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Regularity and projective dimension of some class of well-covered graphs

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Abstract: In this paper we study the Castelnuovo–Mumford regularity of an edge ideal associated with a graph in a special class of well-covered graphs. We show that if G belongs to the class $\mathcal{SQ}$, then the Castelnuovo–Mumford regularity of $R/I(G)$ will be equal to induced matching number of G. For this class of graphs we also compute the projective dimension of the ring $R/I(G)$. As a corollary we describe these invariants in well-covered forests, well-covered chordal graphs, Cohen–Macaulay Cameron–Walker graphs, and simplicial graphs.

Key words: Castelnuovo–Mumford regularity, edge ideal, induced matching, projective dimension, well-covered graph

1. Introduction

Let G be a simple graph (no loops or multiple edges) with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. We can associate to G a square-free monomial ideal $I(G) =< x_ix_j : x_ix_j \in E(G) >$ in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$. The relation between the algebraic invariant of this ideal and combinatorial properties of its corresponding graph is one of the interesting measures for many researchers. The Castelnuovo–Mumford regularity of an ideal I, denoted by $\mathrm{reg}(I)$, is one of these measures for the complexity of I.
Recently, several mathematicians have studied the Castelnuovo–Mumford regularity and projective dimension of edge ideals of graphs. In [18] Zheng described the regularity and projective dimension for trees. Hà and Van Tuyl [3] extended this description for regularity to chordal graphs, Kummini [11] for unmixed bipartite graphs, Mahmoudi et al. [12] for very well-covered graphs, and Khosh-Ahang and Moradi [9] for C_5-free vertex decomposable graphs. Kimura [10] extended the characterization of the projective dimension introduced in [18] to chordal graphs. She introduced d'_G for any graphs and it was shown that for a chordal graph G, one has $\mathrm{pd}(R/I(G)) = d'_G$; see Section 4 for the definition. Van Tuyl [16] also proved that the equality $\mathrm{reg}(G) = \mathrm{im}(G)$ holds for any sequentially Cohen–Macaulay bipartite graph.

A 5-cycle C as a subgraph of a graph G is called basic if it does not contain two adjacent vertices of degree three or more in G. A vertex x is called a *simplicial vertex* if $N_G[x]$ is a clique and $G_{N_G[x]}$, the induced subgraph of G, is called a simplex of G. A graph G is said to be simplicial if every vertex of G belongs to a simplex of G. A 4-cycle Q as a subgraph of a graph G is called basic if it contains two adjacent vertices of degree two and the remaining two vertices belong to a simplex or a basic 5-cycle of G. The graph G is in the class $\mathcal{SQC}$ if there are simplicial vertices $x_1, \ldots, x_m$; basic 5-cycles $C_1, \ldots, C_s$; and basic 4-cycles

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$Q_1, \ldots, Q_t$ such that $\{N_G[x_1], \ldots, N_G[x_m], V(C_1), \ldots, V(C_s), B(Q_1), \ldots, B(Q_t)\}$ is a partition of $V(G)$, where $B(Q_j)$ is the set of two vertices of degree 2 of the basic 4-cycle Q_j. The class $\mathcal{SQC}$ is a subclass of well-covered graphs [14, Theorem 3.1] and every graph in this class is vertex decomposable [4, Theorem 2.3]. We say that a graph G is in the class $\mathcal{SQ}$ if $\{B(Q_1), \ldots, B(Q_t), N_G[x_1], \ldots, N_G[x_m]\}$ is a partition of $V(G)$. This class of graphs is a subclass of well-covered graphs and contains some well-covered classes such as well-covered forests, well-covered chordal graphs, Cohen–Macaulay Cameron–Walker graphs, and simplicial graphs.

This paper is organized as follows: in Section 2, we refer to some definitions and some known results that will be needed later. In Section 3, we show that the regularity of each graph in class $\mathcal{SQ}$ is equal to the induced matching number of it; see Theorem 3.3. In Section 4, we prove that if G belongs to the class $\mathcal{SQ}$, then $\mathrm{pd}(R/I(G)) = d'_G$; see Theorem 4.4.

2. Preliminaries

In this paper we will use some notation on graphs according to [2]. For notation about combinatorics and the algebraic background we refer to [5] and [17].

