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High-order uniformly convergent method for nonlinear singularly perturbed delay differential equations with small shifts

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Abstract: In this paper, we propose and analyze a high-order uniform method for solving boundary value problems (BVPs) for singularly perturbed nonlinear delay differential equations with small shifts (delay and advance). Such types of BVPs play an important role in the modeling of various real life phenomena, such as the variational problem in control theory and in the determination of the expected time for the generation of action potentials in nerve cells. To obtain parameter-uniform convergence, the present method is constructed on a piecewise-uniform Shishkin mesh. The error estimate is discussed and it is shown that the method is uniformly convergent with respect to the singular perturbation parameter. Moreover, a bound of the global error is also derived. The effect of small shifts on the solution behavior is shown by numerical computations. Several numerical examples are presented to support the theoretical results, and to demonstrate the efficiency and the high-order accuracy of the proposed method.

Key words: Singularly perturbed, nonlinear differential equations, high-order method, delay differential equations, small shifts

1. Introduction

In this paper, we consider the following singularly perturbed nonlinear delay differential equation (DDE) with small shifts:

$$Lu(x) \equiv \varepsilon u''(x) + a(x)u'(x) = f\big(x, u(x), u(x-\delta), u(x+\eta)\big), \tag{1.1}$$

on $\Omega = (0,1)$ with the interval conditions

$$\begin{aligned} u(x) &= \phi(x), \qquad -\delta \le x \le 0, \\ u(x) &= \psi(x), \qquad 1 \le x \le 1+\eta, \end{aligned} \tag{1.2}$$

where $0 < \varepsilon \ll 1$ is the singular perturbation parameter, δ and η are called the delay and the advance parameters, sometimes referred to as negative shift and positive shift, respectively, as in [16, 17]; precise assumptions will be given in the next section. This type of differential equation plays an important role in the mathematical modeling of various practical phenomena, for instance, variational problems in control theory [6], description of the so-called human pupil-light reflex [19], evolutionary biology [34], and a variety of models for physiological processes or diseases [21].

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It is well known that the solution of a singularly perturbed differential equation generally exhibits boundary layer behavior. Usually, the standard discretization methods for solving these problems are not useful and fail to give accurate results, especially when the perturbation parameter ε tends to zero. This motivates the need for other methods that have ε-uniform convergence. In general there are two strategies to construct ε-uniform methods. The first one is the fitted operator method, which reflects the qualitative behavior of the solution; such fitted methods can be found in $[3-5, 22, 26, 28]$ and references therein. The second one is the fitted mesh method, which contains finite difference operators on specially designed mesh in the boundary-layer regions, such as Shishkin mesh [5, 22, 27] and grid equidistribution [18, 23, 25].

The linear case of singularly perturbed differential equations with small shifts has been investigated very often, e.g., see $[10-12, 16, 17, 23, 24, 29]$. In contrast, there are few works on singularly perturbed nonlinear DDEs. Lange and Miura [15] considered singularly perturbed nonlinear DDE with layer behavior and discussed the existence and uniqueness of their solutions. Kadalbajoo and Sharma [13, 14] and Kadalbajoo and Kumar [9] studied the numerical solutions of singularly perturbed nonlinear DDE with small negative shift using quasilinearization together with fitted mesh methods. Wang and Ni [33] considered the numerical solution of a singularly perturbed nonlinear DDE with interior layer via a method of boundary function and fractional steps. In most of the previous works, authors used Taylor series expansions for approximating the terms containing these shifts, provided they are of $o(\varepsilon)$. However, this process may lead to a bad approximation in the case when these shifts are of $O(\varepsilon)$.

In this paper, we propose a generic method for solving (1.1) that is useful and effective in both cases when shifts are of $o(\varepsilon)$ or $O(\varepsilon)$. To overcome the defect and weakness of the standard methods, we construct the proposed method on a piecewise-uniform mesh, and to cope with the terms containing shifts, we use cubic interpolation. Both cases, when the boundary layer occurs in the left and right side of the interval, will be studied. We show that this method is useful for obtaining a numerical solution of the considered problem in both cases.

The rest of the paper is organized as follows. Some assumptions on the continuous problem and estimates of the derivatives of its solution are given in Section 2. In Section 3, we describe the piecewise-uniform Shishkin mesh and we present in detail the construction of the numerical method. A study of the convergence analysis of the iterative process is presented in Section 4. Section 5 contains the error and the convergence analysis of the proposed method. Furthermore, we derive a bound of the global error. In Section 6, some numerical examples are presented to show the applicability and the effectiveness of the proposed method. The numerical results are reported with the maximum absolute error and the rate of convergence. Finally, the conclusion is given in Section 7.

2. The continuous problem

Let us consider the problems (1.1) and (1.2) and let us assume that the functions f, a, ϕ, and ψ are smooth and

$$a(x) \geq 2\beta > 0 \quad \text{on } \overline{\Omega}, \qquad f_u(x,u,v,w) > 0, \quad \text{on } \overline{\Omega} \times R^3, \tag{2.1}$$

and we also assume that $f : \overline{\Omega} \times R^3 \to R$ is Lipschitz-continuous with respect to the second, third, and fourth arguments, i.e.

$$\begin{aligned} |f(x,u,v,w) - f(x,\bar{u},\bar{v},\bar{w})| &\leq L_1|u-\bar{u}| + L_2|v-\bar{v}| + L_3|w-\bar{w}|, \\ \forall (x,u,v,w), (x,\bar{u},\bar{v},\bar{w}) &\in \overline{\Omega} \times R^3. \end{aligned} \tag{2.2}$$

In general, solution u has a boundary layer near the origin. The existence and uniqueness of the solution u follow from the arguments given in [2, 31, 32]. Throughout this paper, C is a generic positive constant independent of ε and discretization parameter (N), and for a mesh function $g = \big(g_1, g_2, \cdots, g_N\big)$, we use the simple notation for the discrete maximum norm $\|g\| = \max\limits_{1\leq i\leq N} |g_i|$.

Now we give the required bounds on the solution u that will be used to establish the error estimate.

Lemma 2.1 (Continuous maximum principle) *Let $\psi \in C^2(\Omega)\cap C^0(\overline{\Omega})$ satisfying $\psi(0) \leq 0$ and $\psi(1) \leq 0$. Then $L\psi(x) \geq 0$, for all $x \in \Omega$ implies that $\psi(x) \leq 0$, for all $x \in \overline{\Omega}$.*

Proof Let $x^\star \in \overline{\Omega}$ be such that $\psi(x^\star) =\max\{\psi(x), x \in \overline{\Omega}\}$ and $\psi(x^\star) > 0$. Clearly $x^\star \neq 0, x^\star \neq 1$ and therefore $\psi'(x^\star) = 0$ and $\psi''(x^\star) \leq 0$.
Hence

$$L\psi(x^\star) = \varepsilon\psi''(x^\star) + a(x^\star)\psi'(x^\star) \leq 0,$$

which contradicts the hypothesis that $L\psi(x) \geq 0$. Therefore, $\psi(x^\star) \leq 0$. However, since $x^\star$ was an arbitrary point in $\overline{\Omega}$, $\psi(x) \leq 0$, for all $x \in \overline{\Omega}$. □

