Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2016) 40: 1085 – 1101

doi:10.3906/mat-1507-10

On the solvability of the Riemann boundary value problem in Morrey–Hardy classes

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Received: 04.07.2015 • **Accepted/Published Online:** 11.12.2015 • **Final Version:** 21.10.2016

Abstract: This work considers the Riemann boundary value problem with the piecewise continuous coefficient in Morrey–Hardy classes. Under some conditions on the coefficient, the Fredholmness of this problem is studied and the general solution of homogeneous and nonhomogeneous problems in Morrey–Hardy classes is constructed.

Key words: Morrey–Hardy classes, Riemann problem

1. Introduction

The concept of Morrey space was introduced by Morrey in 1938. Since then, various problems related to this space have been intensively studied. Playing an important role in the qualitative theory of elliptic differential equations (see, for example, [10, 13]), this space also provides a large class of examples of mild solutions to the Navier–Stokes system [9]. In the context of fluid dynamics, Morrey spaces have been used to model fluid flow when vorticity is a singular measure supported on certain sets in R^n [4]. There appeared lately a large number of research works that considered fundamental problems of the theory of differential equations, potential theory, maximal and singular operator theory, approximation theory, etc. in these spaces (see, for example, [3] and the references above). More details about Morrey spaces can be found in [11,14].

In view of the aforesaid, there has recently been a growing interest in the study of various problems in Morrey-type spaces. For example, some problems of harmonic analysis and approximation theory were considered in [1,5–8,12].

In this work, we consider the Riemann boundary value problem in Morrey-type Hardy spaces. We study the solvability of this problem and construct a general solution for both homogeneous and nonhomogeneous problems under some conditions on the coefficient of the problem.

Note that in [1] we treated the Morrey–Hardy and Morrey–Lebesgue classes. We defined the subspaces of these spaces where the shift operator was continuous.

2. Necessary information

In obtaining the main results we will use the following notation. The expression $f(x) \sim g(x)$, $x \in M$, means

$$\exists \delta > 0 : \delta \leq \left| \frac{f(x)}{g(x)} \right| \leq \delta^{-1}, \forall x \in M.$$

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2010 *AMS Mathematics Subject Classification:* 46E30; 30E25.

A similar meaning will be attached to the expression $f(x) \sim g(x),\ x \to a$.

We will also need some facts about the theory of Morrey-type spaces. Let Γ be some rectifiable Jordan curve on the complex plane C. By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$.

By the Morrey-Lebesgue space $M^{p,\alpha}(\Gamma),\ 0 \le \alpha \le 1,\ p \ge 1$, we mean a normed space of all functions $f(\cdot)$ measurable on Γ equipped with a finite norm $\|\cdot\|_{M^{p,\alpha}(\Gamma)}$:

$$\|f\|_{M^{p,\alpha}(\Gamma)} = \sup_B \left(\left| B \bigcap \Gamma \right|_\Gamma^{\alpha-1} \int_{B\bigcap\Gamma} |f(\xi)|^p \, |d\xi| \right)^{1/p} < +\infty.$$

$M^{p,\alpha}(\Gamma)$ is a Banach space and $M^{p,1}(\Gamma) = L_p(\Gamma),\ M^{p,0}(\Gamma) = L_\infty(\Gamma)$. The weighted version of the Morrey–Lebesgue space $M_\mu^{p,\alpha}(\Gamma)$ on Γ with a weight function $\mu(\cdot)$ and a norm $\|\cdot\|_{M_\mu^{p,\alpha}(\Gamma)}$ can be defined in a natural way:

$$\|f\|_{M_\mu^{p,\alpha}(\Gamma)} = \|f\mu\|_{M^{p,\alpha}(\Gamma)}, f \in M_\mu^{p,\alpha}(\Gamma).$$

The embedding $M^{p,\alpha_1}(\Gamma) \subset M^{p,\alpha_2}(\Gamma)$ is valid for $0 \le \alpha_1 \le \alpha_2 \le 1$. Thus, $M^{p,\alpha}(\Gamma) \subset L_1(\Gamma),\ \forall \alpha \in [0,1]$, $\forall p \ge 1$. The case of $\Gamma \equiv [-\pi,\pi]$ will be denoted by $M^{p,\alpha}(-\pi,\pi) \equiv M^{p,\alpha}$, and the norm $\|\cdot\|_{M^{p,\alpha}}$ by $\|\cdot\|_{p,\alpha}$, respectively.

By S_Γ we denote the following singular integral operator:

$$(S_\Gamma f)(\tau) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)\, d\zeta}{\zeta - \tau},\ \tau \in \Gamma.$$

The unit circle centered at $z = 0$ will be denoted by γ with $int\,\gamma = \omega$. Define the Morrey–Hardy space $H_+^{p,\alpha}$ of functions $f(\cdot)$ analytic inside ω with a norm $\|\cdot\|_{H_+^{p,\alpha}}$:

$$\|f\|_{H_+^{p,\alpha}} = \sup_{0<r<1} \left\| f\left(re^{it}\right) \right\|_{p,\alpha}.$$

The following theorem was proved in [1].

Theorem 1. *Function $f(\cdot)$ belongs to $H_+^{p,\alpha}$ only when $\exists f^+ \in M^{p,\alpha}$:*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f^+(\tau)\, d\tau}{\tau - z}.$$

The analog of the Smirnov theorem in Morrey–Hardy classes is also true.

Theorem 2. *Let $f \in H_+^{p_1,\alpha}$, $1 \le p_1 < +\infty$, $0 \le \alpha \le 1$, and $f^+ \in M^{p_2,\alpha}$, where $p_1 < p_2 < +\infty$, f^+ are nontangential boundary values of the function f on γ. Then $f \in H_+^{p_2,\alpha}$.*

Denote by $\tilde{M}^{p,\alpha}$ the linear subspace of $M^{p,\alpha}$ consisting of functions whose shifts are continuous in $M^{p,\alpha}$, i.e. $\|f(\cdot + \delta) - f(\cdot)\|_{p,\alpha} \to 0$ as $\delta \to 0$. The closure of $\tilde{M}^{p,\alpha}$ in $M^{p,\alpha}$ will be denoted by $MC^{p,\alpha}$. In [1] we proved the following:

Theorem 3. *Infinitely differentiable functions on $[0, 2\pi]$ are dense in the space $MC^{p,\alpha}$.*

Consider the following singular operator:

$$(Sf)(\tau) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)\,d\xi}{\xi-\tau}, \tau\in\gamma.$$

Using the results of [5,7,12], it is easy to prove the following:

Theorem 4. *The singular operator* S *acts boundedly in* $\bar{L}^{p,\alpha}(\gamma)$ *when* $0<\alpha\le 1$ *and* $1<p<+\infty$.