Let G be a simple graph with vertex set $V(G)$ and the edge set $E(G)$. A subgraph H of a graph G is said to be *induced* if, for any pair of vertices u and v of H, $uv \in E(H)$ if and only if $uv \in E(G)$. The induced subgraph of G restricted to $A \subseteq V(G)$ is denoted by G_A. The *neighborhood* of a vertex $v \in V$ is the set $N_G(v) = \{u \mid u \in V(G),\ vu \in E(G)\}$. Similarly, for $A \subseteq V(G)$, we have $N_G(A) = \cup_{v \in A} N_G(v)$ and $N_G[A] = A \cup N_G(A)$.

The *complete* graph on n vertices is denoted by K_n. A *bouquet* is the complete bipartite graph $K_{1,n} (n \geq 0)$ consisting of $n+1$ vertices and edges joining one vertex to all the others. A *leaf* in a graph G is a vertex of degree 1; the edge meeting a leaf is called a *leaf* edge or *pendant* edge; the leaf edge is said to be attached to G at its non-leaf vertex. A *pendant triangle* in a graph G is a triangle where two vertices have degree 2 and the third vertex has degree greater than 2; the pendant triangle is said to be attached to G at the vertex with degree greater than 2 and a *star triangle* is a graph joining some triangles at one common vertex.

Recall that a subset $M \subseteq E$ is called a *matching* of G if no two edges in M share a common end and a *maximum matching* is a matching that contains the largest possible number of edges. The *matching number* of G denoted by $\mathrm{m}(G)$ is the cardinality of a maximum matching. Moreover, a matching M of G is an induced matching if it occurs as an induced subgraph of G and the cardinality of a maximum induced matching is called the *induced matching number* of G and denoted by $\mathrm{im}(G)$.

An *independent* set of G is a set of pairwise nonadjacent vertices. A graph G is called *well-covered* (or equivalently *unmixed*) if all its maximal independent sets are of the same size. A *vertex cover* of G is a subset $C \subseteq V(G)$ if it intersects all edges of G and it is called minimal if it has no proper subset that is also a vertex cover of G.

When G is a graph with $V(G) = \{x_1, \ldots, x_n\}$ and $C = \{x_{i_1}, \ldots, x_{i_t}\}$ is a vertex cover of G, by x^C we mean the monomial $x_{i_1} \ldots x_{i_t}$ in the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$. For a monomial ideal $I = \langle x_{1_1} \ldots x_{1_{n_1}}, \ldots, x_{t_1} \ldots x_{t_{n_t}} \rangle$ of the polynomial ring R, the Alexander dual ideal of I, denoted by $I^\vee$, is defined as

$$I^\vee := \langle x_{1_1}, \ldots, x_{1_{n_1}} \rangle \cap \ldots \cap \langle x_{t_1}, \ldots, x_{t_{n_t}} \rangle.$$

One can see that, for a graph G,

$$I(G)^\vee = \langle x^C | C \text{ is a minimal vertex cover of } G \rangle.$$

A graph G is called vertex decomposable if either it is an edgeless graph or it has a vertex x such that:

(i) $G \setminus \{x\}$ and $G \setminus N_G[x]$ are both vertex decomposable.

(ii) For every independent set S of $G \setminus N_G[x]$, there is some vertex $y \in N_G(x)$ such that $S \cup \{y\}$ is an independent set of G.

3. The Castelnuovo–Mumford regularity

In this section we give some descriptions for the regularity of the ring R/I(G) when G is in the class $\mathcal{SQ}$.

Definition 3.1 *Let G be a graph and consider the minimal free graded resolution of $M = R/I(G)$ as an $R-$module.*

$$0 \to \oplus_{j\in\mathbb{Z}} R(-j)^{\beta_{pj}(M)} \to \cdots \to \oplus_{j\in\mathbb{Z}} R(-j)^{\beta_{0j}(M)} \to M \to 0$$

The Castelnuovo–Mumford regularity or simply the regularity of $M = R/I(G)$ is defined as

$$\operatorname{reg}(R/I(G)) := max\{j - i : \beta_{i,j} \neq 0\}.$$

We define $\operatorname{reg}(G) := \operatorname{reg}(R/I(G))$.

Also, the projective dimension of M is defined as

$$\operatorname{pd}(M) := max\{i : \beta_{i,j} \neq 0 \text{ for some } j\}.$$

The following proposition that is proved in [13] is one of our main tools in the study of the regularity of certain graphs.