Lemma 2.2 *Let u be the solution of (1.1) and (1.2) and let (2.1) hold. Then the derivatives of u satisfy the following bounds:*

$$\big|u^{(k)}(x)\big| \leq C\Big[1 + \varepsilon^{-k}\exp\big(-2\beta x/\varepsilon\big)\Big], \qquad 0 \leq k \leq 6. \tag{2.3}$$

Proof The stability inequality given in Theorem 2 in [20] gives $|u(x)| \leq C$ for all $x \in \Omega$. Firstly, we prove

$$\big|u^{(k)}(0)\big| \leq C\varepsilon^{-k}, \qquad 1 \leq k \leq 6. \tag{2.4}$$

By the mean value theorem, there exists a point $\xi \in (0, \varepsilon)$ such that

$$u'(\xi) = \frac{u(\varepsilon) - u(0)}{\varepsilon},$$

and therefore $|\varepsilon u'(\xi)| \leq \|u\|$. Then (2.4) holds for $k = 1$. Using (1.1), we can obtain the required bounds for $k = 1$, and the estimate for $k \geq 2$ follows by induction and differentiation of (1.1).
Now we prove (2.3) for $k \geq 1$. Let $A(x) = \int_0^x a(\xi)d\xi$.
Multiplying both sides of (1.1) by $\exp(A(x)/\varepsilon)$ and integrating over $(0, x)$ and taking the modulus on both sides we get

$$|u'(x)| \leq |u'(0)|\exp(-A(x)/\varepsilon) + \frac{1}{\varepsilon}\int_0^x \|f\|\exp\big((A(\xi) - A(x))/\varepsilon\big)d\xi.$$

Then using (2.4) for $k = 1$ we get

$$\begin{aligned}
|u'(x)| &\leq C\varepsilon^{-1}\bigg(\int_0^x \exp\big((A(\xi) - A(x))/\varepsilon\big)d\xi + \exp(-A(x)/\varepsilon)\bigg)\\
&\leq C\varepsilon^{-1}\bigg(\int_0^x \exp\big(2\beta(\xi - x)/\varepsilon\big)d\xi + \exp(-2\beta x/\varepsilon)\bigg)\\
&\leq C\Big(1 + \varepsilon^{-1}\exp(-2\beta x/\varepsilon)\Big).
\end{aligned}$$

The proof for $k \geq 2$ follows by induction process and differentiation of (1.1). □

3. The discretization

In this section, we derive a fitted mesh method for solving (1.1) on a piecewise-uniform Shishkin mesh. Before constructing the method, we make precise the Shishkin mesh to be considered.

3.1. Shishkin mesh

Shishkin mesh is a piecewise-uniform mesh that is dense in the boundary layer region and coarse in the outer region, as $\varepsilon \to 0$. This is achieved by the use of a transition parameter σ, which depends on ε and N. Thus for given values of ε and N, the interval is divided into two subintervals using $\sigma = \min\{1/2, \sigma_0 \varepsilon \ln N\}$, where the constant σ_0 will be chosen later. In the case of the boundary layer at the left end, the piecewise-uniform Shishkin mesh is constructed by dividing the interval $[0,1]$ into two subintervals $[0,\sigma]$ and $[\sigma,1]$ such that $\Omega^N = \{0 = x_0, x_1, \cdots, x_{N/2} = \sigma, \cdots, x_N = 1\}$. Then each of the subintervals is divided into $N/2$ mesh elements of equal length. Therefore, the mesh points are given by

$$x_i = \begin{cases} 2\sigma i/N, & 0 \leq i \leq N/2, \\ \sigma + 2(1-\sigma)(i - N/2)/N, & N/2 < i \leq N. \end{cases}$$

Similarly, when the boundary layer occurs on the right side of the interval, we partition the interval into two subintervals $[0, 1-\sigma]$ and $[1-\sigma, 1]$, and the mesh points are given by

$$x_i = \begin{cases} 2(1-\sigma)i/N, & 0 \leq i \leq N/2, \\ 1 - \sigma + 2\sigma(i - N/2)/N, & N/2 < i \leq N. \end{cases}$$

In the following, we discuss the case when the solution exhibits a single boundary layer on the left side of the interval, i.e. when $a(x) \geq 2\beta > 0$. The other case, when a boundary layer occurs on the right side of the interval ($a(x) \leq -2\beta < 0$), one can follow the same procedure as we use for the case of the left boundary layer.

Let us denote the local step sizes by $h_i = x_i - x_{i-1}$ in each subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \cdots, N$, and let $h = 2\sigma/N$ and $H = 2(1-\sigma)/N$ be the mesh widths in $[0,\sigma]$ and $[\sigma,1]$ respectively. Then it is easy to see that

$$h = 2\sigma_0 \varepsilon N^{-1} \ln N \quad \text{and} \quad N^{-1} \leq H \leq 2N^{-1}.$$

3.2. Description of the method

We derive a fitted mesh operator compact implicit (FMOCI) method for Eqs. (1.1) and (1.2) as follows:

$$\begin{aligned} &L_N U_i \equiv \frac{\varepsilon}{h_i h_{i+1}(h_i + h_{i+1})} R(U_i) = Q(f_i), \qquad 1 \leq i \leq N-1, \\ &U_0 = \phi(0), \quad U_N = \psi(1), \end{aligned} \tag{3.1}$$

where

$$R(U_i) = r_i^- U_{i-1} + r_i^c U_i + r_i^+ U_{i+1}, \qquad Q(f_i) = q_i^- f_{i-1} + q_i^c f_i + q_i^+ f_{i+1}, \tag{3.2}$$

and

$$f_i = f\big(x_i, U_i, U_{i,\delta}, U_{i,\eta}\big). \tag{3.3}$$

The coefficients $r_i^{-,c,+}$ and $q_i^{-,c,+}$ are unknowns to be determined later as functions of h_i, ε, and $a(x_i)$. It is worthwhile to mention that U_i, $U_{i,\delta}$, and $U_{i,\eta}$ are approximations of $u(x_i)$, $u(x_i-\delta)$, and $u(x_i+\eta)$, respectively. It is assumed that $U_{i,\delta}=\phi(x_i-\delta)$ for $x_i \le \delta$ and $U_{i,\eta}=\psi(x_i+\eta)$ for $x_i \ge 1-\eta$. The delay and advance terms are approximated at nonmesh points by using cubic interpolation defined as

$$u(x_i+\theta h_{i+1}) = \sum_{k=0}^{3} P_{i,k}(\theta) U_{i-1+k}, \qquad 0 \le \theta < 1, \;\; 1 \le i \le N-1,$$

where

$$\begin{aligned}
P_{i,0}(\theta) &= -\frac{\theta^3 h_{i+1}^3 - \theta^2 h_{i+1}^2 (h_{i+1}+d_i) + \theta h_{i+1}^2 d_i}{h_i(h_i+h_{i+1})(h_i+d_i)},\\
P_{i,2}(\theta) &= \frac{\theta^3 h_{i+1}^2 + \theta^2 h_{i+1}(h_i - d_i) - \theta h_i d_i}{(h_i+h_{i+1})(h_{i+1}-d_i)},\\
P_{i,3}(\theta) &= \frac{\theta^3 h_{i+1}^3 + \theta^2 h_{i+1}^2 (h_i - h_{i+1}) - \theta h_i h_{i+1}^2}{d_i(d_i+h_i)(d_i-h_{i+1})},\\
P_{i,1}(\theta) &= 1 - \big(P_{i,0}(\theta) + P_{i,2}(\theta) + P_{i,3}(\theta)\big), \qquad d_i = h_{i+1} + h_{i+2}.
\end{aligned} \tag{3.4}$$

