# On a factorization of operators on finite dimensional Hilbert spaces

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**Abstract:** As is well known, for any operator $T$ on a complex separable Hilbert space, $T$ has the polar decomposition $T = U|T|$, where $U$ is a partial isometry and $|T|$ is the nonnegative operator $(T^*T)^{\frac{1}{2}}$. In 2014, Tian et al. proved that on a complex separable infinite dimensional Hilbert space, any operator admits a polar decomposition in a strongly irreducible sense. More precisely, for any operator $T$ and any $\varepsilon > 0$, there exists a decomposition $T = (U + K)S$, where $U$ is a partial isometry, $K$ is a compact operator with $||K|| < \varepsilon$, and $S$ is strongly irreducible. In this paper, we will answer the question for operators on two-dimensional Hilbert spaces.

**Key words:** Polar decomposition, strongly irreducible operator, Jordan block

## 1. Introduction and preliminaries

Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators acting on $\mathcal{H}$. When $\mathcal{H}$ is a finite dimensional space, the famous Jordan Standard Theorem sufficiently reveals the internal structure of operators. The Jordan Standard Theorem indicates that the eigenvalues and the generalized eigenspaces of an operator determine its complete similarity invariants. When $\mathcal{H}$ is an infinite dimensional space, a fundamental problem in operator theory is how to build up a theorem that is similar to the Jordan Standard Theorem. However, the complexity of infinite dimensional space makes this problem difficult. To replace the notion of a Jordan block, in the 1970s Gilfeather [3] and Jiang [7] proposed the notion of strongly irreducible operator. An operator $T$ in $\mathcal{L}(\mathcal{H})$ is called *strongly irreducible*, denoted by $T \in (SI)$, if there does not exist a nontrivial idempotent operator $P$ in $\mathcal{L}(\mathcal{H})$ such that $PT = TP$. Obviously, strong irreduciblity of the operator is invariant under similarity and a strongly irreducible operator must be irreducible. Moreover, it is easy to show that an operator on a finite dimensional Hilbert space is strongly irreducible if and only if it is similar to a Jordan block. Moreover, Jiang [7] thought that strongly irreducible operators could be viewed as a suitable replacement of the notion of a Jordan block in infinite dimensional space. Indeed, Herrero, Jiang, Wang, and Ji et al. built an "approximate Jordan Theorem" for infinite dimensional space: the operator class

$$\{T \in \mathcal{L}(\mathcal{H}) : T = T_1 \dot{+} T_2 \dot{+} \cdots \dot{+} T_n, \ T_i \in (SI), \ n \in \mathbb{N}\}$$

is dense in $\mathcal{L}(\mathcal{H})$ under the norm topology (see [4–6]).

On the other hand, for an operator, one could consider other decomposition as well. The classical polar decomposition theorem [2] tells us that for an operator $T \in \mathcal{L}(\mathcal{H})$, there exists the decomposition $T = U|T|$ or $U^*T = |T|$, where $U$ is a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$. As a self-adjoint operator, $|T|$ has many reducible subspaces([1]). Hence the polar decomposition theorem says that an operator can always be expressed as a product of a partial isometry and an operator having many reducible subspaces. A natural question can be asked: can we write an operator into a product of a partial isometry and an operator having fewer reducible subspaces? In reference [8], the authors answered the question when $\mathcal{H}$ is complex separable infinite dimensional. More precisely, on a complex separable infinite dimensional Hilbert space, for any operator $T$ and any $\varepsilon > 0$, there exists a decomposition $T = (U+K)S$, where $U$ is a partial isometry, $K$ is a compact operator with $||K|| < \varepsilon$, and $S$ is strongly irreducible.

However, their theorem did not cover the case when $\mathcal{H}$ is finite dimensional. In this paper, we will consider such decomposition on two-dimensional Hilbert spaces. More precisely,

**Theorem 1.1** *For any $T \in \mathcal{L}(\mathcal{H})$, $\dim \mathcal{H} = 2$ and any $\varepsilon > 0$, there exist a partial isometry $U$, an (compact) operator $K$ with $||K|| < \varepsilon$, and a strongly irreducible operator $S$ such that $T = (U + K)S$.*

**Remark 1.2** *The operator $K$ in above theorem cannot be removed in some cases. See Corollary 2.3.*

The proof of the theorem is quite different from the proofs in [8]. We will give it in the next section.