Proposition 3.2 *[13, Theorem 3.14] Let $\mathcal{K}$ be a family of graphs containing any edgeless graph and let $\beta : \mathcal{K} \longrightarrow \mathbb{N}$ be a function satisfying $\beta(G) = 0$ for any edgeless graph G. In addition, for a graph $G \in \mathcal{K}$, with $E(G) \neq \emptyset$, there exists $x \in V(G)$ such that the following two conditions hold:*

(i) $G \setminus \{x\}$ *and* $G \setminus N[x]$ *are in* $\mathcal{K}$.

(ii) $\beta(G \setminus N[x]) < \beta(G)$ *and* $\beta(G \setminus \{x\}) \leq \beta(G)$.

Then $\operatorname{reg}(G) \leq \beta(G)$ *for any* $G \in \mathcal{K}$.

Now we are ready to describe the regularity of R/I(G) when $G \in \mathcal{SQ}$.

Theorem 3.3 *If G belongs to the class $\mathcal{SQ}$, then*

$$\operatorname{reg}(G) = \operatorname{im}(G).$$

Proof We already know that $\operatorname{im}(G) \leq \operatorname{reg}(G)$ for any graph G [8, Lemma 2.2], so it is sufficient to show that $\operatorname{im}(G)$ is an upper bound for $\operatorname{reg}(G)$ when G belongs to the class $\mathcal{SQ}$.

Let $x_1, \ldots, x_m$ be simplicial vertices and $Q_1, \ldots, Q_t$ be basic 4-cycles of G. Then $V(G) = (\dot{\bigcup}_{i=1}^m N_G[x_i]) \dot{\cup} (\dot{\bigcup}_{j=1}^t B(Q_j))$. Assume that $x \in N_G(x_1)$ and without loss of generality $x \in Q_j = xy_jz_jt_jx$ for $1 \leq j \leq r$, for which $\deg(z_j) = \deg(t_j) = 2$.

If we set $G' := G \setminus \{x\}$ and $G'' := G \setminus N_G[x]$, then

$$V(G') = (\dot{\bigcup}_{i=1}^{r} N_{G'}[t_i]) \dot{\cup} (\dot{\bigcup}_{i=1}^{m} N_{G'}[x_i]) \dot{\cup} (\dot{\bigcup}_{i=r+1}^{t} B(Q_i)),$$

for which $N_{G'}[t_j] = t_j z_j \cong k_2$ and $\deg(t_j) = 1$, so t_j for $1 \leq j \leq r$ is a simplicial vertex of G', $N_{G'}[x_1] = N_G[x_1] \setminus \{x\}$ and $N_{G'}[x_i] = N_G[x_i]$ for $i \geq 2$. Therefore, $G' \in \mathcal{SQ}$.

On the other hand,

$$V(G'') = \{z_1, \ldots, z_r\} \dot{\cup} (\dot{\bigcup}_{i=2}^{m} N_{G''}[x_i]) \dot{\cup} (\dot{\bigcup}_{i=r+1}^{t} B(Q_i)),$$

for which $N_{G''}[x_i] = N_G[x_i] \setminus N_G[x]$ for $i \geq 2$, $Q_j \setminus N_G[x] = \{z_j\}$ and $N_{G''}[z_j] = \{z_j\}$, so z_j is a simplicial vertex for $1 \leq j \leq r$.

Therefore, G'' belongs to the class $\mathcal{SQ}$.

The inequality $\operatorname{im}(G \setminus \{x\}) \leq \operatorname{im}(G)$ always holds and for any $x \in V(G)$.

It is clear that if M is an induced matching for $G \setminus N_G[x]$, then $M \cup \{xx_i\}$ will be an induced matching for G, so $\operatorname{im}(G \setminus N[x]) < \operatorname{im}(G)$. Hence, by Proposition 3.2, $\operatorname{reg}(G) \leq \operatorname{im}(G)$.

□

In [6] Herzog et al. characterized Cohen–Macaulay chordal graphs by showing that we can partition the vertex set of every Cohen–Macaulay chordal graph to cliques such that each of the cliques contains a simplicial vertex [6, Theorem 2.1]. This means that these graphs belong to the class $\mathcal{SQ}$. Also, one can see that any well-covered simplicial graph belongs to the class $\mathcal{SQ}$.

Corollary 3.4 *The following statements hold.*

(*a*) *If G is a simplicial graph, then* $\operatorname{reg}(G) = \operatorname{im}(G)$.