The development of the present method is based on computing the local truncation error as follows:

$$\tau_{i,u} = L_N u(x_i) - Q\big(Lu(x_i)\big). \tag{3.5}$$

Since u is sufficiently smooth, and using Taylor expansion, $\tau_{i,u}$ can be written in the form

$$\tau_{i,u} = T_i^0 u(x_i) + T_i^1 u'(x_i) + \cdots + T_i^6 u^{(6)}(x_i) + O(h_m^5), \quad h_m = \max_{1 \le i \le N} h_i,$$

where

$$\begin{aligned}
T_i^0 =& \frac{\varepsilon}{h_i h_{i+1}(h_i+h_{i+1})}\big(r_i^+ + r_i^c + r_i^-\big),\\
T_i^1 =& \frac{\varepsilon}{h_i h_{i+1}(h_i+h_{i+1})}\Big[h_{i+1} r_i^+ - h_i r_i^- - \frac{h_i h_{i+1}(h_i+h_{i+1})}{\varepsilon}\big(q_i^+ a_{i+1} + q_i^c a_i + q_i^- a_{i-1}\big)\Big],\\
T_i^2 =& \frac{\varepsilon}{h_i h_{i+1}(h_i+h_{i+1})}\Big[\frac{h_{i+1}^2}{2} r_i^+ + \frac{h_i^2}{2} r_i^- - h_i h_{i+1}(h_i+h_{i+1})\big(q_i^+ + q_i^c + q_i^-\big)\\
&- \frac{h_i h_{i+1}(h_i+h_{i+1})}{\varepsilon}\big(h_{i+1} q_i^+ a_{i+1} - h_i q_i^- a_{i-1}\big)\Big],
\end{aligned}$$

and

$$\begin{aligned}
T_i^k =& \frac{\varepsilon}{h_i h_{i+1}(h_i+h_{i+1})}\Bigg\{\frac{h_{i+1}^k}{k!} r_i^+ + (-1)^k \frac{h_i^k}{k!} r_i^- - \frac{h_i h_{i+1}(h_i+h_{i+1})}{\varepsilon}\\
&\Big[q_i^+\Big(\frac{h_{i+1}^{k-2}}{(k-2)!}\varepsilon + \frac{h_{i+1}^{k-1}}{(k-1)!} a_{i+1}\Big) + (-1)^k q_i^-\Big(\frac{h_i^{k-2}}{(k-2)!}\varepsilon - \frac{h_i^{k-1}}{(k-1)!} a_{i-1}\Big)\Big]\Bigg\},\\
&k = 3,4,5,6.
\end{aligned}$$

The truncation error is said to be of order p if $\tau_{i,u} = O(h_m^p)$ as $h_m \to 0$ (ε is fixed) for $i = 1, 2, \cdots, N-1$. Here we construct our method by the conditions

$$T_i^k = 0, \qquad k = 0, 1, 2, \tag{3.6}$$

$$T_i^k = O(h_m^4), \qquad k = 3, 4. \tag{3.7}$$

These conditions were first proposed for the case of uniform mesh in [1]. From the conditions (3.6) and (3.7), we get

$$\begin{aligned} r_i^- =& h_{i+1}\left\{2(q_i^+ + q_i^c + q_i^-) + \frac{h_i}{\varepsilon}\left[\alpha_i q_i^+ a_{i+1} - \alpha_i q_i^c a_i - q_i^-(2+\alpha_i)a_{i-1}\right]\right\}, \\ r_i^+ =& h_i\left\{2(q_i^+ + q_i^c + q_i^-) + \frac{h_i}{\varepsilon}\left[(1+2\alpha_i)q_i^+ a_{i+1} + q_i^c a_i - q_i^- a_{i-1}\right]\right\}, \\ r_i^c =& -r_i^+ - r_i^-, \qquad \alpha_i = h_{i+1}/h_i, \qquad 1 \le i \le N-1, \end{aligned} \tag{3.8}$$

and the coefficients q_i^-, q_i^c, and q_i^+ $i = 1, 2, \cdots, N-1$ are defined in two different ways:

(i) For $x_i \in (0, \sigma)$, the coefficients q_i^-, q_i^c, and q_i^+ $i = 1, \cdots, N/2-1$ are given by

$$\begin{aligned} q_i^- =& 1 + \alpha_i - \alpha_i^2 + \frac{1 + 2\alpha_i - 3\alpha_i^2 - 10\alpha_i^3 - 5\alpha_i^4}{6(1+3\alpha_i+\alpha_i^2)}\rho_i + p\rho_i^2, \\ q_i^c =& (1+\alpha_i)\left[\frac{(1+3\alpha_i+\alpha_i^2)}{\alpha_i} + \frac{3p}{(2+\alpha_i)}\rho_i^2 + \frac{p}{2}\rho_i^3\right], \\ q_i^+ =& \frac{\alpha_i^2+\alpha_i-1}{\alpha_i} + \frac{5 + 10\alpha_i + 3\alpha_i^2 - 2\alpha_i^3 - \alpha_i^4}{6(1+3\alpha_i+\alpha_i^2)}\rho_i + \left[\frac{p(1+2\alpha_i)}{(2+\alpha_i)}\right. \\ & \left. + \frac{(\alpha_i^2-1)(1+8\alpha_i+15\alpha_i^2+8\alpha_i^3+\alpha_i^4)}{12(1+2\alpha_i)(1+3\alpha_i+\alpha_i^2)}\right]\rho_i^2 + \frac{p(1+\alpha_i)}{2}\rho_i^3, \end{aligned} \tag{3.9}$$

where

$$p = \frac{(1 + 2\alpha_i - 3\alpha_i^2 - 10\alpha_i^3 - 5\alpha_i^4)^2}{144(1+\alpha_i-\alpha_i^2)(1+3\alpha_i+\alpha_i^2)^2}, \quad \rho_i = a_i h_i/\varepsilon.$$

(ii) For $x_i \in [\sigma, 1)$, we must define two different cases depending on the relation between h_m and ε.

