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Isometric N-Jordan weighted shift operators

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Abstract: A bounded linear operator T on a Hilbert space is an isometric N-Jordan operator if it can be written as $A+Q$, where A is an isometry and Q is a nilpotent of order N such that $AQ=QA$. In this paper, we will show that the only isometric N-Jordan weighted shift operators are isometries. This answers a question recently raised.

Key words: Isometric N-Jordan operator, nilpotent, weighted shift operator

1. Introduction and preliminaries

Let H be a Hilbert space and $B(H)$ stand for the space of all bounded linear operators on H. An operator T in $B(H)$ is called an isometric N-Jordan operator if $T=A+Q$, where A is an isometry and Q is a nilpotent operator of order N, that is, $Q^N=0$ but $Q^{N-1}\neq 0$, and $AQ=QA$. Note that the notions of isometric 1-Jordan and isometry coincide. It follows from Proposition 1.1 of [11] that the operator T is injective. The dynamic and spectral properties of T have been studied in [11]. We note that T^*T is invertible. Indeed, by Corollary 1.2 of [11] the operator T is bounded below, and so for every $h\in H$,

$$\|T^*Th\|\|h\| \geq |\langle T^*Th, h\rangle| = \|Th\|^2 \geq c\|h\|^2$$

for some $c>0$, which implies that T^*T is also bounded below and so is injective and has closed range. However,

$$H = (\ker(T^*T))^{\perp} = \overline{\operatorname{ran}(T^*T)} = \operatorname{ran}(T^*T)$$

implies that T^*T is invertible. It is easy to see that if A is a unitary operator then

$$(T^*T)^{-1} = 3I - 3TT^* + T^2T^{*2}.$$

For a positive integer m an operator $S\in B(H)$ is an m-isometry if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}S^{*k}S^k = 0.$$

The operator S is called a strict m-isometry if it is not an $(m-1)$-isometry. These operators have been introduced by Agler in [1] and have been studied extensively by Agler and Stankus in three papers [2–4].

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Recently, such operators have been considered by several authors. It is proved in [7] that every isometric N-Jordan operator is a strict $(2N-1)$-isometry. The authors asked about the validity of the converse. In this paper, we prove that the answer is negative.

2. Main results

Suppose that H is a separable Hilbert space with orthonormal basis $\{e_n\}_{n\geq 0}(\{e_n\}_{n\in\mathbb{Z}})$. An operator $S\in B(H)$ is called a unilateral (bilateral) weighted forward shift, provided that for every $n\geq 0$ $(n\in\mathbb{Z})$, $Se_n = w_n e_{n+1}$, where $(w_n)_n$ is a sequence of bounded complex numbers. Note that S is injective if and only if $w_n \neq 0$, for every n. It is known that S is an isometry if and only if $|w_n| = 1$ for all n, and is hyponormal if and only if its weight sequence is increasing [9]. Furthermore, m-isometric weighted shifts are discussed in [6, 8, 10]. Recall that S^*, the adjoint of S, is called a unilateral (bilateral) weighted backward shift. In this section, we will show that the only isometric N-Jordan weighted shift operators are isometries.

Theorem 1 *There is no isometric N-Jordan weighted shift operator when $N > 1$.*

Proof In contrast, assume that $T = A + Q$ is an isometric N-Jordan weighted shift operator. In the proof of Theorem 2.2 of [7], it is shown that

$$\sum_{k=0}^{2N-2}(-1)^k\binom{2N-2}{k}\|T^k h\|^2 = \frac{(2N-2)!}{((N-1)!)^2}\|Q^{N-1}h\|^2.$$