(*b*) *If G is a well-covered chordal graph, then* $\operatorname{reg}(G) = \operatorname{im}(G)$.

In [1] Cook and Nagel showed that if we choose a partition $\pi = \{W_1, \ldots, W_t\}$ of cliques for $V(G)$, add new vertices $y_1, \ldots, y_t$, and connect y_i to every vertex in the clique W_i for $1 \leq i \leq t$, then the new graph, clique-whiskered of G, denoted by G^π, will be well-covered and vertex decomposable. Clearly, $W_i = N_{G^\pi}(y_i)$, so y_i is a simplicial vertex for $1 \leq i \leq t$ and $W_1 \cup \{y_1\}, \ldots, W_t \cup \{y_t\}$ is a partition for $V(G^\pi)$. This means that G^π belongs to the class $\mathcal{SQ}$.

Corollary 3.5 *For every graph G and any partition $\pi = \{W_1, \ldots, W_t\}$ of cliques for $V(G)$,*

$$\operatorname{reg}(G^\pi) = \operatorname{im}(G^\pi).$$

Let G be a simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$. Also, let $G_1, \ldots, G_n$ be simple graphs with disjoint vertex sets and $v_i \in V(G_i)$ for $1 \leq i \leq n$. If we set H a graph with $V(H) = V(G) \cup (\bigcup_{i=1}^{n} (V_i \setminus \{v_i\}))$ and $E(H) = E(G) \cup (\bigcup_{i=1}^{n} E(G_i \setminus \{v_i\})) \cup \{x_i v | v_i v \in E(G_i)\}$, then H is the graph obtained by attaching G_i to G on the vertex v_i for all $i = 1, \ldots, n$.

We say that a finite connected simple graph G is a *Cameron–Walker* graph if $\mathrm{im}(G) = \mathrm{m}(G)$ and if G is neither a bouquet nor a star triangle. In [7] Hibi et al. characterized Cohen–Macaulay Cameron–Walker graphs; they showed that a Cameron–Walker graph G is Cohen–Macaulay if and only if G consists of a connected bipartite graph with vertex partition $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_m\}$ such that there is exactly one leaf edge attached to each vertex x_i and that there is exactly one pendant triangle attached to each vertex y_i. It is clear that any Cohen–Macaulay Cameron–Walker graph belongs to the class $\mathcal{SQ}$.

Corollary 3.6 *If G is a Cohen–Macaulay Cameron–Walker graph, then*

$$\mathrm{reg}(G) = \mathrm{im}(G).$$

4. Projective dimension

Let B be a bouquet with $V(B) = \{w, z_1, \dots, z_d\}$ and $E(B) = \{\{w, z_i\} : i = 1, \dots, d\}$. The vertex w is called the *root* of B, the vertices z_i *flowers* of B, and the edges $\{w, z_i\}$ *stems* of B. A subgraph B of G is called a *bouquet of* G if B is a bouquet. Let $\mathcal{B} = \{B_1, B_2, \dots, B_j\}$ be a set of bouquets of G. We set

$F(\mathcal{B}) := \{z \in V : z \text{ is a flower of some bouquet in } \mathcal{B}\}$,

$R(\mathcal{B}) := \{w \in V : w \text{ is a root of some bouquet in } \mathcal{B}\}$,

$S(\mathcal{B}) := \{s \in E(G) : s \text{ is a stem of some bouquet in } \mathcal{B}\}$.

A set $\mathcal{B} = \{B_1, B_2, \dots, B_j\}$ of bouquets of G is said to be *strongly disjoint* in G if the following 2 conditions are satisfied:

(1) $V(B_k) \cap V(B_l) = \emptyset$ for all $k \neq l$.

(2) We can choose a stem s_k from each bouquet $B_k \in \mathcal{B}$ so that $\{s_1, s_2, \dots, s_j\}$ is an induced matching in G.

A set $\mathcal{B} = \{B_1, B_2, \dots, B_j\}$ of bouquets of G is called *semi-strongly disjoint* in G if

(1) $V(B_i) \cap V(B_j) = \emptyset$ for all $i \neq j$,

(2) $R(\mathcal{B})$ is an independent set of G.