In the first case, when $h_m\|a\| < 2\varepsilon$, the coefficients q_i^-, q_i^c, and q_i^+, $i = N/2+1, \cdots, N-1$ are defined by (3.9), and also when $\alpha_{N/2} < (\sqrt{5}+1)/2$ the coefficients $q_{N/2}^-, q_{N/2}^c$, and $q_{N/2}^+$ are defined again by (3.9) and when $\alpha_{N/2} \ge (\sqrt{5}+1)/2$ these coefficients are given by

$$\begin{aligned} q_{N/2}^- =& \rho_{N/2}^2, \quad q_{N/2}^c = \frac{3(1+\alpha_{N/2})}{(2+\alpha_{N/2})}\rho_{N/2}^2 + \frac{1+\alpha_{N/2}}{2}\rho_{N/2}^3, \\ q_{N/2}^+ =& \frac{(1+2\alpha_{N/2})}{(2+\alpha_{N/2})}\rho_{N/2}^2 + \frac{1+\alpha_{N/2}}{2}\rho_{N/2}^3. \end{aligned}$$

While in the second case, when $h_m\|a\| \ge 2\varepsilon$, the coefficients q_i^-, q_i^c, and q_i^+ are given by

$$q_i^- = 0, \quad q_i^c = 1, \quad q_i^+ = a_i/a_{i+1}, \qquad N/2 \le i \le N-1. \tag{3.10}$$

From (3.8), we have

$$\left|r_i^-\right| \le C\frac{h_{i+1}}{\varepsilon}\max\{h_i,\varepsilon\}, \quad \left|r_i^+\right| \le C\frac{h_i}{\varepsilon}\max\{h_i,\varepsilon\}. \tag{3.11}$$

4. Convergence analysis

In this section, we discuss the existence of the approximate solution obtained by the FMOCI method described in the previous section, and we study the convergence of the iterative process. For this purpose, we rewrite the FMOCI method (3.1)–(3.3) in the following matrix form:

$$L_N U = F(U), \qquad U = [U_1, U_2, \cdots, U_{N-1}]^T, \tag{4.1}$$

where L_N is a $(N-1)\times(N-1)$ tridiagonal matrix and $F(U)$ is the right-hand-side vector of order $(N-1)$, which are given by

$$L_N = \begin{pmatrix} \hat{r}_1^c & \hat{r}_1^+ & 0 & 0 & 0 & \cdots & 0 \\ \hat{r}_2^- & \hat{r}_2^c & \hat{r}_2^+ & 0 & 0 & \cdots & 0 \\ 0 & \hat{r}_3^- & \hat{r}_3^c & \hat{r}_3^+ & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \hat{r}_{N-2}^- & \hat{r}_{N-2}^c & \hat{r}_{N-2}^+ \\ 0 & \cdots & 0 & 0 & 0 & \hat{r}_{N-1}^- & \hat{r}_{N-1}^c \end{pmatrix}, \tag{4.2}$$

$$F(U) = \begin{pmatrix} q_1^- f_0 + q_1^c f_1 + q_1^+ f_2 - \phi(0)\hat{r}_1^- \\ q_2^- f_1 + q_2^c f_2 + q_2^+ f_3 \\ \vdots \\ q_{N-2}^- f_{N-3} + q_{N-2}^c f_{N-2} + q_{N-2}^+ f_{N-1} \\ q_{N-1}^- f_{N-2} + q_{N-1}^c f_{N-1} + q_{N-1}^+ f_N - \psi(1)\hat{r}_{N-1}^+ \end{pmatrix},$$

where

$$\hat{r}_i^* = \frac{\varepsilon}{h_i h_{i+1}(h_i + h_{i+1})} r_i^*, \; * = -, c, + \,, \quad 1 \le i \le N-1. \tag{4.3}$$

4.1. Existence of the approximate solution

The existence of the solution of the nonlinear system (4.1) can be proved by the following lemma.

Lemma 4.1 *Let $N \ge N_0$, where N_0 is the smallest positive integer such that*

$$\sigma_0\|a\| \le \frac{N_0}{LnN_0}, \tag{4.4}$$

and also suppose that

$$q_i^c a_i - q_i^+ a_{i+1} - q_i^- a_{i-1} \ge 0, \qquad 1 \le i \le N-1. \tag{4.5}$$

Then the FMOCI method defined by (4.1)–(4.3) *satisfies the following, for* $1 \le i \le N-1$

$$q_i^- \ge 0, \quad q_i^c > 0, \quad q_i^+ \ge 0, \tag{4.6}$$

$$r_i^- \ge 0, \quad r_i^+ > 0, \quad r_i^- + r_i^c + r_i^+ \le 0. \tag{4.7}$$

Proof From (3.9) it is clear that $q_i^+ \ge 0$ and $q_i^c > 0$ for all $\rho_i \in (0,\infty)$. Now we show that $q_i^- \ge 0$. Note that q_i^+ is of the form of quadratic polynomial $a_0 + a_1 x + a_2 x^2$ with $a_0 > 0$; this quadratic is nonnegative on $(0,\infty)$ if and only if: (i) the discriminant is nonpositive when $a_1 < 0$ and (ii) $a_2 \ge 0$ when $a_1 \ge 0$. Using this, we find that $q_i^+ \ge 0$ for all $\rho_i \in (0,\infty)$.

On the other hand, using (3.8) and the condition (4.5), it follows that $r_i^+ > 0$, $1 \le i \le N-1$. To prove that $r_i^- > 0$, three different cases have been considered. Firstly, for $1 < i \le N/2 - 1$, we have

$$r_i^- = 24h_i\Big(1 - \frac{1}{2}\rho_i + \frac{5}{48}\rho_i^2\Big),$$

which is positive (using the above argument). Secondly, for $N/2 \le i \le N-1$ and $h_m\|a\| \ge 2\varepsilon$ holds; the proof is trivial. While in the last case when $h_m\|a\| < 2\varepsilon$, using the coefficients q_i^*, $* = -, c, +$ defined by (3.9) with the condition $h_m\|a\| < 2\varepsilon$, it is straightforward to prove that $r_i^- \ge 0$.

Finally, from (3.8) we have

$$r_i^- + r_i^c + r_i^+ \le 0, \qquad 1 \le i \le N-1.$$

This completes the proof. □

Remark 4.1 *It is important to note that, from Lemma 4.1, inequalities* (4.7) *imply that the triadiagonal matrix* L_N *is diagonally dominant with negative main diagonal elements and positive superdiagonal and subdiagonal elements, and hence* L_N *can be inverted; see [8]. This ensures the existence of the solution of the nonlinear system* (4.1)*. Moreover, the operator* L_N *satisfies a maximum principle.*

Lemma 4.2 *(*Discrete maximum principle)*. Let* ψ_i *be a mesh function and satisfy* $\psi_0 \le 0$, $\psi_N \le 0$, *and* $L_N\psi_i \ge 0$, $i = 1, 2, \cdots, N-1$. *Then* $\psi_i \le 0$, $0 \le i \le N$.

4.2. The iterative process and its convergence

The numerical solution of the nonlinear system (4.1) can be computed by the following iterative process:

$$L_N U^{n+1} = F(U^n), \qquad n = 0, 1, 2, \cdots, \tag{4.8}$$

with the starting vector U^0.

To show the convergence of the iterative process (4.8), we consider the following condition:

$$K\|L_N^{-1}\|_\infty\big(L_1 + M(L_2 + L_3)\big) < 1, \tag{4.9}$$

where

$$K = \max_{1 \le i \le N-1}\Big\{|q_i^-| + |q_i^0| + |q_i^+|\Big\} \quad \text{and} \quad M = \sup_{0 \le \theta < 1}\sum_{k=0}^{3}|P_k(\theta)|. \tag{4.10}$$

A bound of $\|L_N^{-1}\|_\infty$ in the above condition will be computed later, wherein we will show that it is very small especially when $\varepsilon \to 0$. Therefore, it is dominant on the left side of the inequality (4.9).