The following theorem can also be proved.

Theorem 5. *Let* $f\in MC^{p,\alpha}$, $0<\alpha\le 1$, $1<p<+\infty$. *Then*

$$\left\|(\mathrm{K}f)(r\xi)-f^{+}(\xi)\right\|_{p,\alpha}\to 0, r\to 1-0,$$

where $(\mathrm{K}f)(z)$ *is a Cauchy-type integral:*

$$(\mathrm{K}f)(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)\,d\xi}{\xi-z}, z\notin\gamma.$$

A similar assertion is also true for $f^{-}(\xi)$ as $r\to 1+0$.

Consider the space $H_{+}^{p,\alpha}$. Denote by $M_{+}^{p,\alpha}$ the subspace of $M^{p,\alpha}$, generated by the restrictions of the functions from $H_{+}^{p,\alpha}$ to γ. It follows directly from the above results that the spaces $H_{+}^{p,\alpha}$ and $M_{+}^{p,\alpha}$ are isomorphic and $f^{+}(\tau)=(Jf)(z)$, where $f\in H_{+}^{p,\alpha}$, f^{+} are the nontangential boundary values of f on γ, and J performs a corresponding isomorphism. Let $MC_{+}^{p,\alpha}=MC^{p,\alpha}\bigcap M_{+}^{p,\alpha}$. It is clear that $MC_{+}^{p,\alpha}$ is a subspace of $MC^{p,\alpha}$ with regard to the norm $\|\cdot\|_{M^{p,\alpha}}$. Let $\bar{H}_{+}^{p,\alpha}=J^{-1}\left(MC_{+}^{p,\alpha}\right)$. This is a subspace of $H_{+}^{p,\alpha}$. Let $f\in H_{+}^{p,\alpha}$ and f^{+} be its boundary values. It is absolutely clear that the norm $\|f\|_{H_{+}^{p,\alpha}}$ can also be defined as $\|f\|_{H_{+}^{p,\alpha}}=\|f^{+}\|_{p,\alpha}$.

Similar to the classical case, we define the Morrey–Hardy class outside ω. Let $D^{-}=C\backslash\bar{\omega}$, where $\bar{\omega}=\omega\bigcup\gamma$, C - complex plane. We will say that the function f analytic in D^{-} has finite order k at infinity if its Laurent series in a neighborhood of the point at infinity has the following form:

$$f(z)=\sum_{n=-\infty}^{k}a_n z^n, k<+\infty, a_k\neq 0. \tag{1}$$

Thus, when $k>0$, the function $f(z)$ has a pole of order k; when $k=0$, it is bounded; and when $k<0$, it has a zero of order $(-k)$. Let $f(z)=f_0(z)+f_1(z)$, where $f_0(z)$ is the main and $f_1(z)$ is the regular part of expansion (1) for the function $f(z)$. Consequently, if $k\le 0$, then $f_0(z)\equiv 0$. When $k>0$, $f_0(z)$ is a polynomial of degree k. We will say that the function $f(z)$ belongs to the class ${}_mH_{-}^{p,\alpha}$ if f has an order at infinity less than or equal to m, i.e. $k\le m$ and $f_1\left(\frac{1}{z}\right)\in H_{+}^{p,\alpha}$.

Absolutely similar to the case of $\bar{H}_{+}^{p,\alpha}$, we define the class ${}_m\bar{H}_{-}^{p,\alpha}$. In other words, ${}_m\bar{H}_{-}^{p,\alpha}$ is a subspace of functions from ${}_mH_{-}^{p,\alpha}$, whose shifts on a unit circle are continuous with regard to the norm $\|\cdot\|_{p,\alpha(\gamma)}$.

When studying the nonhomogeneous Riemann boundary value problem, we will essentially use the following:

Lemma 1. *Let* $f(\cdot) \in L_\infty$*;* $g(\cdot) \in MC^{p,\alpha} \wedge 1 \le p < +\infty$*,* $0 < \alpha \le 1$*. Then the inclusion* $f(\cdot) g(\cdot) \in MC^{p,\alpha}$ *is valid.*

Proof For $\alpha = 1$, the assertion of lemma is obvious. Let $0 < \alpha < 1$. Consider

$$\Delta_\delta = \| f(\cdot + \delta) g(\cdot + \delta) - f(\cdot) g(\cdot) \|_{p,\alpha} .$$

Take $\forall \varepsilon > 0$. As $g(\cdot) \in MC^{p,\alpha}$, $\exists \varphi(\cdot) \in C[-\pi, \pi]$:

$$\| g(\cdot) - \varphi(\cdot) \|_{p,\alpha} < \frac{\varepsilon}{m}.$$

We have

$$\Delta_\delta = \| f(\cdot + \delta) [g(\cdot + \delta) - \varphi(\cdot + \delta) + \varphi(\cdot + \delta)] -$$

$$- f(\cdot) [g(\cdot) - \varphi(\cdot) + \varphi(\cdot)] \|_{p,\alpha} \le$$

$$\le c_f \| g(\cdot + \delta) - \varphi(\cdot + \delta) \|_{p,\alpha} + \| f(\cdot + \delta) \varphi(\cdot + \delta) -$$

$$- f(\cdot) \varphi(\cdot) \|_{p,\alpha} + c_f \| g(\cdot) - \varphi(\cdot) \|_{p,\alpha} ,$$

where $c_f = \| f(\cdot) \|_{L_\infty}$. It is not difficult to see that

$$\| g(\cdot + \delta) - \varphi(\cdot + \delta) \|_{p,\alpha} = \| g(\cdot) - \varphi(\cdot) \|_{p,\alpha} < \frac{\varepsilon}{m}, \forall \delta \in R.$$

Then the previous inequality implies

$$\Delta_\delta \le \frac{2c_f}{m} \varepsilon + \| f(\cdot + \delta) \varphi(\cdot + \delta) - f(\cdot) \varphi(\cdot) \|_{p,\alpha} .$$

Thus, it suffices to prove that for $\varphi(\cdot) \in C[-\pi, \pi]$ the following is true:

$$\lim_{\delta \to 0} \| f(\cdot + \delta) \varphi(\cdot + \delta) - f(\cdot) \varphi(\cdot) \|_{p,\alpha} = 0.$$

Let $I_\pi = I \bigcap [-\pi, \pi]$. We have

$$\| f(\cdot + \delta) \varphi(\cdot + \delta) - f(\cdot) \varphi(\cdot) \|_{p,\alpha} \le \| f(\cdot + \delta) \varphi(\cdot + \delta) - f(\cdot + \delta) \varphi(\cdot) \|_{p,\alpha} +$$

$$+ \| f(\cdot + \delta) \varphi(\cdot) - f(\cdot) \varphi(\cdot) \|_{p,\alpha}$$

$$\le c_f \| \varphi(\cdot + \delta) - \varphi(\cdot) \|_{p,\alpha} + \| (f(\cdot + \delta) - f(\cdot)) \varphi(\cdot) \|_{p,\alpha} .$$