We set:

$$d_G := max\{\#F(\mathcal{B}) : \mathcal{B} \text{ is a strongly disjoint set of bouquets of } G\}$$

and

$$d'_G := max\{\#F(\mathcal{B}) : \mathcal{B} \text{ is a semi-strongly disjoint set of bouquets of } G\}.$$

Clearly any induced matching in G is a strongly disjoint set of bouquets of G and any strongly disjoint set of bouquets of G is semi-strongly disjoint, so $\mathrm{im}(G) \leq d_G \leq d'_G$.

We use the following proposition for proving our result.

Proposition 4.1 *[15, Theorem 2.1] If I is a square-free monomial ideal, then*

$$\mathrm{pd}(I^{\vee}) = \mathrm{reg}(R/I).$$

Lemma 4.2 *If $G \in \mathcal{SQ}$ and $V(G) = (\dot{\bigcup}_{i=1}^{m} N_G[x_i])\dot{\cup}(\dot{\bigcup}_{j=1}^{t} B(Q_j))$, then $d'_G = |V(G)| - (m+t)$.*

Proof Let $\mathcal{B}$ be a semi-strongly disjoint set of bouquets of G. Clearly, at least one vertex of $N_G[x_i]$ belongs to $R(\mathcal{B})$ for any $1 \leq i \leq m$. Similarly, for any $1 \leq j \leq t$, at least one vertex of $B(Q_j)$ belongs to $R(\mathcal{B})$; this means that $|R(\mathcal{B})| \geq m+t$ and so $|F(\mathcal{B})| \leq |V(G)| - (m+t)$. Hence, $d' \leq |V(G)| - (m+t)$. On the other hand, for any $1 \leq i \leq m$, set B_i as a bouquet of G with root x_i and flower(s) $N_G(x_i)$ and also $B'_j := G_{B(Q_j)}$ for $1 \leq j \leq t$, and so $\mathcal{B} := \{B_1, \ldots, B_m, B'_1, \ldots, B'_t\}$ is a semi-strongly disjoint set of bouquets of G and $|F(\mathcal{B})| = |V(G)| - (m+t)$. Hence, $d'_G = |V(G)| - (m+t)$. □

Now we are ready to characterize the projective dimension of the ring $R/I(G)$ when G belongs to $\mathcal{SQ}$. First we need the following proposition.

Proposition 4.3 *Let $G \in \mathcal{SQ}$. If $x_1, \ldots, x_m$ are simplicial vertices and $Q_1, \ldots, Q_t$ are basic 4-cycles of G, then*

$$\mathrm{pd}(G) \leq d'_G = |V(G)| - (m+t).$$

Proof

By Proposition 4.1 it is enough to show that $\mathrm{reg}(I(G)^{\vee}) \leq d'_G$. We prove the assertion by using induction on $|V(G)|$. For $|V(G)| = 2$, G is totally disconnected or a single edge, so $\mathrm{reg}(I(G)^{\vee}) = 0 \leq 0 = d'_G$ or $\mathrm{reg}(I(G)^{\vee}) = 1 \leq 1 = d'_G$. Now suppose that $G \in \mathcal{SQ}$ for which $|V(G)| \geq 2$ and the result holds for smaller values of $|V(G)|$.

Let $V(G) = (\dot{\bigcup}_{i=1}^{m} N_G[x_i])\dot{\cup}(\dot{\bigcup}_{j=1}^{t} B(Q_j))$ and $B(Q_j) = z_j t_j$ for $1 \leq j \leq t$. Assume that $x \in N_G(x_1)$ and without loss of generality $x \in Q_j := x y_j z_j t_j x$ for $1 \leq j \leq r$.

As in the proof of Theorem 3.3, if $G' := G \setminus \{x\}$ and $G'' := G \setminus N_G[x]$, then G' and $G'' \in \mathcal{SQ}$ and

$$V(G') = (\dot{\bigcup}_{i=1}^{r} N_{G'}[t_i])\dot{\cup}(\dot{\bigcup}_{i=1}^{m} N_{G'}[x_i])\dot{\cup}(\dot{\bigcup}_{i=r+1}^{t} B(Q_i))$$

for which $N_{G'}[t_j] = t_j z_j \cong k_2$ and $\deg(t_j) = 1$, so t_j is a simplicial vertex for $1 \leq j \leq r$, $N_{G'}[x_1] = N_G[x_1] \setminus \{x\}$, and $N_{G'}[x_i] = N_G[x_i]$ for $i \geq 2$. We consider the following two cases:

Case 1: Let $|N_G(x_1)| > 1$. For any $1 \leq i \leq m$, if $B_i := G_{N_{G'}[x_i]}$, then B_i is a bouquet of G' with root x_i and flower(s) $N_{G'}(x_i)$. Also, if $B'_j := G'_{B(Q_j)}$, then B'_j is a bouquet of G' with root t_j and flower z_j for $1 \leq j \leq t$, so by Lemma 4.2, $\mathcal{B} := \{B_1, \ldots, B_m, B'_1, \ldots, B'_t\}$ is the maximum semi-strongly disjoint set of bouquets of G'.