Lemma 4.3 *Assume that the function f satisfies the Lipschitz condition (2.2) and that the condition (4.9) holds. Then the sequence $\{U^n\}_{n=0}^{\infty}$ generated by (4.8) converges to the solution of the nonlinear system (4.1).*

Proof From (4.8) and (4.10), we have

$$\|U^{n+1} - U^n\|_\infty \le \|L_N^{-1}\|_\infty \|F(U^n) - F(U^{n-1})\|_\infty,$$

and

$$\|F(U^n) - F(U^{n-1})\|_\infty \le K \max_{1\le i\le N-1} \big(L_1|U_i^n - U_i^{n-1}| + L_2\zeta_i + L_3\xi_i\big),$$

where

$$\zeta_i = |U_{i,\delta}^n - U_{i,\delta}^{n-1}| \qquad \text{and} \qquad \xi_i = |U_{i,\eta}^n - U_{i,\eta}^{n-1}|.$$

Again, using (4.10) and the fact that $U_j^\ell = \sum_{k=0}^{3} P_k(\theta)U_{i-1+k}^l$, $\ell = n-1, n$, $x_j - \delta \in [x_i, x_{i+1})$, we obtain

$$\zeta_i \le M\|U^n - U^{n-1}\|_\infty \qquad \text{and} \qquad \xi_i \le M\|U^n - U^{n-1}\|_\infty.$$

Therefore, since $U_0^n - U_0^{n-1} = U_N^n - U_N^{n-1} = 0$

$$\|F(U^n) - F(U^{n-1})\|_\infty \le K\big(L_1 + M(L_2 + L_3)\big)\|U^n - U^{n-1}\|_\infty.$$

Hence

$$\|U^{n+1} - U^n\|_\infty \le K\|L_N^{-1}\|_\infty\big(L_1 + M(L_2 + L_3)\big)\|U^n - U^{n-1}\|_\infty.$$

Using the condition (4.9), the sequence $\{U^n\}_{n=0}^{\infty}$ converges. It is clear that the solution of (4.1) is the limit of this sequence. □

We repeat the above process with suitable initial value until the solution profiles do not differ from iteration to iteration within a desired accuracy. For computational purposes, the iterative process (4.8) stops at the nth iteration if the following condition is satisfied

$$\|U^{n+1} - U^n\| < \text{Tol.},$$

where Tol. is a given tolerance.

5. Error estimates

In the previous section, we proved that the iterative process (4.8) converges to the solution of the nonlinear system (4.1). Here, we analyze the ε-uniform error estimate of the FMOCI method (4.1)–(4.3), and we derive a bound on global error. To estimate ε-uniform convergence of the present method, we need more precise bounds on the exact solution of the problem (1.1) rather than those in Lemma 2.2. To obtain these bounds, we decompose the solution u^{n+1} into regular and singular components at the $(n+1)$th iteration as follows:

$$u^{n+1} = v^{n+1} + w^{n+1},$$

where v^{n+1} and w^{n+1} are the regular and singular components, respectively. We further express the regular component v^{n+1} in the form

$$v^{n+1} = \sum_{i=0}^{5} v_i, \tag{5.1}$$

where the functions v_i, $0 \le i \le 4$ are defined to be the solutions for the following first order problems

$$\begin{cases} a v_0' = f, & v_0(1) = u(1), \\ a v_i' = -v_{i-1}'', & v_i(1) = 0, \qquad 1 \le i \le 4, \end{cases} \tag{5.2}$$

and the last function v_5 satisfies the second order problem

$$\varepsilon v_5'' + a v_5' = -v_4'', \quad v_5(0) = 0, \qquad v_5(1) = 0. \tag{5.3}$$

Thus, v^{n+1} and w^{n+1} satisfy

$$L v^{n+1} = f, \qquad v^{n+1}(0) = \sum_{i=0}^{5} v_i(0), \qquad v^{n+1}(1) = u(1), \tag{5.4}$$

$$L w^{n+1} = 0, \quad w^{n+1}(0) = u(0) - v^{n+1}(0), \quad w^{n+1}(1) = 0. \tag{5.5}$$

Theorem 5.1 *Let u^{n+1} be the solution of the problem (1.1) at the nth iteration and let $u^{n+1} = v^{n+1} + w^{n+1}$. Then for k, $0 \le k \le 6$ and for all $x \in \overline{\Omega}$, the regular component v^{n+1} and the singular component w^{n+1}, defined in (5.4) and (5.5), respectively, satisfy*

$$\left| \frac{d^k v^{n+1}(x)}{dx^k} \right| \le C \Big[1 + \varepsilon^{5-k} \exp\big(-2\beta x/\varepsilon \big) \Big], \tag{5.6}$$

$$\left| \frac{d^k w^{n+1}(x)}{dx^k} \right| \le C \varepsilon^{-k} \exp\big(-2\beta x/\varepsilon \big), \tag{5.7}$$

for some constant C independent of ε.

Proof Firstly, the bounds on the regular component v^{n+1} and its derivatives are proved as follows

From (5.2), since the solutions $v_i, i = 0, 1, 2, 3, 4$ are independent of ε, we obtain

$$\big| v_i^{(k)} \big| \le C, \quad 0 \le k \le 6, \quad \text{for} \quad i = 0, 1, 2, 3, 4. \tag{5.8}$$

Furthermore, v_5 is the solution of the problem similar to (1.1); hence using Lemma 2.2 it follows that

$$\big| v_5^{(k)} \big| \le C \Big[1 + \varepsilon^{-k} \exp\big(-2\beta x/\varepsilon \big) \Big], \qquad 0 \le k \le 6. \tag{5.9}$$

Therefore, combining (5.8), (5.9), and (5.1), we obtain the required estimates for v^{n+1} and its derivatives. To obtain the required bounds on the singular component w^{n+1}, define

$$\Psi \pm (x) = (v^{n+1}(0) - \kappa) \exp\big(-2\beta x/\varepsilon \big) \pm w^{n+1}(x), \qquad x \in \overline{\Omega},$$

where κ is a positive constant. Then, for a sufficiently large value of κ, and using the bounds on v^{n+1}, we have $\Psi^{\pm}(0) \leq 0$, $\Psi^{\pm}(1) \leq 0$ and $L_\varepsilon \Psi^{\pm}(x) \geq 0$, for all $x \in \Omega$. Therefore, using the maximum principle given in Lemma 2.1, we get $\Psi^{\pm}(x) \leq 0$, for all $x \in \overline{\Omega}$, which gives

$$\big|w^{n+1}(x)\big| \leq C \exp\big(-2\beta x/\varepsilon\big), \qquad C = \kappa - v(0).$$

To find the bound on the derivatives of w^{n+1}, we introduce the function

$$\varphi(x) = \frac{\int_0^x \exp(-A(t)/\varepsilon)dt}{\int_0^1 \exp(-A(t)/\varepsilon)dt}, \quad \text{where} \quad A(t) = \int_0^t a(s)ds.$$

