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A characterization of derivations on uniformly mean value Banach algebras

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Abstract: In this paper, a uniformly mean value Banach algebra (briefly UMV-Banach algebra) is defined as a new class of Banach algebras, and we characterize derivations on this class of Banach algebras. Indeed, it is proved that if $\mathcal{A}$ is a unital UMV-Banach algebra such that either $a = 0$ or $b = 0$ whenever $ab = 0$ in $\mathcal{A}$, and if $\delta : \mathcal{A} \to \mathcal{A}$ is a derivation such that $a\delta(a) = \delta(a)a$ for all $a \in \mathcal{A}$, then the following assertions are equivalent:
(i) δ is continuous;
(ii) $\delta(e^a) = e^a\delta(a)$ for all $a \in \mathcal{A}$;
(iii) δ is identically zero.

Key words: Derivation, mean value property, uniformly mean value property, classical mean value theorem, Gelfand transform

1. Introduction

Throughout this paper, $\mathcal{A}$ denotes an associative complex Banach algebra. If the algebra $\mathcal{A}$ is unital, then $\mathbf{1}$ stands for its unit element. An algebra $\mathcal{A}$ is said to be a domain if $\mathcal{A} \neq \{0\}$, and either $a = 0$ or $b = 0$, whenever $ab = 0$ in $\mathcal{A}$. A commutative algebra that is also a domain is called an integral domain. Recall that a derivation of an algebra $\mathcal{A}$ is a linear mapping δ from a subalgebra $D(\delta)$, the domain of δ, into $\mathcal{A}$ that satisfies the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all pairs $a, b \in D(\delta)$. If $\mathcal{A}$ contains the unit element $\mathbf{1}$, we will always assume $\mathbf{1} \in D(\delta)$ (see [18]). Now we offer the concepts and symbols that will be used in the coming pages. Let $\mathcal{G}$ be an open subset of $\mathbb{C}$. A map $f : \mathcal{G} \subseteq \mathbb{C} \to \mathcal{A}$ is said to be differentiable at point z_0 of $\mathcal{G}$ if $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. This limit is called the derivative of f at the point z_0 and is denoted by $f^{'}(z_0)$. For example, the function $f_a : [\alpha, \beta] \subset \mathbb{R} \to \mathcal{A}$ defined by $f_a(t) = e^{ta}$, where a is an arbitrary fixed element of $\mathcal{A}$, is continuous on $[\alpha, \beta]$ and also is differentiable on open interval (α, β) such that $f^{'}_a(t) = ae^{ta}$. By $C^*(a)$ we denote the C^*-subalgebra generated by $\{a\}$. Let a be an arbitrary element of $\mathcal{A}$. Then the spectrum of a is denoted by $\mathfrak{S}(a)$ and is defined as the set of all complex numbers λ such that $\lambda\mathbf{1} - a$ is not invertible in $\mathcal{A}$. If a is a self-adjoint element of $\mathcal{A}$, then $\mathfrak{S}(a) \subseteq [-\|a\|, \|a\|] \subset \mathbb{R}$. The set of all continuous functions from $\mathfrak{S}(a)$ into $\mathbb{C}$ is denoted by $C(\mathfrak{S}(a))$ and it is well known that if a is a normal element, then the Gelfand transform $G : C^*(a) \to C(\mathfrak{S}(a))$ is an isometric $*-isomorphism$. If $f \in C(\mathfrak{S}(a))$, then $f(a) = G^{-1}(f)$ and furthermore, if $I : \mathfrak{S}(a) \to \mathbb{C}$ is the inclusion map, then $G^{-1}(I) = a$. The unit element of $C(\mathfrak{S}(a))$ is denoted by $\mathfrak{i}$ and we have $\mathfrak{i}(x) = 1$ for all $x \in \mathfrak{S}(a)$. The mentioned definitions and concepts can all be found in [6, 18], and the

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reader is referred to these sources for more general information on Banach algebras and C^*-algebras. As is well known, the class of derivations is a very important class of linear mappings both in theory and applications and was studied intensively. Recently, a number of authors [1, 4, 8, 9, 14] have studied various generalized notions of derivations in the context of Banach algebras. Such mappings have been extensively studied in pure algebra; cf. [1, 3, 9, 16, 25, 26]. In 1955, Singer and Wermer [20] obtained a fundamental result that started investigation into the range of derivations on Banach algebras. The result states that every continuous derivation on a commutative Banach algebra maps the algebra into its Jacobson radical. In the same paper, they conjectured that the assumption of continuity is superfluous. This is called the Singer–Wermer conjecture. In 1988, Thomas [22] proved the conjecture. According to this result, every derivation on a commutative, semisimple Banach algebra is zero. Since then, a number of authors have presented many noncommutative versions of the Singer–Wermer theorem (e.g., see [4, 12, 13, 23]). A result of Johnson and Sinclair [10] states that every derivation on a semisimple Banach algebra is continuous and hence the Singer–Wermer theorem implies that it must be zero. The question of under which conditions all derivations are zero on a given Banach algebra has attracted much attention of authors (for instance, see [5, 7, 12, 13, 15, 16, 19, 24, 26]). The current research is focused on this topic. Indeed, this study is an attempt to offer a new approach for investigation of this subject. This article introduces a new type of Banach algebras and it will be shown that, under certain conditions, derivations are zero on such Banach algebras.

An element a of $\mathcal{A}$ has the mean value property (MV-property, briefly) if for every closed interval $[\alpha, \beta] \subset \mathbb{R}$ there exists a function $h_{\alpha,\beta} : \mathcal{A} \to \mathcal{A}$ such that

$$\sum_{n=1}^{\infty} \frac{(\beta a)^n}{n!} - \sum_{n=1}^{\infty} \frac{(\alpha a)^n}{n!} = (\beta - \alpha) \sum_{n=1}^{\infty} \frac{h_{\alpha,\beta}(a)^{n-1} a^n}{(n-1)!}.$$

In the case that $\mathcal{A}$ is unital the above-mentioned formula turns to $e^{\beta a} - e^{\alpha a} = (\beta - \alpha) a e^{a h_{\alpha,\beta}(a)}$. A Banach algebra $\mathcal{A}$ is called MV-Banach algebra if every element of $\mathcal{A}$ has the MV-property.