It is absolutely clear that $\lim_{\delta \to 0} \| \varphi(\cdot + \delta) - \varphi(\cdot) \|_{p,\alpha} = 0$. Therefore, $\exists \delta_1 > 0$:

$$\| \varphi(\cdot + \delta) - \varphi(\cdot) \|_{p,\alpha} < \frac{\varepsilon}{m}, \forall \delta : |\delta| < \delta_1.$$

Hence, we get

$$\Delta_\delta \leq \frac{3c_f}{m}\varepsilon + \Delta_\delta(\varphi), \forall\delta \in (-\delta_1, \delta_1),$$

where

$$\Delta_\delta(\varphi) = \|(f(\,\cdot\, + \delta) - f(\,\cdot\,))\,\varphi(\,\cdot\,)\|_{p,\alpha}.$$

Let $\nu > 0$ be an arbitrary number. We have $\Delta_\delta(\varphi) = \max\left\{\Delta_\delta^{(1)}(\nu); \Delta_\delta^{(2)}(\nu)\right\}$, where

$$\Delta_\delta^{(1)}(\nu) = \sup_{I:|I_\pi|\geq\nu}\left(|I_\pi|^{\alpha-1}\int_{I_\pi}|(f(t+\delta) - f(t))\,\varphi(t)|^p\,dt\right)^{1/p},$$

$$\Delta_\delta^{(2)}(\nu) = \sup_{I:|I_\pi|\leq\nu}\left(|I_\pi|^{\alpha-1}\int_{I_\pi}|(f(t+\delta) - f(t))\,\varphi(t)|^p\,dt\right)^{1/p}.$$

Regarding $\Delta_\delta^{(1)}(\nu)$, we have

$$\Delta_\delta^{(1)}(\nu) \leq \nu^{\frac{\alpha-1}{p}}\sup_{I:|I_\pi|\geq\nu}\left(\int_{I_\pi}|(f(t+\delta) - f(t))\,\varphi(t)|^p\,dt\right)^{1/p} \leq$$

$$\leq \nu^{\frac{\alpha-1}{p}}\left(\int_{-\pi}^{\pi}|(f(t+\delta) - f(t))\,\varphi(t)|^p\,dt\right)^{1/p} \leq c_\varphi\nu^{\frac{\alpha-1}{p}}\|f(\,\cdot+\delta) - f(\,\cdot\,)\|_p,$$

where $c_\varphi = \|\varphi(\,\cdot\,)\|_{L_\infty}$ and $\|f\|_p = \left(\int_{-\pi}^{\pi}|f|^p\,dt\right)^{1/p}$. As

$$\lim_{\delta\to 0}\|f(\,\cdot+\delta) - f(\,\cdot\,)\|_p = 0,$$

it is clear that $\exists\delta_2 > 0$:

$$\|f(\,\cdot+\delta) - f(\,\cdot\,)\|_p < \nu^{\frac{1}{p}}, \forall\delta \in (-\delta_2, \delta_2).$$

Regarding $\Delta_\delta^{(2)}(\nu)$, we have

$$\Delta_\delta^{(2)}(\nu) \leq 2c_f c_\varphi \sup_{I:|I_\pi|\leq\nu}\left(|I_\pi|^{\alpha-1}\int_{I_\pi}1dt\right)^{1/p} \leq 2c_f c_\varphi \nu^{\frac{\alpha}{p}}.$$

Let $c_{f\varphi} = \max\{c_\varphi; 2c_f c_\varphi\}$. Hence, we get $\Delta_\delta(\varphi) \leq c_{f\varphi}\nu^{\frac{\alpha}{p}}$.

It should be noted that the constant $c_{f\varphi}$ does not depend on ν. Now let us take $\nu : \nu < \left(\frac{\varepsilon}{c_{f\varphi}m}\right)^{\frac{p}{\alpha}}$. Consequently, $\Delta_\delta(\varphi) \leq \frac{\varepsilon}{m}$. Hence, we get

$$\Delta_\delta(\varphi) \leq \frac{3c_f}{m}\varepsilon + \frac{\varepsilon}{m} = \frac{3c_f + 1}{m}\varepsilon, \forall\delta \in (-\delta_3, \delta_3),$$

where $\delta_3 = \min\{\delta_1; \delta_2\}$. Taking $m = 3c_f + 1$, we have $\Delta_\delta \leq \varepsilon$, $\forall\delta \in (-\delta_3, \delta_3)$, i.e. $\Delta_\delta \to 0$, $\delta \to 0$. The lemma is proved.

We will also use the following concepts. Let $\Gamma \subset C$ be some bounded rectifiable curve and $t = t(\sigma)$, $0 \leq \sigma \leq l$ be its parametric representation with respect to the length of arc σ, where l is the length of Γ. Let $d\mu(t) = d\sigma$, i.e. $\mu(\cdot)$ is a linear measure on Γ. Let

$$\Gamma_t(r) = \{\tau \in \Gamma : |\tau - t| < r\}, \Gamma_{t(s)}(r) = \{\tau(\sigma) \in \Gamma : |\sigma - s| < r\}.$$

It is absolutely clear that $\Gamma_{t(s)}(r) \subset \Gamma_t(r)$.

Definition 1. *Curve Γ is called a Carleson curve if $\exists c > 0$:*

$$\sup_{t \in \Gamma} \mu(\Gamma_t(r)) \leq cr, \ \forall r > 0.$$

Curve Γ is said to satisfy the *chord-arc* condition at the point $t_0 = t(s_0) \in \Gamma$, if there exists a constant $m > 0$, independent of t, such that

$$|s - s_0| \leq m |t(s) - t(s_0)|, \forall t(s) \in \Gamma.$$

Γ satisfies the chord-arc condition uniformly on Γ if

$$\exists m > 0 : |s - \sigma| \leq m |t(s) - t(\sigma)|, \ \forall t(s), t(\sigma) \in \Gamma.$$

Let us state the following lemma from [12], which is interesting in itself.

Lemma 2 [12]. *Let Γ be a bounded rectifiable curve. If the exponential function $|t - t_0|^{\gamma}$, $t_0 \in \Gamma$, belongs to the space $M^{p,\alpha}(\Gamma)$, $1 \leq p < \infty$, $0 < \alpha < 1$, then $\gamma \geq -\frac{\alpha}{p}$. If Γ is a Carleson curve, then this condition is also sufficient.*

We will essentially use the following theorem of Samko [12].