It is clear that $L_\varepsilon \varphi \leq 0, \varphi(0) = 0, \varphi(1) = 1$ and $0 \leq \varphi(x) \leq 1$. Therefore, w^{n+1} can be written as

$$w^{n+1}(x) = C_1\varphi(x) + C_2\big(1 - \varphi(x)\big).$$

Using (5.5) and imposing the boundary value of φ at 0 and 1, it follows that

$$w^{n+1}(x) = \big(u(0) - v(0)\big)\big(1 - \varphi(x)\big).$$

Hence

$$\left|\frac{dw^{n+1}(x)}{dx}\right| \leq C|\varphi'(x)| \leq C\varepsilon^{-1} \exp\big(-2\beta x/\varepsilon\big).$$

Now, using (5.5) and the above estimates, we obtain the bound on d^2w^{n+1}/dx^2. The proof for $k \geq 3$ follows by differentiating (5.5) and using the bounds on the derivatives obtained previously. This completes the proof. □

It is easy to show that the local truncation error (3.5) can be written as

$$\begin{aligned}\tau_{i,u} =& T_i^3 u''' + T_i^4 u^{(4)} + \left[\frac{\varepsilon}{5!}\left(\frac{h_{i+1}^5 r_i^+ - h_i^5 r_i^-}{h_i h_{i+1}(h_i + h_{i+1})}\right) + \frac{\varepsilon}{3!}\big(h_{i+1} q_i^- - h_{(i+1)}^3 q_i^+\big)\right] u^{(5)} \\ &- a_{i+1} R_3(x_i, x_{i+1}, u') q_i^+ - a_{i-1} R_3(x_i, x_{i-1}, u') q_i^- \\ &+ \frac{\varepsilon}{h_i h_{i+1}(h_i + h_{i+1})}\Big(R_5(x_i, x_{i+1}, u) r_i^+ + R_5(x_i, x_{i-1}, u) r_i^-\Big) \\ &- \varepsilon\Big(R_3(x_i, x_{i+1}, u'') q_i^+ + R_3(x_i, x_{i-1}, u'') q_i^-\Big),\end{aligned} \tag{5.10}$$

where

$$R_n(a, b, f) = \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) = \frac{1}{n!}\int_a^b (b-\xi)^n f^{(n+1)} d\xi, \tag{5.11}$$

where $a < \xi < b$. In the following, when $h_i \leq \varepsilon$ we use the derivative form of R_n, and when $\varepsilon \leq h_i$, the integral form will be used.

Lemma 5.2 *Let the hypothesis (4.4) of Lemma 4.1 be satisfied. Then the local truncation error given by (5.10) satisfies the following:*

$$\big|\tau_{i,u}\big| \leq \begin{cases} C\Big[\varepsilon^{-2}h^4 + \varepsilon^{-5}h^4 \exp\big(-2\beta x_i/\varepsilon\big)\Big], & 1 \leq i < N/2, \\ C\Big[h_i^2 + \dfrac{1}{\max\{h_i, \varepsilon\}} \exp\big(-2\beta x_{i-1}/\varepsilon\big)\Big], & N/2 \leq i \leq N-1. \end{cases}$$

Proof The estimate of the truncation error depends on the location of the mesh point x_i and the relation between the step size mesh h_i and ε, so that we consider different cases as follows.
Case1 (Inner region). For $1 \leq i < N/2$, we have $h_i = h$. Using (5.10) and the bounds (3.11) for r_i^- and r_i^+, we obtain

$$\big|\tau_{i,u}\big| \leq C\Big(\varepsilon^{-2}h^4\big|u'''\big| + \varepsilon^{-1}h^4\big|u^{(4)}\big| + h^4\big|u^{(5)}\big| + \varepsilon h^4\big|u^{(6)}\big|\Big).$$

Using the bounds (2.3) for derivatives of u, it follows that

$$\big|\tau_{i,u}\big| \leq C\Big[\varepsilon^{-2}h^4 + \varepsilon^{-5}h^4\exp\big(-2\beta x_i/\varepsilon\big)\Big], \qquad 1 \leq i < N/2.$$

Case2 (Outer region). For $N/2 \leq i \leq N-1$, the truncation error is split into two parts $\tau_{i,v}$ and $\tau_{i,w}$ corresponding to v and w.
Thus

$$\big|\tau_{i,u}\big| \leq \big|\tau_{i,v}\big| + \big|\tau_{i,w}\big|.$$

Here we consider two subcases depending on the relation between h_m and ε.

(i) In the case $h_m\|a\| < 2\varepsilon$, using (3.11), (5.10) and the bounds on the derivatives of v given in Theorem 5.1, we have

$$\big|\tau_{i,v}\big| \leq C\Big[\varepsilon^{-2}h_i^4 + h_i^4\exp\big(-2\beta x_i/\varepsilon\big)\Big] \leq C\Big[h_i^2 + h_i^4\exp\big(-2\beta x_i/\varepsilon\big)\Big].$$

Similarly,

$$\big|\tau_{i,w}\big| \leq C\varepsilon^{-5}h_i^4\exp\big(-2\beta x_i/\varepsilon\big) \leq C\varepsilon^{-1}\exp\big(-2\beta x_{i-1}/\varepsilon\big),$$

using $y^k e^{-y} \leq C$, for all $y \geq 0$, k is a positive integer, in the last inequality.

(ii) For the case $h_m\|a\| \geq 2\varepsilon$, we use the following estimate for the local truncation error:

$$\begin{aligned}\tau_{i,u} =& \frac{\varepsilon}{h_i h_{i+1}(h_i + h_{i+1})}\Big(R_2(x_i, x_{i+1}, u)r_i^+ - R_2(x_i, x_{i-1}, u)r_i^-\Big)\\ &+ \Big(\varepsilon R_0(x_i, x_{i+1}, u'') - R_1(x_i, x_{i+1}, u')a_{i+1}\Big)q_i^+\\ &- \Big(\varepsilon R_0(x_i, x_{i-1}, u'') + R_1(x_i, x_{i-1}, u')a_{i-1}\Big)q_i^-.\end{aligned} \tag{5.12}$$

Using (3.11) and (5.7) in (5.11) and integration by parts, we deduce that

$$\begin{aligned}\left|\frac{\varepsilon R_2(x_i, x_{i-1}, w)r_i^-}{h_i h_{i+1}(h_i + h_{i+1})}\right| \leq& \frac{C}{h_i}\left|\int_{x_{i-1}}^{x_i}(x_{i-1} - \xi)^2\varepsilon^{-3}\exp\big(-2\beta\xi/\varepsilon\big)d\xi\right|\\ \leq& \frac{C}{h_i}\left[\Big(\frac{h_i}{\varepsilon}\Big)^2\exp\big(-2\beta x_i/\varepsilon\big)\right]\\ \leq& \frac{C}{h_i}\exp\big(-2\beta x_{i-1}/\varepsilon\big).\end{aligned}$$

In a similar way, we obtain

$$\left|\frac{\varepsilon R_2(x_i,x_{i+1},w)r_i^+}{h_ih_{i+1}(h_i+h_{i+1})}\right| \le \frac{C}{h_i}\exp\big(-2\beta x_i/\varepsilon\big),$$

$$\left|\varepsilon R_0(x_i,x_{i+1},w'')q_i^+\right| \le C\varepsilon^{-1}\exp\big(-2\beta x_i/\varepsilon\big),$$

$$\left|R_1(x_i,x_{i+1},w')q_i^+\right| \le C\varepsilon^{-1}\exp\big(-2\beta x_i/\varepsilon\big).$$