An element a of $\mathcal{A}$ has the uniformly mean value property (UMV-property, briefly) if for every closed interval $[\alpha, \beta] \subset \mathbb{R}$ there exists a real number $c_{\alpha,\beta} \in (\alpha, \beta)$ such that

$$\sum_{n=1}^{\infty} \frac{(\beta a)^n}{n!} - \sum_{n=1}^{\infty} \frac{(\alpha a)^n}{n!} = (\beta - \alpha) \sum_{n=1}^{\infty} \frac{c_{\alpha,\beta}^{n-1} a^n}{(n-1)!}.$$

It is evident that, if $\mathcal{A}$ is unital, then the previous formula is turns to $e^{\beta a} - e^{\alpha a} = (\beta - \alpha) a e^{c_{\alpha,\beta} a}$. A Banach algebra $\mathcal{A}$ is called UMV-Banach algebra if every element of $\mathcal{A}$ has the UMV-property. As a proposition, we prove that if a is an element of a unital Banach algebra $\mathcal{A}$ such that the function $R_a(z) = (z\mathbf{1} - a)^{-1}$ satisfies $R_a(\beta) - R_a(\alpha) = (\beta - \alpha) R_a^{'}(c)$ for some $c \in (\alpha, \beta) \subset \mathbb{R}$, then there exists a real number t_0 such that $a = t_0 \mathbf{1}$. The main purpose of this study is to prove the following result:
Let $\mathcal{A}$ be a unital domain and δ be a derivation on $\mathcal{A}$. Furthermore, assume that a is an element of $\mathcal{A}$ with the UMV-property such that $e^{c_{0,1} a} \delta(a) = \delta(a) e^{c_{0,1} a}$, where $c_{0,1} \in (0, 1) \subset \mathbb{R}$ is achieved from UMV-property for a. If $\delta(e^a) = e^a \delta(a)$ and $\delta(e^{c_{0,1} a}) = c_{0,1} e^{c_{0,1} a} \delta(a)$, then $\delta(a) = 0$.

Using the above-mentioned result, the following corollary can be achieved:
Let the UMV-Banach algebra $\mathcal{A}$ be a unital domain and $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation. If $a\delta(a) = \delta(a)a$ for all $a \in \mathcal{A}$, then the following assertions are equivalent.
(i) δ is continuous;

(ii) $\delta(e^a) = e^a\delta(a)$ for all $a \in \mathcal{A}$;
(iii) δ is identically zero.

2. Main results

Theorem 2.1 *Let $C^* -$ algebra $\mathcal{A}$ be unital, and $\delta : D(\delta) \subseteq \mathcal{A} \to \mathcal{A}$ be a closed derivation. Suppose a is a self-adjoint element of $D(\delta)$ such that $C^*(a) \subseteq D(\delta)$ and $a\delta(a) = \delta(a)a$. Furthermore, assume that $f(a)\delta(g(a)) = 0$, where f, g are two functions with respect to a, implies that either $f(a) = 0$ or $\delta(g(a)) = 0$. Then there exists a continuous, nondifferentiable function $h : \mathfrak{S}(a)\bigcup\{0\} \subseteq [-\|a\|, \|a\|] \to (0,1)$ satisfying $e^x - 1 = xe^{xh(x)}$ such that $\delta(h(a)) = 0$ if and only if $\delta(a) = 0$.*

Proof If $a = 0$, then there is nothing to be proved. Let a be a nonzero self-adjoint element of $D(\delta)$ such that $C^*(a) \subseteq D(\delta)$. For $x \in \mathfrak{S}(a) - \{0\}$, define the map $f_x : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ by $f_x(t) = e^{tx}$. Evidently, f_x is continuous on $[\alpha, \beta]$ and is differentiable on open interval (α, β). Hence, by the classical mean value theorem for f_x on [0,1], there exists an element c_x in (0,1) such that $f_x(1) - f_x(0) = xe^{xc_x}$, i.e. $e^x - 1 = xe^{xc_x}$. Now we define $h : \mathfrak{S}(a) - \{0\} \to (0,1)$ by $h(x) = c_x$. In the next step, it is shown that h is well-defined. Let $x_1 = x_2 (\in \mathfrak{S}(a) - \{0\})$. We have $x_1e^{x_1c_{x_1}} = e^{x_1} - 1 = e^{x_2} - 1 = x_2e^{x_2c_{x_2}}$. Thus, $c_{x_1} = c_{x_2}$, i.e. $h(x_1) = h(x_2)$. The equality $e^x - 1 = xe^{xh(x)}$ shows that $h(x) = \frac{1}{x}\ln(\frac{e^x-1}{x})$. Obviously, h is continuous on $\mathfrak{S}(a) - \{0\}$. Note that

$$\begin{aligned}
\lim_{x\to 0} h(x) &= \lim_{x\to 0} \frac{1}{x}\ln(\frac{e^x - 1}{x}) = \lim_{x\to 0} \frac{\frac{xe^x - e^x + 1}{x^2}}{\frac{e^x - 1}{x}} \\
&= \lim_{x\to 0} \frac{xe^x - e^x + 1}{x(e^x - 1)} = \lim_{x\to 0} \frac{e^x + xe^x - e^x}{e^x - 1 + xe^x} \\
&= \lim_{x\to 0} \frac{xe^x}{e^x - 1 + xe^x} = \lim_{x\to 0} \frac{e^x + xe^x}{e^x + e^x + xe^x} \\
&= \frac{1}{2}.
\end{aligned}$$