Theorem 6 [12]. *Let the curve Γ satisfy the chord-arc condition and the weight $\rho(\cdot)$ be defined as follows:*

$$\rho(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}; \ \{t_k\}_1^m \subset \Gamma, \ t_i \neq t_j, i \neq j. \tag{2}$$

Singular operator S_Γ is bounded in the weighted space $M_\rho^{p,\alpha}(\Gamma)$, $1 < p < +\infty$, $0 \leq \alpha < 1$, if the following inequalities are valid:

$$-\frac{\alpha}{p} < \alpha_k < -\frac{\alpha}{p} + 1, \ k = \overline{1,m}. \tag{3}$$

Moreover, if Γ is smooth in some neighborhoods of the points t_k, $k = \overline{1,m}$, then the validity of the inequalities (3) is necessary for the boundedness of S_Γ in $M_\rho^{p,\alpha}(\Gamma)$.

In what follows, as Γ we will consider a unit circle $\gamma = \partial\omega$. Consider the weighted space $M_\rho^{p,\alpha}(\gamma) =: M_\rho^{p,\alpha}$ with the weight $\rho(\cdot)$. Let the weight $\rho(\cdot)$ satisfy the condition (3). Then, by Theorem 6, the operator S is bounded in $M_\rho^{p,\alpha}$, i.e. $\exists c > 0$:

$$\|Sf\|_{M_\rho^{p,\alpha}} \leq C \|f\|_{M_\rho^{p,\alpha}}, \forall f \in M_\rho^{p,\alpha}.$$

Let us show that $MC_{\rho}^{p,\alpha}$ is an invariant subspace with respect to the singular operator S if the inequalities (3) are fulfilled. It is absolutely clear that to do so it suffices to prove the continuity of the shift of S. Take $\forall\delta \in R$ and consider

$$(Sf)\left(\tau e^{i\delta}\right) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)\,d\xi}{\xi - \tau e^{i\delta}}.$$

We have

$$(Sf)\left(e^{i\delta}\tau\right) = \frac{1}{2\pi i}\int_{\gamma}\frac{f\left(e^{-i\delta}\xi e^{i\delta}\right)d\left(e^{-i\delta}\xi\right)}{\xi e^{i\delta} - \tau} =$$

$$= \frac{1}{2\pi i}\int_{\gamma}\frac{f\left(\xi e^{i\delta}\right)d(\xi)}{\xi - \tau}.$$

It follows that

$$(Sf)\left(e^{i\delta}\tau\right) - (Sf)(\tau) = \frac{1}{2\pi i}\int_{\gamma}\frac{f\left(\xi e^{i\delta}\right) - f(\xi)}{\xi - \tau}d\xi =$$

$$= \left(S\left(f\left(\cdot\, e^{i\delta}\right) - f(\cdot)\right)\right)(\tau).$$

Let $f \in MC_{\rho}^{p,\alpha}$. Then Theorem 8 of [12] immediately implies

$$\left\|(Sf)\left(\tau e^{i\delta}\right) - (Sf)(\tau)\right\|_{MC_{\rho}^{p,\alpha}} = \left\|\left(S\left(f\left(\cdot\, e^{i\delta}\right) - f(\cdot)\right)\right)(\tau)\right\|_{MC_{\rho}^{p,\alpha}} \le$$

$$\le C\left\|f\left(\cdot\, e^{i\delta}\right) - f(\cdot)\right\|_{MC_{\rho}^{p,\alpha}} \to 0,\ \delta \to 0.$$

Thus, the following theorem is valid.

Theorem 7. *Let the weight function $\rho(\cdot)$ be defined by (2) with $\Gamma \equiv \gamma$. If the inequalities (3) are fulfilled, then the singular operator S acts boundedly in $M!_{\rho}^{p,\alpha}$.*

Let I be some interval and $f \in M^{p,\alpha}(I)$, $g \in M^{q,\alpha}(I)$; hereinafter $\frac{1}{p} + \frac{1}{q} = 1$. We have

$$\int_I |fg|\,dt \le |I|^{1-\alpha}\sup_{x\in I, r>0} r^{\alpha-1}\int_{I_r(x)}|fg|\,dt = |I|^{1-\alpha}\|fg\|_{1,\alpha},$$

where $|I|$ is a Lebesgue measure of I, $I_r(x) \equiv I\bigcap(x-r, x+r)$. Applying Hölder's inequality, we obtain

$$\int_I |fg|\,dt \le |I|^{1-\alpha}\sup_{x\in I, r>0}\left(r^{\alpha-1}\int_{I_r(x)}|f|^p\,dt\right)^{\frac{1}{p}} \times$$

$$\times\left(r^{\alpha-1}\int_{I_r(x)}|g|^q\,dt\right)^{\frac{1}{q}} \le |I|^{1-\alpha}\sup_{x\in I, r>0}\left(r^{\alpha-1}\int_{I_r(x)}|f|^p\,dt\right)^{\frac{1}{p}} \times$$

$$\times \sup_{x\in I, r>0}\left(r^{\alpha-1}\int_{I_r(x)}|g|^q\,dt\right)^{\frac{1}{q}} = |I|^{1-\alpha}\|f\|_{p,\alpha}\|g\|_{q,\alpha}.$$

Thus, the following lemma is valid.

Lemma 3. *Let* $f\in M^{p,\alpha}(I)\wedge g\in M^{q,\alpha}(I), \frac{1}{p}+\frac{1}{q}=1,\ p\in[1,+\infty)$. *Then the following Hölder inequality holds:*

$$\|fg\|_{L_1}\le |I|^{1-\alpha}\|fg\|_{1,\alpha}\le |I|^{1-\alpha}\|f\|_{p,\alpha}\|g\|_{q,\alpha}.$$

In the sequel, we will often use the following obvious lemma.

Lemma 4. *Let* $|f(t)|\le|g(t)|$ *for almost every* $t\in[-\pi,\pi]$. *Then* $\|f\|_{M_\rho^{p,\alpha}}\le\|g\|_{M_\rho^{p,\alpha}}$.

To obtain our main result, we will also use the following lemma that follows directly from Lemma 2 of [12].

Lemma 5. *Let* $\{t_k\}_1^m\subset[-\pi,\pi]$. *The finite product* $\omega(t)=\prod_{k=1}^m|t-t_k|^{\alpha_k}$ *belongs to the space* $M^{p,\alpha}$ *if the inequalities* $\alpha_k\ge-\frac{\alpha}{p}, \forall k$, *are valid, where* $0<\alpha<1,\ 1<p<+\infty$.