## 2. Proof of main results

**Lemma 2.1** *Let $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$. Then*

$$T = \begin{bmatrix} \alpha & \beta \\ & \alpha \end{bmatrix}$$

*is strongly irreducible.*

**Proof** Indeed, if $TP = PT$ for a certain (one-rank) projection $P$ then $T(I - P) = (I - P)T$ as well, that is, T is diagonalizable. Therefore, $\beta = 0$; otherwise the geometric multiplicity of $\alpha$ would be 1, which is less than its algebraic one. □

**Proof of Theorem 1.1.** From classical polar decomposition theorem, $T = U|T|$, where $U$ is a unitary operator and $|T| = (T^*T)^{\frac{1}{2}}$. Since $|T|$ is self-adjoint, we have under some orthonormal basis $\{v_1, v_2\}$,

$$|T| = \begin{bmatrix} \alpha & \\ & \beta \end{bmatrix} \begin{matrix} v_1 \\ v_2 \end{matrix}, \alpha, \beta \geq 0.$$

The case $\alpha = \beta = 0$ is trivial. Consider the case of $\alpha = \beta > 0$. Let

$$K_1 = \begin{bmatrix} 1 & \epsilon \\ & 1 \end{bmatrix} \begin{matrix} v_1 \\ v_2 \end{matrix}$$

and

$$S = \begin{bmatrix} \alpha & -\alpha\epsilon \\ & \alpha \end{bmatrix} \begin{matrix} v_1 \\ v_2 \end{matrix},$$

then $|T| = K_1 S$.

Let

$$K = U \begin{bmatrix} 0 & \epsilon \\ & 0 \end{bmatrix} \begin{matrix} v_1 \\ v_2 \end{matrix},$$

then

$$T = U|T| = UK_1 S = (U + K)S.$$

It is not hard to show that $||K|| \leq \epsilon$. $S$ is strongly irreducible from Lemma 2.1.

Consider the case of $\alpha \neq \beta$ and assume that $\beta > \alpha \geq 0$. Let

$$V = (\alpha + \beta)^{-1} \begin{bmatrix} 2\sqrt{\alpha\beta} & -(\alpha - \beta) \\ \alpha - \beta & 2\sqrt{\alpha\beta} \end{bmatrix} \begin{matrix} v_1 \\ v_2 \end{matrix}$$

and

$$\begin{aligned} S & = V^* D = (\alpha + \beta)^{-1} \begin{bmatrix} 2\sqrt{\alpha\beta} & \alpha - \beta \\ -(\alpha - \beta) & 2\sqrt{\alpha\beta} \end{bmatrix} \begin{matrix} v_1 \\ v_2 \end{matrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{matrix} v_1 \\ v_2 \end{matrix} \\ & = (\alpha + \beta)^{-1} \begin{bmatrix} 2\alpha\sqrt{\alpha\beta} & \beta(\alpha - \beta) \\ -\alpha(\alpha - \beta) & 2\beta\sqrt{\alpha\beta} \end{bmatrix} \begin{matrix} v_1 \\ v_2 \end{matrix}. \end{aligned}$$

Then $|T| = VS$ and $T = U|T| = UVS$.

It is easy to check that $UV$ is unitary. Therefore, it suffices to show that $S$ is strongly irreducible. Notice that in this case we do not need $K$.

The characteristic polynomial of $S$ is $\lambda^2 - 2\sqrt{\alpha\beta}\lambda + \alpha\beta$. Hence, $\sqrt{\alpha\beta}$ is the only spectrum of $S$.

Obviously, $S - \sqrt{\alpha\beta} \neq 0$ and so $S$ is similar to

$$\begin{bmatrix} \sqrt{\alpha\beta} & r \\ & \sqrt{\alpha\beta} \end{bmatrix}, r \neq 0.$$

By Lemma 2.1 and as strong irreduciblity of the operator is invariant under similarity, $S$ is strongly irreducible.

We complete the proof.

**Proposition 2.2** *Every normal operator in $\mathcal{L}(\mathcal{H})$ ($1 < \dim\mathcal{H} < \infty$) cannot be strongly irreducible.*

**Proof** Every normal operator in $\mathcal{L}(\mathcal{H})$ for $1 < \dim(\mathcal{H}) < \infty$ cannot be strongly irreducible, for it is a diagonalizable operator. □

**Corollary 2.3** *$\alpha I$ cannot be decomposed into $US$, where $U$ is a unitary operator and $S$ is strongly irreducible.*

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