Hence, we define the function h by

$$h(x) = \begin{cases} \frac{1}{x}\ln(\frac{e^x-1}{x}) & x \neq 0 \\ \frac{1}{2} & x = 0. \end{cases}$$

It is clear that h is a nondifferentiable, continuous function on $\mathfrak{S}(a) \cup \{0\} \subseteq [-\|a\|, \|a\|]$. If $f(x) = 1 + \sum_{n=1}^{\infty} \frac{(tx)^n}{n!} = e^{tx}$, then $G^{-1}(f) = \mathbf{1} + \sum_{n=1}^{\infty} \frac{(ta)^n}{n!} = e^{ta}$, where G is the Gelfand transform. Furthermore, we have

$$\begin{aligned}
e^{hI}(x) &= \sum_{n=0}^{\infty} \frac{h^n I^n}{n!}(x) \\
&= \sum_{n=0}^{\infty} \frac{(h(x)x)^n}{n!} \\
&= e^{h(x)x}.
\end{aligned}$$

It follows from the equality $e^x - 1 = xe^{h(x)x}$ that $(e^I - \mathfrak{i})(x) = (Ie^{hI})(x)$, and hence

$$e^I - \mathfrak{i} = Ie^{Ih}.$$

By applying the Gelfand transform on the previous equality, we have $e^a - \mathbf{1} = ae^{ah(a)}$. Therefore, $\delta(e^a - \mathbf{1}) = \delta(ae^{ah(a)})$ and it implies that

$$\delta(e^a) = \delta(a)e^{ah(a)} + a\delta(e^{ah(a)}). \tag{2.1}$$

Since $a\delta(a) = \delta(a)a$, it is seen that $\delta((ah(a))^n) = n(ah(a))^{n-1}\delta(ah(a))$. Therefore, we have

$$\delta(e^{ah(a)}) = e^{ah(a)}\delta(ah(a)) = e^{ah(a)}[a\delta(h(a)) + h(a)\delta(a)] \quad (*).$$

However, $\delta(e^{ah(a)})$ can also be calculated in a different way. Let $P(x)$ be a polynomial of variable x. Then it is easily seen that $P(a) \in D(\delta)$ and $\delta(P(a)) = P^{'}(a)\delta(a)$, where $P^{'}$ is the derivative of P. A straightforward verification shows that the function

$$e^{xh(x)} = \begin{cases} \frac{e^x - 1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

is differentiable. Indeed,

$$e^{Ih} \in C^{'}(\mathfrak{S}(a) \cup \{0\}).$$

For e^{Ih}, take a sequence $\{P_n\}$ of polynomials on $\mathfrak{S}(a) \cup \{0\}$ such that $\|P_n - e^{Ih}\| \to 0$ and $\|P_n^{'} - (e^{Ih})^{'}\| \to 0$. So, $\|P_n(a) - e^{ah(a)}\| \to 0$ and $\|\delta(P_n(a)) - (ah(a))^{'}e^{ah(a)}\delta(a)\| \to 0$. The closedness of δ implies that $e^{ah(a)} \in D(\delta)$ and

$$\delta(e^{ah(a)}) = (ah(a))^{'}e^{ah(a)}\delta(a) \quad (**).$$

Comparing $(*)$ and $(**)$, we obtain

$$e^{ah(a)}(a\delta(h(a)) + h(a)\delta(a)) = e^{ah(a)}(ah(a))^{'}\delta(a).$$

Thus, we have

$$a\delta(h(a)) = ((ah(a))^{'} - h(a))\delta(a) \quad (***).$$

If $\delta(a) = 0$, then it follows from $(***)$ that $a\delta(h(a)) = 0$, and it is obtained from our assumption in this theorem that $\delta(h(a)) = 0$. Now we are going to show that $\delta(h(a)) = 0$ implies that $\delta(a) = 0$. If $\delta(h(a)) = 0$, then it follows from $(*)$ that $\delta(e^{ah(a)}) = e^{ah(a)}h(a)\delta(a)$. Having put $e^{ah(a)}h(a)\delta(a)$ instead of $\delta(e^{ah(a)})$ in (2.1), we conclude that $(e^a - ah(a)e^{ah(a)} - e^{ah(a)})\delta(a) = 0$. By reusing our assumption, it can be concluded that $\delta(a) = 0$ or $e^a - ah(a)e^{ah(a)} - e^{ah(a)} = 0$. Suppose that $e^a - ah(a)e^{ah(a)} - e^{ah(a)} = 0$. Hence, $k = G(e^a - ah(a)e^{ah(a)} - e^{ah(a)}) = e^I - Ihe^{Ih} - e^{Ih}$ is a zero function on $\mathfrak{S}(a) \cup \{0\} \subseteq [-\|a\|, \|a\|]$. It means that

$$k(x) = e^x - xh(x)e^{xh(x)} - e^{xh(x)} = 0,$$

for all $x \in \mathfrak{S}(a) \cup \{0\}$. Clearly, $k(0) = 0$. Moreover, we have

$$k(x) = e^x - (\frac{e^x - 1}{x}) - (\frac{e^x - 1}{x})\ln(\frac{e^x - 1}{x}),$$

for all $x \in \mathfrak{S}(a) - \{0\}$. By using MATLAB software, the function k on the closed interval [-2,2] was drawn, and it was observed that this function is not zero. Hence,

$$e^a - e^{h(a)a} - ah(a)e^{h(a)a} \neq 0$$

and consequently, $\delta(a) = 0$. This completes the proof of the theorem. □

Definition 2.2 *An element a of $\mathcal{A}$ has the mean value property (MV- property, briefly) if for every closed interval $[\alpha, \beta] \subset \mathbb{R}$ there exists a function $h_{\alpha,\beta} : \mathcal{A} \to \mathcal{A}$ such that*

$$\sum_{n=1}^{\infty} \frac{(\beta a)^n}{n!} - \sum_{n=1}^{\infty} \frac{(\alpha a)^n}{n!} = (\beta - \alpha) \sum_{n=1}^{\infty} \frac{h_{\alpha,\beta}(a)^{n-1} a^n}{(n-1)!}.$$

In the case that $\mathcal{A}$ is unital the above-mentioned formula turns to $e^{\beta a} - e^{\alpha a} = (\beta - \alpha)ae^{ah_{\alpha,\beta}(a)}$. A Banach algebra $\mathcal{A}$ is called MV-Banach algebra if every element of $\mathcal{A}$ has the MV-property.