3. The homogeneous Riemann problem in Morrey–Hardy classes

Let us consider the following homogeneous Riemann problem in classes $\left(H_+^{p,\alpha};\ {}_mH_-^{p,\alpha}\right)$:

$$\begin{cases} F^+(\tau)-G(\tau)F^-(\tau)=0,\ \tau\in\gamma, \\ F^+(z)\in H_+^{p,\alpha};\ F^-(z)\in\ {}_mH_-^{p,\alpha}, \end{cases} \tag{4}$$

where

$$G\left(e^{it}\right)=\left|G\left(e^{it}\right)\right|e^{i\theta(t)},\ \ \theta(t)=\arg G\left(e^{it}\right), t\in[-\pi,\pi).$$

We will treat the problem (4) with the help of the method developed by Daniluk in [2]. Introduce the following functions $X_i^{\pm}(z)$ analytic inside (with the sign +) and outside (with the sign -) the unit circle:

$$X_1(z)\equiv\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\left|G\left(e^{it}\right)\right|\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_2(z)\equiv\exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi}\theta(t)\frac{e^{it}+z}{e^{it}-z}dt\right\}.$$

Define

$$Z_i(z)\equiv\begin{cases} X_i(z),\ |z|<1, \\ \left[X_i(z)\right]^{-1},\ |z|>1. \end{cases}$$

Sokhotski–Plemelj formulas yield

$$\left|G\left(e^{it}\right)\right| = \frac{Z_1^+\left(e^{it}\right)}{Z_1^-\left(e^{it}\right)}, e^{i\theta(t)} = \frac{Z_2^+\left(e^{it}\right)}{Z_2^-\left(e^{it}\right)}.$$

Denoting $Z(z) \equiv Z_1(z) Z_2(z)$, we have

$$Z^+(\tau) - G(\tau) Z^-(\tau) = 0, \tau \in \gamma. \tag{5}$$

Following the classics, we call the function $Z(z)$ *a canonical solution of the problem* (4).

Substituting $G(\tau)$ from (5) in (4), we get

$$\frac{F^+(\tau)}{Z^+(\tau)} = \frac{F^-(\tau)}{Z^-(\tau)}, \tau \in \gamma.$$

Put

$$\Phi(z) \equiv \frac{F(z)}{Z(z)},$$

and let

$$\Phi(z) \equiv \begin{cases} \Phi^+(z), & |z| < 1, \\ \Phi^-(z), & |z| > 1. \end{cases}$$

It is not difficult to see that the function $Z(z)$ has neither a zero nor a pole if $z \notin \gamma$. Therefore, the functions $\Phi(z)$ and $F(z)$ have the same order at infinity. The results of [2] immediately imply that the function $\Phi(z)$ belongs to the Hardy class $H_\delta^\pm$ for sufficiently small $\delta > 0$. Let us show that $\Phi(z) \in H_1^\pm$. To do so, it suffices to prove that $\Phi^\pm(\tau) \in L_1(\gamma)$. The rest will immediately follow from the Smirnov theorem.

The relation $F^- \in {}_mH_-^{p,\alpha}$, true by the definition of the solution, immediately implies $F^- \in M^{p,\alpha}$. Therefore, by Lemma 2, to prove the validity of the inclusion $\Phi^- \in L_1$ it suffices to show that $\left[Z^-(\tau)\right]^{-1} \in M^{q,\alpha}$.

In the sequel, we will assume that the function $\theta(\cdot)$ is of bounded variation and has a representation $\theta(t) = \theta_0(t) + \theta_1(t)$, where $\theta_0(\cdot)$ is a continuous part of $\theta(\cdot)$ in $[-\pi, \pi]$, and $\theta_1(\cdot)$ is a jump function

$$\theta_1(-\pi) = 0, \theta_1(s) = \sum_{s_k : -\pi < s_k < s} h_k,$$

where $h_k = \theta(s_k + 0) - \theta(s_k - 0)$, $k = \overline{1, r}-$are the jumps of the function $\theta(\cdot)$ at the points of discontinuity $\{s_k\}_1^r : -\pi < s_1 < ... < s_r < \pi$. Let

$$h_0 = \theta(-\pi) - \theta(\pi), \quad h_0^{(0)} = \theta(\pi) - \theta(-\pi),$$

and

$$u_0(t) \equiv \left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0^{(0)}}{2\pi}} \exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi} \theta_0(\tau)\, ctg\frac{t-\tau}{2} d\tau\right\}.$$

Denote

$$u(t)=\prod_{k=1}^{r}\left\{\sin\left|\frac{t-s_k}{2}\right|\right\}^{\frac{h_k}{2\pi}}.$$

According to the results of [2], $\left|Z_2^-(\tau)\right|$ is expressed by the formula

$$\left|Z_2^-\left(e^{it}\right)\right|=u_0(t)\,u^{-1}(t)\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0}{2\pi}}.$$

The Sokhotski–Plemelj formula directly implies that

$$\sup_{(-\pi,\pi)} vrai\left\{\left|Z_1^-\left(e^{it}\right)\right|^{\pm1}\right\}<+\infty.$$

Thus, we have the following representation for $\left|Z^-\left(e^{it}\right)\right|^{-1}$:

$$\left|Z^-\left(e^{it}\right)\right|^{-1}=\left|Z_1^-\left(e^{it}\right)\right|^{-1}\left|u_0(t)\right|^{-1}\left|u(t)\right|\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{\frac{h_0}{2\pi}}. \tag{6}$$

According to [2], we have

$$\sup_{(-\pi,\pi)} vrai\left|u_0(\cdot)\right|^{\pm1}<+\infty.$$

Applying Lemma 2 of [12] to (6) and taking into account Lemma 5, we obtain that the function $\left|Z^-\left(e^{it}\right)\right|^{-1}$ belongs to the space $M^{q,\alpha}$ if the following inequalities are true:

$$\frac{h_k}{2\pi}\geq-\frac{\alpha}{q},\quad k=\overline{0,r}. \tag{7}$$

Thus, we obtain that if the inequalities (7) are true, then the function $\Phi^-\left(e^{it}\right)$ belongs to L_1. This follows directly from Lemma 3. Then the uniqueness theorem of [2] (Lemma 19.1, p. 194) implies that $\Phi(z)$ is a polynomial $P_m(z)$ of degree $k\leq m$. Hence, we have $F(z)\equiv Z(z)\,P_m(z)$.