Using (5.12) and the previous bounds, we obtain

$$\begin{aligned}|\tau_{i,w}| &\le \frac{C}{h_i}\Big[1+\frac{h_i}{\varepsilon}\exp\big(-2\beta h_i/\varepsilon\big)\Big]\exp\big(-2\beta x_{i-1}/\varepsilon\big)\\ &\le \frac{C}{h_i}\exp\big(-2\beta x_{i-1}/\varepsilon\big), \quad N/2<i\le N-1.\end{aligned}$$

Likewise

$$|\tau_{i,v}| \le C\Big[h_i^2+\varepsilon^3 h_i\exp\big(-2\beta x_{i-1}/\varepsilon\big)\Big], \quad N/2<i\le N-1.$$

Now, at the transition point $x_{N/2}=\sigma$, following the above argument, we get

$$|\tau_{N/2,v}| \le C\Big[H^2+\varepsilon^3 H\exp\big(-2\beta x_{N/2}/\varepsilon\big)\Big].$$

For the layer component, since $W_{N/2}$ is the solution of a homogeneous difference equation $L_N W_{N/2}=0$, it follows that

$$\begin{aligned}\left|\tau_{N/2,w}\right| =&\frac{\varepsilon}{hH(h+H)}\left|(r_{N/2}^-+r_{N/2}^c+r_{N/2}^+)w_{N/2}\right| + \left|R_0(x_{N/2},x_{N/2+1},w)r_{N/2}^+\right|\\ &+\left|R_0(x_{N/2},x_{N/2-1},w)r_{N/2}^-\right|.\end{aligned}$$

From (3.11) and (5.11), we deduce that

$$\begin{aligned}\left|\tau_{N/2,w}\right| \le& C\Big[\frac{1}{(h+H)}\Big(1-\exp\big(-2\beta H/\varepsilon\big)\Big)\\ &+\frac{\varepsilon}{h(h+H)}\Big(\exp\big(2\beta h/\varepsilon\big)-1\Big)\Big]\exp\big(-2\beta x_{N/2}/\varepsilon\big)\\ \le& C\Big[\frac{1}{(h+H)}\Big]\exp\big(-2\beta x_{N/2}/\varepsilon\big)\\ \le& \frac{C}{H}\exp\big(-2\beta x_{N/2}/\varepsilon\big),\end{aligned}$$

where we use $\exp(\phi)\ge 0$ and $\exp(\phi)\le 1+C\phi$ in bounded intervals of ϕ in the above inequality. Hence, combining $|\tau_{i,v}|$ and $|\tau_{i,w}|$ in the above two cases, we have

$$\left|\tau_{i,u}\right| \le \begin{cases} C\Big[h_i^2+\varepsilon^{-1}\exp\Big(-2\beta x_i/\varepsilon\Big)\Big], & i=N/2,\\ C\Big[h_i^2+\dfrac{1}{\max\{h_i,\varepsilon\}}\exp\big(-2\beta x_{i-1}/\varepsilon\big)\Big], & N/2<i\le N-1.\end{cases}$$

This completes the proof. □

To prove the uniform convergence of the present method, we will use the discrete maximum principle with the following barrier functions:

$$Z_i = 1 + x_i, \qquad \Phi_i(\mu) = \begin{cases} \prod_{j=1}^{i} \big(1+\frac{\mu h_j}{\varepsilon}\big)^{-1}, & 1 \le i \le N-1, \\ 1, & i = 0, \end{cases}$$

where μ is a positive constant.

Lemma 5.3 *Let the assumptions in Lemma 4.1 be satisfied and let $\mu \le \beta$. Then for some constant $C(\mu)$, we have*

$$-L_N \Phi_i(\mu) \ge \frac{C(\mu)}{\max\{\varepsilon, h_i\}} \Phi_i(\mu), \qquad 1 \le i \le N-1.$$

Proof Applying the operator L_N to the discrete function Φ_i, we obtain

$$\begin{aligned} -L_N\Phi_i(\mu) &= -\big(\hat{r}_i^- \Phi_{i-1}(\mu) + \hat{r}_i^c \Phi_i(\mu) + \hat{r}_i^+ \Phi_{i+1}(\mu)\big) \\ &= -\Big[\Big(1+\frac{\mu h_i}{\varepsilon}\Big)\hat{r}_i^- + \hat{r}_i^c + \Big(1+\frac{\mu h_{i+1}}{\varepsilon}\Big)^{-1}\hat{r}_i^+\Big]\Phi_i(\mu) \\ &= \frac{\mu}{\varepsilon+\mu h_{i+1}}\Big[h_{i+1}\hat{r}_i^+ - h_i\hat{r}_i^- - \frac{\mu h_i h_{i+1}}{\varepsilon}\hat{r}_i^-\Big]\Phi_i(\mu). \end{aligned}$$

Thus, using (3.8), (4.5), and that $\mu \le \beta$, we obtain the desired results. □

Lemma 5.4 *For each $1 \le i \le N-1$ and $0 < \mu \le \beta$, we have*

$$\exp\big(-\beta x_i/\varepsilon\big) \le \Phi_i(\mu). \tag{5.13}$$

Moreover, for the Shishkin mesh defined in the previous section, we have

$$\Phi_{N/2}(\mu) \le C N^{-\mu\sigma_0}, \tag{5.14}$$

for some positive constant C.

Proof For $1 \le i \le N-1$ and using that $\mu \le \beta$, we have

$$\exp\big(-\beta x_i/\varepsilon\big) = \exp\Big((-\beta/\varepsilon)\sum_{j=1}^{i} h_j\Big) = \prod_{j=1}^{i}\exp\big(-\beta h_j/\varepsilon\big) \le \prod_{j=1}^{i}\exp\big(-\mu h_j/\varepsilon\big),$$

and using the inequalities

$$\exp\big(-\mu h_j/\varepsilon\big) = \Big(\exp\big(\mu h_j/\varepsilon\big)\Big)^{-1} \le \Big(1+\frac{\mu h_j}{\varepsilon}\Big)^{-1}, \quad 1 \le j \le i,$$

yields the desired estimate (5.13).