An element a of $\mathcal{A}$ has the uniformly mean value property (UMV-property, briefly) if for every closed interval $[\alpha, \beta] \subset \mathbb{R}$ there exists a real number $c_{\alpha,\beta} \in (\alpha, \beta)$ such that

$$\sum_{n=1}^{\infty} \frac{(\beta a)^n}{n!} - \sum_{n=1}^{\infty} \frac{(\alpha a)^n}{n!} = (\beta - \alpha) \sum_{n=1}^{\infty} \frac{c_{\alpha,\beta}^{n-1} a^n}{(n-1)!}.$$

It is evident that, if $\mathcal{A}$ is unital, then the previous formula becomes $e^{\beta a} - e^{\alpha a} = (\beta - \alpha)ae^{c_{\alpha,\beta} a}$. A Banach algebra $\mathcal{A}$ is called UMV-Banach algebra if every element of $\mathcal{A}$ has the UMV-property.

Below we offer some examples about the uniformly mean value property and mean value property.
Let a be an idempotent element of a unital Banach algebra $\mathcal{A}$, i.e. $a^2 = a$. We have

$$\begin{aligned} e^{ta} &= \sum_{n=0}^{\infty} \frac{t^n a^n}{n!} = \mathbf{1} + \sum_{n=1}^{\infty} \frac{t^n a}{n!} \\ &= \mathbf{1} + \sum_{n=0}^{\infty} \frac{t^n a}{n!} - a \\ &= e^t a - a + \mathbf{1} \end{aligned}$$

for all $t \in \mathbb{R}$. Hence,

$$e^{\beta a} - e^{\alpha a} = e^{\beta} a - a + 1 - (e^{\alpha} a - a + \mathbf{1}) = (e^{\beta} - e^{\alpha})a.$$

According to the classical mean value theorem for the function $f(t) = e^t$ on $[\alpha, \beta]$, there exists an element $c_{\alpha,\beta} \in (\alpha, \beta)$ such that $e^{\beta} - e^{\alpha} = (\beta - \alpha)e^{c_{\alpha,\beta}}$. So, $e^{\beta a} - e^{\alpha a} = (\beta - \alpha)e^{c_{\alpha,\beta}} a$. Now we show that $e^{c_{\alpha,\beta}} a = ae^{c_{\alpha,\beta} a}$. We have

$$ae^{c_{\alpha,\beta} a} = a(e^{c_{\alpha,\beta}} a - a + \mathbf{1}) = e^{c_{\alpha,\beta}} a^2 - a^2 + a = e^{c_{\alpha,\beta}} a - a + a = e^{c_{\alpha,\beta}} a.$$

Thus, $e^{\beta a} - e^{\alpha a} = (\beta - \alpha)ae^{c_{\alpha,\beta} a}$, and this means that a has the UMV-property.

As another example, let α, β, t_0 be arbitrary fixed real numbers. We define the function $f_{t_0}:[\alpha,\beta]\subset\mathbb{R}\to\mathbb{R}$ by $f_{t_0}(x)=t_0x$. Clearly, f_{t_0} is a continuous function and so $f_{t_0}\in C([\alpha,\beta])$, where $C([\alpha,\beta])$ denotes the set of all continuous functions from $[\alpha,\beta]$ into $\mathbb{C}$. It is well known that if $h,g\in C([\alpha,\beta])$, then $(h+g)(x)=h(x)+g(x)$ and $(hg)(x)=h(x)g(x)$ for all $x\in[\alpha,\beta]$. We therefore have $e^{sf_{t_0}}=\sum_{n=0}^{\infty}\frac{s^nf_{t_0}^n}{n!}(x)=\sum_{n=0}^{\infty}\frac{s^n{t_0}^nx^n}{n!}=e^{st_0x}$ for all $s\in\mathbb{R}$. Now we define the function $F_{t_0x}:[\alpha,\beta]\to\mathbb{R}$ by $F_{t_0x}(s)=e^{st_0x}$. It is evident that F_{t_0x} is a continuous function on $[\alpha,\beta]$ and differentiable on (α,β). It follows from the classical mean value theorem that there exists a number $c\in(\alpha,\beta)$ such that $F_{t_0x}(\beta)-F_{t_0x}(\alpha)=(\beta-\alpha)F'_{t_0x}(c)$. Hence, $e^{t_0x\beta}-e^{t_0x\alpha}=(\beta-\alpha)t_0xe^{t_0xc}$. In fact, we have $e^{\beta f_{t_0}}(x)-e^{\alpha f_{t_0}}(x)=(\beta-\alpha)(f_{t_0}e^{cf_{t_0}})(x)$ and since x was an arbitrary element of $[\alpha,\beta]$, $e^{\beta f_{t_0}}-e^{\alpha f_{t_0}}=(\beta-\alpha)f_{t_0}e^{cf_{t_0}}$. Thus, f_{t_0} has the UMV-property in $C([\alpha,\beta])$.