Let J be the set $\mathbb{N}\cup\{0\}$ or $\mathbb{Z}$ and suppose that the operator T is a forward shift operator with weight sequence $(w_n)_n$. Put $Q^{N-1}e_0 = \sum_{n\in J} c_n e_n$. Thus the above equality shows that

$$\begin{aligned}
0 &= \sum_{k=0}^{2N-2}(-1)^k\binom{2N-2}{k}\|T^k(Q^{N-1}e_0)\|^2\\
&= \sum_{k=0}^{2N-2}(-1)^k\binom{2N-2}{k}\Big\|\sum_{n\in J} c_n T^k e_n\Big\|^2\\
&= \sum_{k=0}^{2N-2}(-1)^k\binom{2N-2}{k}\Big\|\sum_{n\in J} c_n(\prod_{i=0}^{k-1} w_{n+i})e_{n+k}\Big\|^2\\
&= \sum_{k=0}^{2N-2}(-1)^k\binom{2N-2}{k}\sum_{n\in J}|c_n|^2\Big|\prod_{i=0}^{k-1} w_{n+i}\Big|^2\\
&= \sum_{n\in J}|c_n|^2\sum_{k=0}^{2N-2}(-1)^k\binom{2N-2}{k}\Big\|T^k e_n\Big\|^2\\
&= \sum_{n\in J}|c_n|^2\frac{(2N-2)!}{((N-1)!)^2}\Big\|Q^{N-1}e_n\Big\|^2.
\end{aligned}$$

On the other hand, for every $n\in J$

$$Q^{N-1}Ae_n = w_n Q^{N-1}e_{n+1},$$

and so

$$\|Q^{N-1}e_n\| = |w_n|\,\|Q^{N-1}e_{n+1}\|. \tag{1}$$

Therefore, if $Q^{N-1}e_0 = 0$ then $Q^{N-1}e_n = 0$ for every $n \in J$; hence $Q^{N-1} \equiv 0$, which is a contradiction. Moreover, if $Q^{N-1}e_0$ is nonzero then there is $n_0 \in J$ such that $c_{n_0} \neq 0$ and the previous argument shows that $Q^{N-1}e_{n_0} = 0$. Thus, (1) shows that $Q^{N-1}e_n = 0$ for every $n \in J$; hence $Q^{N-1} \equiv 0$, which is again a contradiction. Now suppose that $Te_n = w_n e_{n-1}$ $(n \in \mathbb{Z})$ is a bilateral backward shift operator. Define the unitary operator U on H by $U\left(\sum_{n\in\mathbb{Z}} \beta_n e_n\right) = \sum_{n\in\mathbb{Z}} \beta_n e_{-n}$. It is easily seen that $SU = UT$, where S is the bilateral forward shift defined by $Se_n = w_{-n}e_{n+1}$. Put $B = UAU^{-1}$ and $P = UQU^{-1}$; therefore, $S = B + P$ is an isometric N-Jordan operator which is impossible. Lastly, since every unilateral backward shift is not injective, we conclude that T cannot be a unilateral weighted backward shift. □

For a positive integer m let T be the unilateral weighted shift with weight sequence $w_n = \sqrt{\frac{n+m}{n+1}}$, $n \geqslant 0$. It is known that T is a strict m- isometric operator (see [5, Proposition 8]). Moreover, it is proved in [8] that for every odd number m, there is an invertible bilateral weighted shift that is a strict m-isometry. Thus, we have the following corollary that answers the question posed in [7].

Corollary 1 *For a fixed $m > 1$, there is a strict m-isometric operator T so that it is not an isometric N-Jordan operator for every $N \geq 1$.*

Recall that an operator is a co-isometry if its adjoint is an isometry.

Corollary 2 *If the operator $S = B + P$ is a weighted shift where B is a co-isometry, P is a nilpotent operator and $BP = PB$; then $P = 0$.*

Proof Apply the preceding theorem for $S^* = B^* + P^*$. □

Note that the commutativity of A and Q is essential in the preceding theorem as the following example shows.

Example 1 *Let $\{e_n\}_n$ be an orthonormal basis for the Hilbert space H. Define the isometric operator A by $Ae_n = e_{n+1}$ for all n and the weighted shift operator Q by $Qe_n = v_n e_{n+1}$, where $v_{2n} = \frac{1}{2n+1}$ and $v_{2n-1} = 0$. Note that $Q^2 = 0$ and $AQ \neq QA$. Moreover, $T = A + Q$ is a forward weighted shift with weight sequence $w_{2n} = 1 + \frac{1}{2n+1}$ and $w_{2n+1} = 1$.*

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