Let us show that the function $F(\cdot)$ belongs to the class $H_{\pm}^{p,\alpha}$. To do so, it suffices to prove that $Z^-\left(e^{it}\right)$ belongs to $M^{p,\alpha}$. From (6) we obtain the representation

$$\left|Z^-\left(e^{it}\right)\right|=\left|Z_1^-\left(e^{it}\right)\right|\left|u_o(t)\right|\left|u(t)\right|^{-1}\left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0}{2\pi}}.$$

Using Lemma 5 again, we obtain that the inclusion $Z^-\in M^{p,\alpha}$ is true if and only if the following inequalities are fulfilled:

$$\frac{h_k}{2\pi}\leq\frac{\alpha}{p},\quad k=\overline{0,r}. \tag{8}$$

Hence, we obtain that if the inequalities (7) and (8) are fulfilled, then the general solution of the homogeneous problem (4) in classes $H_{+}^{p,\alpha} \times {}_{m}H_{-}^{p,\alpha}$ has the form $F(z) \equiv Z(z) P_m(z)$, where $P_m(\cdot)$ is an arbitrary polynomial of degree $k \le m$. Therefore, the following theorem is true.

Theorem 8. *Let the coefficient $G(\cdot)$ of the problem (4) satisfy the following conditions:*

i) $G^{\pm 1} \in L_{\infty}(\gamma)$;

ii) $\theta(t) \equiv \arg G(e^{it})$ is piecewise continuous in $[-\pi, \pi]$, $\{s_k\}_1^r : -\pi < s_1 < \ldots < s_r < \pi$ are the points of discontinuity, $h_k = \theta(s_k + 0) - \theta(s_k - 0)$, $k = \overline{1, r}$, are the corresponding jumps, $h_0 = \theta(-\pi) - \theta(\pi)$.

If the inequalities

$$-\frac{\alpha}{q} \le \frac{h_k}{2\pi} \le \frac{\alpha}{p}, k = \overline{0, r} \tag{9}$$

are fulfilled, then the homogeneous Riemann problem (4) has a general solution in classes $H_{+}^{p,\alpha} \times {}_{m}H_{-}^{p,\alpha}$ of the form $F(z) \equiv Z(z) P_m(z)$, where $Z(\cdot)$ is a canonical solution, and $P_m(\cdot)$ is an arbitrary polynomial of degree $k \le m$.

This theorem has the following corollary.

Corollary 1. *Let all the conditions of Theorem 8 be fulfilled. Then the homogeneous Riemann problem (4) has only a trivial solution in classes $H_{+}^{p,\alpha} \times {}_{m}H_{-}^{p,\alpha}$ when $m \le -1$.*

Note that in the case where the conditions i), ii) are satisfied with respect to the coefficient $G(\cdot)$, the solution of the homogeneous problem (4) belongs to the class $\bar{H}_{+}^{p,\alpha} \times {}_{m}\bar{H}_{-}^{p,\alpha}$. In fact, it follows from the expression of the solution that it suffices to show that the boundary values of $Z^{\pm}(\cdot)$ belong to the space $MC^{p,\alpha}$. We have $Z^{\pm}(\cdot) = Z_1^{\pm}(\cdot) \times Z_2^{\pm}(\cdot)$. As $Z_1^{\pm} \in L_{\infty}$, it follows from Lemma 1 that it suffices to prove the validity of inclusion $Z_2^{\pm} \in MC^{p,\alpha}$. Lemma 1 directly implies the validity of inclusion $L_{\infty} \subset M!^{p,\alpha}$. As $\theta(\cdot) \in L_{\infty}$, applying Stokhotski–Plemelj formulas to $Z_2(z)$, we obtain from Theorem 9 that the inclusion $Z_2^{\pm} \in M!^{p,\alpha}$ is valid. Thus, the following statement is true.

Statement 1. *Let all the conditions of Theorem 8 be satisfied. Then the solution of the problem (4) belongs to the class $\bar{H}_{+}^{p,\alpha} \times {}_{m}\bar{H}_{-}^{p,\alpha}$.*

Remark 1. It should be noted that for $\alpha \to 1 - 0$ the inequalities (9) become

$$-\frac{1}{q} < \frac{h_k}{2\pi} < \frac{1}{p},\ k = \overline{0, r}, \tag{10}$$

which are sufficient for finding the general solution of the homogeneous Riemann problem (4) in Hardy classes $H_{+}^{p} \times {}_{m}H_{-}^{p}$. For this case, the theory of the Riemann problem has been well developed by Daniluk [2]. We obtain that if the inequalities (9) are true for some $\alpha \in (0, 1)$, then the general solution of the homogeneous Riemann problem (4) in Hardy classes $H_{+}^{p} \times {}_{m}H_{-}^{p}$ has the form $F(z) \equiv Z(z) P_m(z)$, where $Z(\cdot)$ is a canonical solution, and $P_m(\cdot)$ is an arbitrary polynomial of degree $k \le m$.

On the contrary, if the inequalities (10) are true, then it is clear that there exists $\alpha \in (0, 1)$ such that the inequalities (9) are also true. Hence, the validity of the assertion of Theorem 8 follows.

4. The nonhomogeneous Riemann problem in Morrey–Hardy classes

In this section we consider the nonhomogeneous Riemann boundary value problem,

$$F^{+}(\tau) - G(\tau) F^{-}(\tau) = f(\arg \tau), \; \tau \in \partial\omega, \tag{11}$$

in Morrey–Hardy classes $H_{+}^{p,\alpha} \times {}_{m}H_{-}^{p,\alpha}$, $\alpha \in (0, 1)$, $1 < p < +\infty$, where $f \in M^{p,\alpha}$ is some given function.

Let $Z(\cdot)$ be a canonical solution of a homogeneous problem corresponding to the problem (11). Consider the integral

$$F_1(z) = \frac{Z(z)}{2\pi} \int_{-\pi}^{\pi} \left[Z^{+}\left(e^{it}\right)\right]^{-1} K_z(t) f(t)\, dt \tag{12}$$

with Cauchy kernel $K_z(t) \equiv \frac{e^{it}}{e^{it}-z}$. By Stokhotski–Plemelj formulas, from (12) we obtain

$$F_1^{\pm}(\tau) = Z^{\pm}(\tau) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z^{+}(e^{it})} \frac{e^{it}dt}{e^{it}-z}\right]_{\gamma}^{\pm} =$$

$$= Z^{\pm}(\tau) \left(\pm\frac{1}{2}\left[Z^{+}(\tau)\right]^{-1} f(\arg \tau) - \left[Z^{+}(\tau)\right]^{-1} (K_\rho f)(\tau)\right),$$

where the expression $[\cdot]_{\gamma}^{\pm}$ means boundary values on γ from inside ($sign$ "$+$") and ($sign$ "$-$") outside of ω, respectively, and K_ρ is a singular Cauchy operator

$$(K_\rho f)(\tau) = \frac{Z^{+}(\tau)}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z^{+}(e^{it})} \frac{e^{it}dt}{e^{it}-\tau}, \tau \in \gamma.$$

We have

$$\frac{F_1^{+}(\tau)}{Z^{+}(\tau)} - \frac{F_1^{-}(\tau)}{Z^{-}(\tau)} = \frac{f(\arg\tau)}{Z^{+}(\tau)}, \tau \in \gamma. \tag{13}$$