In the following, we provide an example of the MV-property. We denote the vector space of all 2×2 matrices with real entries by $\mathcal{M}_2(\mathbb{R})$. Clearly, $\mathcal{M}_2(\mathbb{R})$ is a Banach algebra with the norm $\|\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\|=|a_{11}|+|a_{12}|+|a_{21}|+|a_{22}|$. Let $A=\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}$ be an element of $\mathcal{M}_2(\mathbb{R})$. Obviously, $A^n=\begin{bmatrix} a^n & 0\\ 0 & b^n\end{bmatrix}$ for each positive integer n. Hence, if t is a real number, then

$$e^{tA}=\sum_{n=0}^{\infty}\frac{t^nA^n}{n!}=\sum_{n=0}^{\infty}\frac{1}{n!}\begin{bmatrix} t^na^n & 0\\ 0 & t^nb^n\end{bmatrix}$$
$$=\begin{bmatrix} \sum_{n=0}^{\infty}\frac{t^na^n}{n!} & 0\\ 0 & \sum_{n=0}^{\infty}\frac{t^nb^n}{n!}\end{bmatrix}=\begin{bmatrix} e^{ta} & 0\\ 0 & e^{tb}\end{bmatrix}.$$

Define $f_a(x)=e^{ax}$ and $g_b(x)=e^{bx}$. Applying the classical mean value theorem on f_a and g_b, we have $f_a(\beta)-f_a(\alpha)=(\beta-\alpha)f'_a(c_a)$ and $g_b(\beta)-g_b(\alpha)=(\beta-\alpha)g'_b(c_b)$, where c_a and c_b are two elements in (α,β). It means that $e^{\beta a}-e^{\alpha a}=(\beta-\alpha)ae^{c_aa}$ and $e^{\beta b}-e^{\alpha b}=(\beta-\alpha)be^{c_bb}$. Therefore,

$$e^{\beta A}-e^{\alpha A}=\begin{bmatrix} e^{\beta a}-e^{\alpha a} & 0\\ 0 & e^{\beta b}-e^{\alpha b}\end{bmatrix}=\begin{bmatrix} (\beta-\alpha)ae^{c_aa} & 0\\ 0 & (\beta-\alpha)be^{c_bb}\end{bmatrix}. \quad (*)$$

Now we define the function $h:\mathcal{M}_2(\mathbb{R})\to\mathcal{M}_2(\mathbb{R})$ as follows:

$$h(\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix})=\begin{bmatrix} c_a & 0\\ 0 & c_b\end{bmatrix}.$$

Thus,

$$(\beta-\alpha)Ae^{Ah(A)} = (\beta-\alpha)\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} e^{\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} c_a & 0 \\ 0 & c_b \end{bmatrix}\right)}$$

$$= (\beta-\alpha)\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} e^{\begin{bmatrix} ac_a & 0 \\ 0 & bc_b \end{bmatrix}}$$

$$= (\beta-\alpha)\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} e^{ac_a} & 0 \\ 0 & e^{bc_b} \end{bmatrix}$$

$$= (\beta-\alpha)\begin{bmatrix} ae^{ac_a} & 0 \\ 0 & be^{bc_b} \end{bmatrix}$$

$$= \begin{bmatrix} (\beta-\alpha)ae^{c_a a} & 0 \\ 0 & (\beta-\alpha)be^{bc_b} \end{bmatrix}. \qquad (**)$$

Comparing (*) and (**), we conclude that

$$e^{\beta A} - e^{\alpha A} = (\beta-\alpha)Ae^{Ah(A)}.$$

It means that A has the mean value property (or MV-property).

Next we will present a UMV-Banach algebra. Let $\mathcal{E}$ be a Banach algebra. The *annihilator* of $\mathcal{E}$ is denoted by $ann(\mathcal{E})$, and $ann(\mathcal{E}) = \{b \in \mathcal{E} \mid \mathcal{E}b = \{0\} = b\mathcal{E}\}$. Let $\mathcal{A}$ be a unital Banach algebra. Set

$$\mathcal{B} = \begin{bmatrix} \mathbb{R} & \mathcal{A} \\ \mathcal{A} & \mathbb{R} \end{bmatrix} = \left\{\begin{bmatrix} r & a \\ b & s \end{bmatrix} \ : \ a, b \in \mathcal{A} \ \ and \ \ r, s \in \mathbb{R}\right\}.$$

We should consider $\mathcal{B}$ as a Banach algebra with point-wise addition, scalar multiplication, product, and norm, which are defined as follows.
$\begin{bmatrix} r & a \\ b & s \end{bmatrix} \bullet \begin{bmatrix} t & c \\ d & u \end{bmatrix} = \begin{bmatrix} rt & ac \\ bd & su \end{bmatrix}$ and $\| \begin{bmatrix} r & a \\ b & s \end{bmatrix} \| = |r| + |s| + \|a\| + \|b\|$.
It is well known that the $ann(\mathcal{A})$ is a closed bi-ideal of $\mathcal{A}$. Hence,

$$\mathcal{D} = \left\{\begin{bmatrix} r & a \\ b & r \end{bmatrix} \ : \ a, b \in ann(\mathcal{A}) \ and \ r \in \mathbb{R}\right\}$$

is a Banach subalgebra of $\mathcal{B}$. Clearly, if $X = \begin{bmatrix} r & a \\ b & r \end{bmatrix}$ then $X^n = \begin{bmatrix} r^n & 0 \\ 0 & r^n \end{bmatrix}$ for all natural number $n \geq 2$. Suppose that $[\alpha, \beta] \subset \mathbb{R}$. Then $\sum_{n=1}^{\infty} \frac{(\beta X)^n}{n!} = \begin{bmatrix} \sum_{n=1}^{\infty} \frac{(\beta r)^n}{n!} & \beta a \\ \beta b & \sum_{n=1}^{\infty} \frac{(\beta r)^n}{n!} \end{bmatrix} = \begin{bmatrix} e^{\beta r} - 1 & \beta a \\ \beta b & e^{\beta r} - 1 \end{bmatrix}$. Similarly, $\sum_{n=1}^{\infty} \frac{(\alpha X)^n}{n!} = \begin{bmatrix} e^{\alpha r} - 1 & \alpha a \\ \alpha b & e^{\alpha r} - 1 \end{bmatrix}$. According to the classical mean value theorem, there exists an element $c \in (\alpha, \beta)$ such that