Case 2: If $|N_G(x_1)| = 1$, then $\mathcal{B} := \{B_2, \ldots, B_m, B'_1, \ldots, B'_t\}$ will be the maximum semi-strongly disjoint set of bouquets of G'. Thus, in any case

$$|F(\mathcal{B})| = (|V(G)| - 1) - (m+t) = |V(G)| - (m+t) - 1,$$

and hence $d'_{G'} = d'_G - 1$.

On the other hand,

$$V(G'') = \{z_1, \ldots, z_r\}\dot{\cup}(\dot{\bigcup}_{i=2}^{m} N_{G''}[x_i])\dot{\cup}(\dot{\bigcup}_{i=r+1}^{t} B(Q_i)).$$

If $B_i := G_{N_{G''}[x_i]}$ for any $2 \leq i \leq m$ and $B'_j := G''_{B(Q_j)}$ for $r+1 \leq j \leq t$, then by Lemma 4.2, $\mathcal{B} := \{B_2, \ldots, B_m, B'_{r+1}, \ldots, B'_t\}$ is the semi-strongly disjoint set of bouquets of G''. Hence,

$$|F(\mathcal{B})| = (|V(G)| - |N_G[x]| - r) - (m - 1 + (t - r)) = |V(G)| - (m + t) - |N_G[x]| + 1$$

and $d'_{G''} = d'_G - |N_G[x]| + 1 = d'_G - |N_G(x)|$.

By [9, Corollary 2.2] we know that

$\mathrm{reg}(I(G)^\vee) \leq max\{\mathrm{reg}(I(G')^\vee) + 1, reg(I(G'')^\vee) + |N_G(x)|\}$.

Thus, by induction hypothesis,

$\mathrm{reg}(I(G)^\vee) \leq max\{d'_{G'} + 1, d'_{G''} + |N_G(x)|\} = d'_G$.

□

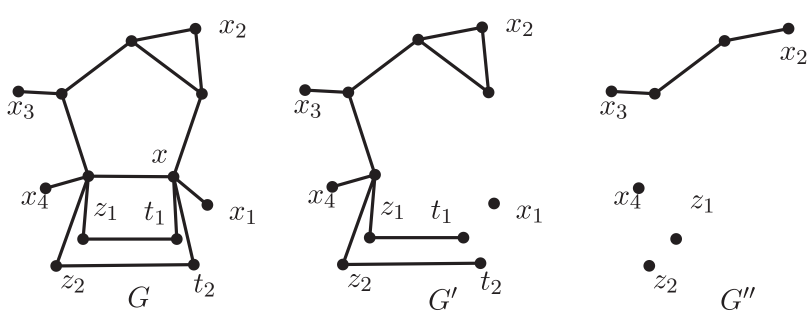
Theorem 4.4 *If G belongs to the class $\mathcal{SQ}$ such that $x_1, \ldots, x_m$ are simplicial vertices and $Q_1, \ldots, Q_t$ are basic 4-cycles of G, then:*

$$\mathrm{pd}(G) = d'_G = |V(G)| - (m + t).$$

Proof In [9, Corollary 2.8] it is shown that for any graph G, $\mathrm{pd}(G) \geq d'_G$. Hence, by Proposition 4.3 the equality holds. □

The following example is remarkable.

Example 4.5 In the Figure, $V(G) = N_G[x_1] \cup N_G[x_2] \cup N_G[x_3] \cup N_G[x_4] \cup B(Q_1) \cup B(Q_2)$ for which $B(Q_1) = \{t_1, z_1\}$ and $B(Q_2) = \{t_2, z_2\}$, so G belongs to $\mathcal{SQ}$. We can see that $G' := G \setminus \{x\}$ and $G'' := G \setminus N_G[x]$ belong to the class $\mathcal{SQ}$. It is clear that G is not chordal, bipartite, very well-covered, or C_5-free.



Figure

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