Taking into account the fact that $Z(\cdot)$ satisfies the homogeneous boundary condition

$$Z^{+}(\tau) - G(\tau) Z^{-}(\tau) = 0, \tau \in \gamma,$$

we have

$$\frac{Z^{+}(\tau)}{Z^{-}(\tau)} = G(\tau) \text{ for almost every } \tau \in \gamma.$$

Substituting in (13), we get

$$F_1^{+}(\tau) - G(\tau) F_1^{-}(\tau) = f(\arg \tau) \text{ for almost every } \tau \in \gamma.$$

Thus, the boundary values $F_1^{\pm}(\cdot)$ satisfy the relation (11). Let us show that the function $F_1(\cdot)$ belongs to the required class, i.e. the inclusion

$$\left(F_1^{+}(z);\, F_1^{-}(z)\right) \in H_{+}^{p,\alpha} \times {}_{m}H_{-}^{p,\alpha},$$

is true. Acting just as in the previous section, we get

$$\left|Z^{-}\left(e^{it}\right)\right| = \left|Z_1^{-}\left(e^{it}\right)\right| \left|u_0\left(t\right)\right| \prod_{k=0}^{r} \left|\frac{t-s_k}{2}\right|^{-\frac{h_k}{2\pi}},$$

where

$$u_0\left(t\right) = \left\{\sin\left|\frac{t-\pi}{2}\right|\right\}^{-\frac{h_0^{(0)}}{2\pi}} \exp\left\{-\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta_0\left(\xi\right) ctg\frac{t-\xi}{2}d\xi\right\}.$$

It is not difficult to see that the following relation is true:

$$\left|Z^{+}\left(\tau\right)\right| \sim \left|Z^{-}\left(\tau\right)\right|, \tau\in\gamma.$$

For $F_1^{+}\left(\cdot\right)$ we have

$$F_1^{+}\left(\tau\right) = \frac{1}{2}f\left(\arg\tau\right) - \left(K_\rho f\right)\left(\tau\right) \text{ for almost every } \tau\in\gamma.$$

It follows that the inclusion $F_1^{+}\left(\cdot\right)\in M^{p,\alpha}$ is true if and only if the inclusion $\left(K_\rho f\right)\left(\cdot\right)\in M^{p,\alpha}$ is true. It is absolutely clear that

$$\left|Z^{\pm}\left(e^{it}\right)\right| \sim \prod_{k=0}^{r}\left\{\sin\left|\frac{t-s_k}{2}\right|\right\}^{-\frac{h_k}{2\pi}}, t\in\left[-\pi,\pi\right].$$

We have

$$\sin\left|\frac{t-s_k}{2}\right| \sim \left|t-s_k\right|, \ \ t\in\left[-\pi,\pi\right], \ \ k=\overline{1,r}.$$

For $s_0=\pi$ the following relation is true:

$$\sin\left|\frac{t-\pi}{2}\right| \sim \left|t-\pi\right|\left|t+\pi\right|, \ \ t\in\left[-\pi,\pi\right].$$

Taking into account these relations, we obtain

$$\left|Z^{\pm}\left(e^{it}\right)\right| \sim \left|t-\pi\right|^{-\frac{h_0}{2\pi}}\left|t+\pi\right|^{-\frac{h_0}{2\pi}}\prod_{k=1}^{r}\left|t-s_k\right|^{-\frac{h_k}{2\pi}}, \ \ t\in\left[-\pi,\pi\right].$$

Put

$$\rho\left(t\right) = \left|t^2-\pi^2\right|^{-\frac{h_0}{2\pi}}\prod_{k=1}^{r}\left|t-s_k\right|^{-\frac{h_k}{2\pi}}, \ \ t\in\left[-\pi,\pi\right].$$

Let

$$\hat{\hat{f}}\left(t\right) = \frac{f\left(t\right)}{\rho\left(t\right)}, \ \ t\in\left[-\pi,\pi\right].$$

We have an inclusion $\hat{\hat{f}}\rho = f \in M^{p,\alpha}$, i.e. we have by definition $\hat{\hat{f}} \in M^{p,\alpha}$. Consider the following singular operator:

$$\left(\hat{S}_\rho \hat{\hat{f}}\right)(\tau) = \int_{-\pi}^{\pi} \frac{\hat{f}(t)\, e^{it}}{e^{it} - \tau} dt, \tau \in \gamma.$$

It is not difficult to see that the operator K_ρ is bounded in $M^{p,\alpha}$ if and only if it is bounded in $M^{p,\alpha}_\rho$. Applying Theorem 6 of [12] to the operator K_ρ, we obtain that if the inequalities

$$-\frac{\alpha}{p} < -\frac{h_k}{2\pi} < -\frac{\alpha}{p} + 1, \quad \Leftrightarrow$$

$$\frac{\alpha}{p} - 1 < \frac{h_k}{2\pi} < \frac{\alpha}{p}, \quad k = \overline{0, r} \tag{14}$$

are true, then the singular operator K_ρ acts boundedly in $M^{p,\alpha}$. Hence, we have that if the inequalities (14) are true, then the function $F_1^+(\cdot)$ belongs to the space $M^{p,\alpha}$. Then it follows from Theorem 2 that $F_1 \in H^{p,\alpha}_+$. As $F_1(\infty) = 0$, it is clear that $F_1 \in {}_{-1}H^{p,\alpha}_-$, and consequently $\left(F_1^+(z)\,;\, F_1^-(z)\right) \in H^{p,\alpha}_+ \times {}_{-1}H^{p,\alpha}_-$. Thus, the following statement is true.

Statement 2. *Let the coefficient $G(\cdot)$ of the problem (11) satisfy the conditions i), ii) and the inequalities (14) be true. Then the inclusion*

$$\left(F_1^+(z)\,;\, F_1^-(z)\right) \in H^{p,\alpha}_+ \times {}_{-1}H^{p,\alpha}_-,$$

is valid, where the particular solution $F_1(\cdot)$ is defined by (12).