$$e^{\beta r} - e^{\alpha r} = (\beta-\alpha)re^{cr}.$$

Hence, we have

$$\sum_{n=1}^{\infty}\frac{(\beta X)^n}{n!}-\sum_{n=1}^{\infty}\frac{(\alpha X)^n}{n!}=\begin{bmatrix} e^{\beta r}-e^{\alpha r} & (\beta-\alpha)a \\ (\beta-\alpha)b & e^{\beta r}-e^{\alpha r}\end{bmatrix}$$
$$=\begin{bmatrix} (\beta-\alpha)re^{cr} & (\beta-\alpha)a \\ (\beta-\alpha)b & (\beta-\alpha)re^{cr}\end{bmatrix}$$
$$=(\beta-\alpha)\begin{bmatrix} re^{cr} & a \\ b & re^{cr}\end{bmatrix}.$$

At this point, we show that $(\beta-\alpha)\sum_{n=1}^{\infty}\frac{c^{n-1}X^n}{(n-1)!}=(\beta-\alpha)\begin{bmatrix} re^{cr} & a \\ b & re^{cr}\end{bmatrix}$. Note that

$$(\beta-\alpha)\sum_{n=1}^{\infty}\frac{c^{n-1}X^n}{(n-1)!}=(\beta-\alpha)(\begin{bmatrix} r & a \\ b & r\end{bmatrix}+\begin{bmatrix} cr^2 & 0 \\ 0 & cr^2\end{bmatrix}+\begin{bmatrix} \frac{c^2r^3}{2!} & 0 \\ 0 & \frac{c^2r^3}{2!}\end{bmatrix}+...)$$
$$=(\beta-\alpha)\begin{bmatrix} r+cr^2+\frac{c^2r^3}{2!}+... & a \\ b & r+cr^2+\frac{c^2r^3}{2!}+...\end{bmatrix}$$
$$=(\beta-\alpha)\begin{bmatrix} re^{cr} & a \\ b & re^{cr}\end{bmatrix}.$$

Thus, every element of $\mathcal{D}$ has the mean value property and it means that $\mathcal{D}$ is a UMV-Banach algebra.

In the following proposition, we characterize the unital Banach algebras for which the resolvent function $R_a(z)=(z\mathbf{1}-a)^{-1}$ satisfies the classical mean value theorem for real numbers.

Proposition 2.3 *Let a be an element of the unital Banach algebra $\mathcal{A}$ such that the resolvent function $R_a(z)=(z\mathbf{1}-a)^{-1}$ has the following property:*

$$R_a(\beta)-R_a(\alpha)=(\beta-\alpha)R_a^{'}(c)$$

for some $c\in(\alpha,\beta)\subset\mathbb{R}$. Then there exists a real number t_0 such that $a=t_0\mathbf{1}$.

Proof It is evident that the derivative of resolvent function R_a at point $z_0\in\mathbb{C}-\mathfrak{S}(a)$ is

$$R_a^{'}(z_0)=-(z_0\mathbf{1}-a)^{-2}.$$

By hypothesis, there exists an element c in open interval (α,β) such that

$$(\beta\mathbf{1}-a)^{-1}-(\alpha\mathbf{1}-a)^{-1}=(\beta-\alpha)R_a^{'}(c),$$

and it means that

$$(\beta\mathbf{1}-a)^{-1}-(\alpha\mathbf{1}-a)^{-1}=-(\beta-\alpha)(c\mathbf{1}-a)^{-2}.$$

This equation with the fact that $A^{-1}-B^{-1}=A^{-1}(B-A)B^{-1}$ implies that

$$(\beta\mathbf{1}-a)^{-1}(\alpha-\beta)(\alpha\mathbf{1}-a)^{-1}=(\alpha-\beta)(c\mathbf{1}-a)^{-2}.$$

This equation together with the fact that $(AB)^{-1} = B^{-1}A^{-1}$ implies that

$$((\alpha\mathbf{1} - a)(\beta\mathbf{1} - a))^{-1} = ((c\mathbf{1} - a)^2)^{-1}.$$

Hence,

$$\alpha\beta\mathbf{1} - \alpha a - \beta a + a^2 = c^2\mathbf{1} - 2ca + a^2.$$

Consequently,

$$a = \frac{\alpha\beta - c^2}{\alpha + \beta - 2c}\mathbf{1}.$$

□

Theorem 2.4 *Let $\mathcal{A}$ be a unital domain and δ be a derivation on $\mathcal{A}$. Furthermore, assume that a is an element of $\mathcal{A}$ with the UMV-property satisfying $e^{c_{0,1}a}\delta(a) = \delta(a)e^{c_{0,1}a}$, where $c_{0,1} \in (0,1) \subset \mathbb{R}$ is obtained from UMV-property for a. If $\delta(e^a) = e^a\delta(a)$ and $\delta(e^{c_{0,1}a}) = c_{0,1}e^{c_{0,1}a}\delta(a)$, then $\delta(a) = 0$.*

