = B_j > 0,$$

a contradiction. Similarly, if $\mathbf{B}$ has a negative maximum at some $j \in [2, n-1]_{\mathbb{Z}}$, we can also get a contradiction. Thus, $\mathbf{B} \equiv 0$ is a unique solution of (15). Since the homogeneous problem only has the trivial solution, the standard linear theory implies that (14) has a unique solution.

Moreover, if $h_k > 0$ for all $k \in [2, n-1]_{\mathbb{Z}}$ and $\mathbf{A}$ has a nonpositive minimum at some $j \in [2, n-1]_{\mathbb{Z}}$, then $\Delta^2 A_{j-1} \geq 0$, and

$$0 \leq D_j\Delta^2 A_{j-1} = A_j - h_j < 0,$$

a contradiction. Consequently, $A_k > 0, k \in [1, n]_{\mathbb{Z}}$. □

From Lemma 6, for each function $\mathbf{h} = (h_2, \cdots, h_{n-1})$, let us define the linear operator

$$K : \mathbb{R}^{n-2} \to V^{n-2}, \tag{16}$$

such that $\mathbf{A} = K\mathbf{h}$ is the unique solution of (14).

Let u and v be unique solutions of initial value problems

$$-\Delta^2 u_{k-1} + w^2 u_k = 0, \quad k \in [2, n-1]_{\mathbb{Z}}, \quad u_{n-1} = 1, \ \Delta u_{n-1} = 0 \tag{17}$$

and

$$-\Delta^2 v_{k-1} + w^2 v_k = 0, \quad k \in [2, n-1]_{\mathbb{Z}}, \quad v_1 = 1, \ \Delta v_1 = 0 \tag{18}$$

respectively. By a simple computation, we have

$$u_k = \frac{1}{\varphi(1,0)}[\varphi(T+1, k) - \varphi(T, k)], \quad k \in [1, n]_{\mathbb{Z}},$$

$$v_k = \frac{1}{\varphi(1,0)}[\varphi(k, 0) - \varphi(k-1, 0)], \quad k \in [1, n]_{\mathbb{Z}},$$

where

$$\varphi(k, j) = \rho^{k-j} - \rho^{j-k}, \quad \rho = \frac{2 + w^2 + w\sqrt{w^2+4}}{2} > 1.$$

Thus, we can easily get the following standard result and the proof is omitted.

Lemma 7 Let u and v be unique solutions of initial value problems (17) and (18), respectively. Then:

(i) $u_k > 0,\ k \in [1,n]_{\mathbb{Z}}$, and $\Delta u_k < 0,\ k \in [1,n-2]_{\mathbb{Z}}$;

(ii) $v_k > 0,\ k \in [1,n]_{\mathbb{Z}}$, and $\Delta v_k > 0,\ k \in [2,n-1]_{\mathbb{Z}}$. □

Lemma 8 For each function $\mathbf{h} = (h_2, \cdots, h_{n-1})$, the problem

$$-\Delta^2 N_{k-1} + w^2 N_k = h_k,\ k \in [2,n-1]_{\mathbb{Z}}, \quad \Delta N_1 = 0 = \Delta N_{n-1}$$

has a unique solution $\mathbf{N}$ given by

$$N_k = \sum_{j=2}^{n-1} G(k,j) h_j, \quad k \in [1,n]_{\mathbb{Z}},$$

where

$$G(k,j) = \frac{1}{\Delta v(n-1)} \begin{cases} v_j u_k, & 2 \le j \le k \le n, \\ u_j v_k, & 1 \le k \le j \le n-1. \end{cases}$$

Proof. The proof of Lemma 8 is standard and therefore is omitted. □

An immediate consequence of Lemma 8 is the following existence result.

Corollary 1 The problem (7) has the unique solution $N \equiv 1$. □

4. Proof of Theorem 1 and its applications

As mentioned in the introduction, our approach to the search of positive solutions of (1) is based on the Brouwer degree. Accordingly, we transform (1) into a fixed-point problem for a associated operator. To this end, we present now the vector space $E := V^{n-2} \times V^{n-2}$ with the usual norm $\|(\mathbf{A},\mathbf{N})\|_E = \|\mathbf{A}\| + \|\Delta\mathbf{A}\| + \|\mathbf{N}\|$. Let us define the operator

$$\mathcal{F} : \{(\mathbf{A},\mathbf{N},\lambda) \in E \times [0,1] \mid A_k > 0, N_k \ge 0 \text{ with } N_k \not\equiv 0,\ k \in [2,n-1]_{\mathbb{Z}}\} \to E$$

by

$$\mathcal{F}(\mathbf{A},\mathbf{N},\lambda) = \begin{pmatrix} K[-\mathbf{D}\Delta^2\mathbf{A}^0 + \mathbf{A}^0 + \lambda\mathbf{N} f(\cdot,\mathbf{A})], \\ -\lambda \sum_{j=2}^{n-1} \Delta[G(k,j)] g(j, A_j, \Delta A_{j-1}) N_j + w^2 \sum_{j=2}^{n-1} G(k,j) \end{pmatrix}.$$

Lemma 9 For any fixed $\lambda \in [0,1], (\mathbf{A},\mathbf{N})$ is a positive solution of problems (4)–(5) if and only if $(\mathbf{A},\mathbf{N})$ is a fixed point of the continuous operator $\mathcal{F}$.