Let us find the general solution of nonhomogeneous problem (11) in classes $H^{p,\alpha}_+ \times {}_mH^{p,\alpha}_-$. We first consider the case $m \geq -1$. It is clear that in this case the particular solution (12) belongs to the class $H^{p,\alpha}_+ \times {}_mH^{p,\alpha}_-$. Denote by $F_0(\cdot)$ the general solution of homogeneous problem (4). Applying Theorem 8, we obtain that if the inequalities (9) are true, then $F_0(\cdot)$ has the form $F_0(z) = Z(z)\, P_m(z)$, where $Z(\cdot)$ is a canonical solution, and $P_m(\cdot)$ is a polynomial of degree $\leq m$. Comparing inequalities (9) and (14), we obtain

$$-\frac{\alpha}{q} \leq \frac{h_k}{2\pi} < \frac{\alpha}{p}, k = \overline{0, r}. \tag{15}$$

Now consider the case $m < -1$. In this case, as it follows from Corollary 1, the homogeneous problem is only trivially solvable if the inequalities (15) are true. In this case, the nonhomogeneous problem (11) is solvable in the class $H^{p,\alpha}_+ \times {}_mH^{p,\alpha}_-$ if and only if the following $(-m-1)$ orthogonality relations are true:

$$\int_{-\pi}^{\pi} \frac{f(t)}{Z^+(e^{it})} e^{ikt} dt = 0, k = \overline{1, -m-1}. \tag{16}$$

These relations follow directly from the Taylor series expansion for the Cauchy type integral

$$K(z) \equiv \int_{-\pi}^{\pi} \left[Z^+\left(e^{it}\right)\right]^{-1} K_z(t)\, f(t)\, dt,$$

with respect to the degrees of z in a neighborhood of the point at infinity $z=\infty$:

$$K(z)\equiv -\frac{1}{z}\int_{-\pi}^{\pi}\frac{f(t)}{Z^{+}(e^{it})}\frac{e^{it}dt}{1-e^{it}z^{-1}}=-\sum_{k=1}^{\infty}\int_{-\pi}^{\pi}\frac{f(t)}{Z^{+}(e^{it})}e^{ikt}dtz^{-k}.$$

As we have $|Z^{-}(\infty)|^{\pm}<+\infty$, it is clear that the order of the function $F_1(\cdot)$ at infinity coincides with the order of the Cauchy type integral $K(z)$. It is absolutely clear that if the orthogonality conditions (16) are true, then the nonhomogeneous problem (11) is uniquely solvable in classes $H_{+}^{p,\alpha}\times {}_{m}H_{-}^{p,\alpha}$. Thus, the following theorem is true.

Theorem 9. *Let the coefficient $G(\cdot)$ of the problem (11) satisfy the conditions $i)$, $ii)$ and*

$$h_k=\theta(s_k+0)-\theta(s_k-0),\ k=\overline{1,r}$$

be the jumps of the function $\theta(t)\equiv\arg G(e^{it})$ at the points of discontinuity

$$\{s_k\}_1^r\subset(-\pi,\pi);h_0=\theta(-\pi)-\theta(\pi).$$

Assume that the following inequalities are fulfilled:

$$-\frac{\alpha}{q}\leq\frac{h_k}{2\pi}<\frac{\alpha}{p},k=\overline{0,r}. \tag{17}$$

Then the following assertions are true with regard to the solvability of nonhomogeneous problem (11) in the class $H_{+}^{p,\alpha}\times {}_{m}H_{-}^{p,\alpha}$:

$\alpha)$ when $m\geq-1$, the problem (11) has a general solution $F(\cdot)$ of the form

$$F(z)=Z(z)P_m(z)+F_1(z),$$

where $Z(\cdot)$ is a canonical solution of the homogeneous problem (4), $P_m(\cdot)$ is an arbitrary polynomial of degree $k\leq m$, $F_1(\cdot)$ is a particular solution of the form

$$F_1(z)=\frac{Z(z)}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{Z^{+}(e^{it})}K_z(t)\,dt, \tag{18}$$

$K_z(\cdot)$ is a Cauchy kernel, and $f\in M^{p,\alpha}$ is an arbitrary function;

$\beta)$ when $m<-1$, the problem (11) is solvable if and only if the orthogonality conditions

$$\int_{-\pi}^{\pi}\frac{f(t)}{Z^{+}(e^{it})}e^{ikt}dt=0,k=\overline{1,-m-1}, \tag{19}$$

are true, and $F(z)\equiv F_1(z)$ - is a unique solution of this problem.

This theorem has the following corollary.

Corollary 2. *Let all the conditions of Theorem 9 be fulfilled. Then the nonhomogeneous problem (11) with arbitrary $f\in L^{p,\alpha}$ has a unique solution $F_1(\cdot)$ in the class $H_{+}^{p,\alpha}\times {}_{-1}H_{-}^{p,\alpha}$, which can be represented in the form of a Cauchy-type integral (18).*

Let us consider the case where the right-hand side of the problem (11) belongs to the space $\overline{L^{p,\alpha}}$. It follows directly from Theorem 9 that the boundary values $F_1^{\pm}(\cdot)$ of the function $F_1(z)$ defined by (12) also belong to $MC^{p,\alpha}$ if the inequalities (3) are true. Then the condition $i)$ and Lemma 1 imply that the product $G(\cdot)F_1^{-}(\cdot)$ belongs to $MC^{p,\alpha}$. Consequently, similar to the proof of Theorem 9, we get the validity of the following theorem.

Theorem 10. *Let all the conditions of Theorem 9 be fulfilled. Then the following assertions are true with regard to the solvability of the problem (11) with a right-hand side* $f(\cdot)\in MC^{p,\alpha}$ *in the class* $\overline{H_+^{p,\alpha}}\times\overline{{}_mH_-^{p,\alpha}}$*:*

$\alpha)$ *when* $m\geq -1$*, the problem (11) has a general solution* $F(\cdot)$ *of the form*

$$F(z)=Z(z)P_m(z)+F_1(z),$$

where $Z(\cdot)$ *is a canonical solution,* $P_m(\cdot)$ *is a polynomial of degree* $\leq m$*, and* $F_1(\cdot)$ *is a particular solution of the form (18);*

$\beta)$ *when* $m<-1$*, the problem (11) is solvable if and only if the orthogonality conditions (19) are true.*

Remark 2. Again it should be noted that for $\alpha\to 1-0$ the inequalities (17) become

$$-\frac{1}{q}<\frac{h_k}{2\pi}<\frac{1}{p}, k=\overline{0,r}. \tag{20}$$

The inequalities (20) are sufficient for finding the general solution of nonhomogeneous problem (11) in classical Hardy classes $H_+^p\times{}_mH_-^p$. The theory of this problem was developed by Daniluk [2]. If the inequalities (17) hold, then the assertions $\alpha)$ and $\beta)$ of Theorem 9 are true with regard to the solvability of nonhomogeneous problem (11) in the class $H_+^p\times{}_mH_-^p$. On the contrary, if the inequalities (20) hold, then there exists $\alpha\in(0,1)$ such that the inequalities (17) are true, and hence the assertions of Theorem 9 are valid.

Acknowledgment

This work was supported by the Research Program Competition launched by the National Academy of Sciences of Azerbaijan (Program: Frame Theory Applications of Wavelet Analysis to Signal Processing in Seismology and Other Fields).

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