Proof If $a = 0$, then there is nothing to be proved. Let a be a nonzero element of $\mathcal{A}$ with the UMV-property. Hence, there exists an element $c_{0,1} = c$ of (0,1) such that

$$e^a - \mathbf{1} = ae^{ca}. \tag{2.2}$$

By assumption, $\delta(e^{ca}) = ce^{ca}\delta(a)$; therefore, we have $e^a\delta(a) - \delta(\mathbf{1}) = \delta(a)e^{ca} + a(ce^{ca}\delta(a))$. This equality together with the fact that $e^{ca}\delta(a) = \delta(a)e^{ca}$ implies that

$$(e^a - e^{ca} - cae^{ca})\delta(a) = 0.$$

Using the fact that $\mathcal{A}$ is a domain, we conclude that either $\delta(a) = 0$ or $e^a - cae^{ca} - e^{ca} = 0$. We will show that if $e^a - cae^{ca} - e^{ca} = 0$, then $\delta(a) = 0$. Reusing the UMV-property for a on the closed interval $[c, 1]$, we obtain an element $c_{c,1} = c_1$ in $(c, 1)$ such that

$$e^a - e^{ca} = (1 - c)ae^{c_1 a}.$$

Thus,

$$e^a - e^{ca} - ae^{c_1 a} + cae^{c_1 a} = 0. \tag{2.3}$$

The previous equation together with the fact that $e^a - e^{ca} = cae^{ca}$ implies that

$$cae^{ca} - ae^{c_1 a} + cae^{c_1 a} = 0. \tag{2.4}$$

Replacing c_1 by $c + h$ in (2.4), we get

$$cae^{ca} - ae^{(c+h)a} + cae^{(c+h)a} = 0,$$

and hence,

$$ae^{ca}[c\mathbf{1} - e^{ha} + ce^{ha}] = 0.$$

Since $\mathcal{A}$ is a domain and ae^{ca} is nonzero, $c\mathbf{1} - e^{ha} + ce^{ha} = 0$. Hence, we have $e^{ha} = \frac{c}{1-c}\mathbf{1}$. Based on the spectral mapping theorem, it is achieved that $\mathfrak{S}(e^{ha}) = e^{\mathfrak{S}(ha)}$. First note that $\mathfrak{S}(e^{ha}) = \mathfrak{S}(\frac{c}{1-c}\mathbf{1}) = \{\frac{c}{1-c}\}$. So,

$$\{\frac{c}{1-c}\} = \mathfrak{S}(e^{ha}) = e^{h\mathfrak{S}(a)}. \tag{2.5}$$

If λ is an arbitrary element of $\mathfrak{S}(a)$, then the previous relation implies that $e^{h\lambda} = \frac{c}{1-c} \in \mathbb{R}$ and it shows that λ is a real number. Since λ was arbitrary, $\mathfrak{S}(a) \subset \mathbb{R}$. Suppose that β_1, $\beta_2 \in \mathfrak{S}(a)$. It follows from (2.5) that $e^{h\beta_1} = \frac{c}{1-c} = e^{h\beta_2}$ and consequently, $\beta_1 = \beta_2$. It means that $\mathfrak{S}(a)$ contains only one element such as $\beta \in \mathbb{R}$. Thus, $\mathfrak{S}(ha) = h\mathfrak{S}(a) = \{h\beta\}$. Additionally, it follows from (2.5) that $\frac{c}{1-c} = e^{h\beta}$ and thus, $e^{ha} = e^{h\beta}\mathbf{1}$. It is clear that $\mathfrak{S}(ha)$ is contained in the open strip: $-\pi < Im(\lambda) < \pi$. So, by Proposition 2.10 of [21], we obtain that $log(exp(ha)) = ha$, i.e. $log(e^{ha}) = ha$. We know that $e^{ha} = e^{h\beta}\mathbf{1}$. Therefore,

$$ha = log(e^{ha}) = log(e^{h\beta}\mathbf{1}) = h\beta\mathbf{1}.$$

This equation demonstrates that $a = \beta\mathbf{1}$ and consequently, $\delta(a) = 0$. □

An immediate but noteworthy corollary to Theorem 2.4 is:

Corollary 2.5 *Let the UMV-Banach algebra $\mathcal{A}$ be a unital domain and $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation. If $a\delta(a) = \delta(a)a$ for all $a \in \mathcal{A}$, then the following assertions are equivalent:*
(i) δ is continuous;
(ii) $\delta(e^a) = e^a\delta(a)$ for all $a \in \mathcal{A}$;
(iii) δ is identically zero.

Proof $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ are clear. According to Theorem 2.4, (iii) is an immediate conclusion from (ii). □

In the next results, like most authors, we denote the commutator $ab - ba$ by $[a, b]$ for all pairs $a, b \in \mathcal{A}$.

Corollary 2.6 *Let $\mathcal{A}$ be a unital domain and a be an element of $\mathcal{A}$ with the UMV-property satisfying $e^{c_{0,1}a}[a, x] = [a, x]e^{c_{0,1}a}$ for some $x \in \mathcal{A}$ and for $c_{0,1} \in (0, 1)$, which is obtained from the UMV-property for a. If $[e^a, x] = e^a[a, x]$ and $[e^{c_{0,1}a}, x] = c_{0,1}e^{c_{0,1}a}[a, x]$, then $[a, x] = 0$.*

Proof Define $\delta_x : \mathcal{A} \to \mathcal{A}$ by $\delta_x(a) = [a, x]$. Obviously, δ_x is a continuous derivation. At this point, Theorem 2.4 is just what we need to complete the proof. □

If δ is a continuous derivation on $\mathcal{A}$ such that $a\delta(a) = \delta(a)a$ for all $a \in \mathcal{A}$, then a straightforward verification shows that $\delta(e^a) = e^a\delta(a)$. In Corollary 2.5, under certain circumstances, the converse of this result has been investigated. This allows us to offer the following problem.