Proof. Using Lemma 6, together with (16), (4) is equivalent to

$$\mathbf{A} = K[-\mathbf{D}\Delta^2\mathbf{A}^0 + \mathbf{A}^0 + \lambda\mathbf{N} f(\cdot,\mathbf{A})].$$

On the other hand, using Lemma 8, combining Neumann boundary conditions with (H1), (5) is equivalent to

$$\begin{aligned}
N_k =& \sum_{j=2}^{n-1} G(k,j)\big(\lambda\Delta[g(j,A_j,\Delta A_{j-1})N_j]+w^2\big)\\
=&\lambda\sum_{j=2}^{n-1}\Delta[G(k,j)g(j,A_j,\Delta A_{j-1})N_j]-\lambda\sum_{j=2}^{n-1}\Delta[G(k,j)]g(j,A_j,\Delta A_{j-1})N_j+w^2\sum_{j=2}^{n-1}G(k,j)\\
=&-\lambda\sum_{j=2}^{n-1}\Delta[G(k,j)]g(j,A_j,\Delta A_{j-1})N_j+w^2\sum_{j=2}^{n-1}G(k,j).
\end{aligned}$$

Consequently, the proof of Lemma 9 is complete. □

Proof of Theorem 1. Fix $0<R_0<\underline{A},\ R_1>\overline{A},\ R_2>M, R_3>\overline{N}$. Let us consider the bounded set $\Omega\subset E$ defined by

$$\Omega=\{(\mathbf{A},\mathbf{N})\in E:\ R_0<A_k<R_1,\ |\Delta A_k|<R_2, 0\le N_k<R_3\ (k\in[2,n-1]_{\mathbb{Z}})\}.$$

It follows from Lemmas 1–5 and 9 that, for any $\lambda\in[0,1]$ and the possible fixed point $(\mathbf{A},\mathbf{N})$ of $\mathcal{F}$, one has $(\mathbf{A},\mathbf{N})\notin\partial\Omega$. Indeed, any possible solution in $\overline{\Omega}$ belongs to Ω. Therefore, using the homotopy invariance of the topological degree, we conclude that

$$deg[I-\mathcal{F}(\cdot,1),\Omega,0]=deg[I-\mathcal{F}(\cdot,0),\Omega,0].$$

Since

$$I-\mathcal{F}(\cdot,0)=I-\big(K(-\mathbf{D}\Delta^2\mathbf{A}^0+\mathbf{A}^0),\quad w^2\sum_{j=2}^{n-1}G(\cdot,j)\big),$$

and $\big(K(-\mathbf{D}\Delta^2\mathbf{A}^0+\mathbf{A}^0),\quad w^2\sum_{j=2}^{n-1}G(\cdot,j)\big)\in\Omega$, we have

$$deg[I-\mathcal{F}(\cdot,0),\Omega,0]=1,$$

and so

$$deg[I-\mathcal{F}(\cdot,1),\Omega,0]=1.$$

Thus, $\mathcal{F}(\cdot,1)$ has at least one fixed point in Ω. By virtue of Lemma 9 with $\lambda=1$, the systems (1) have at least one positive solution in Ω. □

Finally, we provide an application of the existence results of the positive solution obtained in Theorem 1 to the search for positive solutions for the model problem

$$\begin{aligned}
&D_k\Delta^2(A_{k-1}-A_{k-1}^0)-(A_k-A_k^0)+\psi_kN_kA_k(1-A_k)=0,\quad k\in[2,n-1]_{\mathbb{Z}},\\
&\Delta^2N_{k-1}+\Delta[\frac{\Delta A_{k-1}}{A_k}N_k]-w^2(N_k-1)=0,\quad k\in[2,n-1]_{\mathbb{Z}},\\
&\Delta A_1=0=\Delta A_{n-1},\quad \Delta N_1=0=\Delta N_{n-1},
\end{aligned}\tag{19}$$

where $\psi_k>0, k\in[2,n-1]_{\mathbb{Z}}$.

Corollary 2 Assume that D_k and w satisfy all of the conditions in Theorem 1. Furthermore, let $A_k^0 > 0$, $k \in [1, n]_{\mathbb{Z}}$ and one of the following conditions,

(i) $osc\mathbf{A}^0 < 1$;

(ii) $\min(\mathbf{A}^0 - \mathbf{D}\Delta^2\mathbf{A}^0) > 0$,

hold. Then the systems (19) have at least one positive solution.

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