Problem 2.7 *Let $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation such that $\delta(e^a) = e^a\delta(a)$ for all $a \in \mathcal{A}$. Is δ a continuous operator?*

Theorem 2.8 *Let $\mathcal{A}$ be a unital domain and $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation such that $a\delta(a) = \delta(a)a$ and $\delta(e^a) = e^a\delta(a)$ for all $a \in \mathcal{A}$. If there exists a continuous, injective linear mapping from $\mathcal{A}$ into $\mathbb{R}$, then δ is identically zero.*

Proof Let a be a nonzero fixed element of $\mathcal{A}$. We define a function $f_a : [\alpha, \beta] \subset \mathbb{R} \to \mathcal{A}$ by $f_a(t) = e^{ta}$. It is well known that f_a is continuous on $[\alpha, \beta]$ and is differentiable on (α, β). Let $F : \mathcal{A} \to \mathbb{R}$ be a continuous, injective linear mapping. Put $H = Fof_a$ to get a function from $[\alpha, \beta]$ into $\mathbb{R}$. Let x_0 be an arbitrary element of $[\alpha, \beta]$, and then $\lim_{x\to x_0} H(x) = \lim_{x\to x_0}(Fof_a)(x) = F(\lim_{x\to x_0} f_a(x)) = H(x_0)$. It means that H is continuous. Moreover, we have

$$\begin{aligned} H^{'}(x_0) &= \lim_{x\to x_0} \frac{H(x) - H(x_0)}{x - x_0} \\ &= \lim_{x\to x_0} \frac{F(f_a(x)) - F(f_a(x_0))}{x - x_0} \\ &= F(\lim_{x\to x_0} \frac{f_a(x) - f_a(x_0)}{x - x_0}) \\ &= (Fof_a^{'})(x_0), \end{aligned}$$

for all $x_0 \in (\alpha, \beta)$. It means that H is differentiable on (α, β). Since $H = Fof_a$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β), the classical mean value theorem ensures that there is an element $c_{\alpha,\beta} \in (\alpha, \beta)$ such that $H(\beta) - H(\alpha) = (\beta - \alpha)H^{'}(c_{\alpha,\beta})$, i.e. $F(f_a(\beta) - f_a(\alpha) - (\beta - \alpha)f_a^{'}(c_{\alpha,\beta})) = 0$. Since F is injective, $f_a(\beta) - f_a(\alpha) = (\beta - \alpha)f_a^{'}(c_{\alpha,\beta})$. So, we have $e^{\beta a} - e^{\alpha a} = (\beta - \alpha)ae^{c_{\alpha,\beta}a}$. It means that a has the UMV-property and since a was arbitrary, $\mathcal{A}$ is a UMV-Banach algebra. Finally, Corollary 2.5 completes the proof. □

In the following two theorems, we present some results on the range of a derivation.

Theorem 2.9 *Let $\delta : \mathcal{A} \to \mathcal{A}$ be a Jordan derivation and $\mathcal{P}$ be a primitive ideal of $\mathcal{A}$. If $[a, \delta(a)] \in \mathcal{P}$ for all $a \in \mathcal{A}$ and $\delta(\mathcal{P}) \subseteq \mathcal{P}$, then $\delta(\mathcal{A}) \subseteq \mathcal{P}$.*

Proof Let us define $\Delta : \frac{\mathcal{A}}{\mathcal{P}} \to \frac{\mathcal{A}}{\mathcal{P}}$ by $\Delta(a + \mathcal{P}) = \delta(a) + \mathcal{P}$. One can easily show that Δ is a Jordan derivation. By Proposition 1.4.34 (ii) of [6], $\mathcal{P}$ is closed and so $\frac{\mathcal{A}}{\mathcal{P}}$ is a semisimple Banach algebra. Note that every Jordan derivation on a semisimple Banach algebra is an ordinary derivation (see Corollary 5 of [2]). So, Δ is a derivation. Since $[\Delta(x), x] = 0$ for all $x \in \frac{\mathcal{A}}{\mathcal{P}}$ is equivalent to $[\Delta(x), y] = 0$ for all $x, y \in \frac{\mathcal{A}}{\mathcal{P}}$ by [[13], Proposition 2], we see that Δ is a left derivation on semisimple Banach algebra $\frac{\mathcal{A}}{\mathcal{P}}$. The proof is completed by Corollary 3.7 of [11]. □

Theorem 2.10 *Suppose that $\mathcal{A}$ is a unital, commutative UMV-Banach algebra, and $\delta : \mathcal{A} \to \mathcal{A}$ is a continuous derivation. Then $\delta(\mathcal{A}) \subseteq \mathcal{P}$, where $\mathcal{P}$ is an arbitrary minimal prime closed ideal of $\mathcal{A}$.*

Proof We know that minimal prime closed ideals in commutative algebras are invariant under derivations (see [17]). Clearly, $\frac{\mathcal{A}}{\mathcal{P}}$ is a UMV-Banach algebra and further, it is an integral domain. A linear mapping $\Delta : \frac{\mathcal{A}}{\mathcal{P}} \to \frac{\mathcal{A}}{\mathcal{P}}$ defined by $\Delta(a+\mathcal{P}) = \delta(a)+\mathcal{P}$ is a continuous derivation, and it follows from Corollary 2.5 that Δ is identically zero. It implies that $\delta(a) \in \mathcal{P}$ for all $a \in \mathcal{A}$, and consequently, $\delta(\mathcal{A}) \subseteq \mathcal{P}$. □

In the next theorem, we will add another statement to the equivalent assertions below, which have been stated in [17].

Theorem 2.11 *The following statements are equivalent:*
(i) Every derivation on a UMV-Banach algebra has a nilpotent separating ideal;

(ii) Every derivation on a semiprime UMV-Banach algebra is continuous;

(iii) Every derivation on a prime UMV-Banach algebra is continuous;

(iv) Every derivation on an integral domain UMV-Banach algebra is continuous;

(v) Every derivation on an integral domain UMV-Banach algebra is identically zero.

Conjecture 2.12 *Let $\mathcal{A}$ be a unital Banach algebra and a be an element of $\mathcal{A}$ with the UMV-property. Then $\mathfrak{S}(a) \subset \mathbb{R}$. It seems that the same is also true for the MV-property.*

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