Moreover, using that $h_i = 2\sigma_0\varepsilon \ln N/N$, $1 \le i \le N/2$, we have

$$\Phi_{N/2}(\mu) = \prod_{j=1}^{N/2}\Big(1+\frac{\mu h_j}{\varepsilon}\Big)^{-1} = \prod_{j=1}^{N/2}\Big(1+\frac{2\mu\sigma_0 \ln N}{N}\Big)^{-1} = \Big(1+\frac{2\mu\sigma_0\ln N}{N}\Big)^{-N/2}, \tag{5.15}$$

and using the inequality [5]

$$\Big(1+\frac{2\sigma_0\ln N}{N}\Big)^{-N/2}\le CN^{-\sigma_0},\quad \text{if}\quad 2\sigma_0\ln N<N,$$

we deduce that

$$\Big(1+\frac{2\mu\sigma_0\ln N}{N}\Big)^{-N/2}\le CN^{-\mu\sigma_0},$$

where the condition (4.4) and that $\mu\le\beta$ imply that $2\mu\sigma_0\ln N<N$. □

Theorem 5.5 *Let u and U be the exact and the discrete solutions of (1.1) and (4.1), respectively, and assume that N satisfies the conditions (4.4). Then if $\mu\le\beta$, we have the following ε-uniform error estimate*

$$\big|u(x_i)-U_i\big|\le\begin{cases} C\big(N^{-2\mu\sigma_0}+N^{-4}\sigma_0^4\ln^4 N\big), & 1\le i<N/2,\\ C\big(N^{-2}+N^{-2\mu\sigma_0}\big), & N/2\le i\le N-1.\end{cases}$$

Proof We begin with the outer region. Thus, for $N/2\le i\le N-1$ we consider the following mesh function:

$$\Psi_i(\mu)=-C\Big[h_i^2Z_i+\Phi_{N/2}(\mu)\Phi_{i-1}(\mu)\Big].$$

Using (5.13) and Lemmas 5.2 and 5.3, we have

$$\big|\tau_{i,u}\big|\le L_\varepsilon^N\Psi_i(\mu),\qquad N/2\le i\le N-1.$$

Thus, applying the discrete maximum principle to $\Psi_i(\mu)\pm\big(u_i-U_i\big)$, we have

$$\big|u_i-U_i\big|\le\Psi_i(\mu)\le C\Big[H^2+\Phi_{N/2}(\mu)\Phi_{i-1}(\mu)\Big]\le C\Big[H^2+\Big(1+\frac{\mu h}{\varepsilon}\Big)\Big(\Phi_{N/2}(\mu)\Big)^2\Big].$$

It follows from (4.4) that $\mu h/\varepsilon<1$. Using this and (5.14), we obtain

$$\big|u_i-U_i\big|\le C\big(N^{-2}+N^{-2\mu\sigma_0}\big),\quad N/2\le i\le N-1.$$

Now, in the inner region, we consider the following barrier function:

$$\Psi_i(\mu)=-C\Big[\big(\varepsilon^{-2}h^4+N^{-2\mu\sigma_0}\big)Z_i+h^4\varepsilon^{-4}\Phi_i(\mu)\Big],\quad 1\le i<N/2.$$

Again, applying the discrete maximum principle and using $h=2\sigma_0\varepsilon N^{-1}\ln N$, we get

$$\big|u_i-U_i\big|\le C\big(N^{-2\mu\sigma_0}+N^{-4}\sigma_0^4\ln^4 N\big),\quad 1\le i<N/2.$$

Thus the proof is complete. □

Remark 5.1 *From Theorem 5.5, we can see that the present method is ε-uniform convergent of order $(N^{-4}\sigma_0^4\ln^4 N)$ in the boundary layer region, and it is second order ε-uniform convergent outside the boundary layer region provided that $\mu\sigma_0\ge 2$.*

Now we investigate a bound on the global error. Before analyzing the error bound, we introduce some notation. Denote by $\omega(x)$ the error of the piecewise cubic interpolation for $u(x)$ defined as

$$\omega(x_i + \theta h_{i+1}) = u(x_i + \theta h_{i+1}) - \sum_{k=0}^{3} P_k(\theta) u(x_{i-1+k}), \quad \theta \in [0, 1).$$

Define

$$E_i = u(x_i) - U_i \qquad \text{and} \qquad \Omega = \max\big\{|\omega(x)| : x \in [0,1]\big\},$$

where u and U are the solutions of (1.1) and (4.1), respectively. It is easy to verify that $\Omega = O(N^{-3})$. The main result on error estimate is given by the following theorem.

Theorem 5.6 *Assume that the function f satisfies Lipschitz condition* (2.2) *and that the condition* (4.9) *holds. Assume that N satisfies the conditions* (4.4)*. Then if $\mu \le \beta$ and $\sigma_0 \ge 2/\mu$, the method* (4.1)–(4.3) *is convergent and the following error estimate holds:*

$$\|E\| \le \begin{cases} C\Big[D\Big(N^{-4}\sigma_0^4 \ln^4 N\Big) + \Big(DK(L_2+L_3)+1\Big)N^{-3}\Big], & 1 \le i < N/2, \\ C\Big[DN^{-2} + \Big(DK(L_2+L_3)+1\Big)N^{-3}\Big], & N/2 \le i \le N-1, \end{cases}$$

where

$$D = \frac{M\|L_N^{-1}\|}{1 - K\|L_N^{-1}\|\Big(L_1 + M(L_2+L_3)\Big)}.$$

Proof Let $\mathbf{u} = \big[u(x_1), u(x_2), \cdots, u(x_{N-1})\big]^T$ and $U = \big[U_1, U_2, \cdots, U_{N-1}\big]^T$. Then since

$$E = L_N^{-1}\Big(F(\mathbf{u}) - F(U) + \tau_u\Big),$$

we get the following:

$$\big\|E\big\| \le \big\|L_N^{-1}\big\|\big\|F(\mathbf{u}) - F(U)\big\| + \big\|L_N^{-1}\big\|\big\|\tau_u\big\|,$$

and we have

$$\big\|F(\mathbf{u}) - F(U)\big\| \le K \max_{1 \le i \le N-1}\Big(L_1|u(x_i) - U_i| + L_2\zeta_i + L_3\xi_i\Big),$$

where

$$\zeta_i = \big|u(x_i - \delta) - U_{i,\delta}\big| \qquad \text{and} \qquad \xi_i = \big|u(x_i + \eta) - U_{i,\eta}\big|.$$

It is clear that, if $x_i \le \delta$, then $\zeta_i = 0$. Otherwise, $x_i \in [x_j + \delta, x_{j+1} + \delta)$ for some $0 \le j \le N-1$. Thus, $x_i - \delta = x_j + \theta h_{j+1}, 0 \le \theta < 1$. Using the cubic interpolation, we get

$$\begin{aligned}\zeta_i \le& \Big|u(x_i - \delta) - \sum_{k=0}^{3} P_k(\theta)u(x_{j-1+k})\Big| + \Big|\sum_{k=0}^{3} P_k(\theta)u(x_{j-1+k}) - \sum_{k=0}^{3} P_k(\theta)U_{j-1+k}\Big| \\ \le& \Omega + M\big\|E\big\|.\end{aligned}$$

Similarly, we obtain

$$\xi_i \le \Omega + M\big\|E\big\|.$$

Thus

$$\|E\| \le K\|L_N^{-1}\|\Big(L_1+M(L_2+L_3)\Big)\|E\| + \|L_N^{-1}\|\Big(\|\tau_u\|+K(L_2+L_3)\Omega\Big),$$

which, using the condition (4.9), is equivalent to

$$\|E\| \le \frac{\|L_N^{-1}\|\Big(\|\tau_u\|+K(L_2+L_3)\Omega\Big)}{1-K\|L_N^{-1}\|\Big(L_1+M(L_2+L_3)\Big)}. \tag{5.16}$$

To obtain a bound on E, let $U_{i,\theta}$ be the approximation of $u(x_i+\theta h_{i+1})$; then we have

$$\begin{aligned}\Big|u(x_i+\theta h_{i+1})-U_{i,\theta}\Big| \le& \Big|u(x_i+\theta h_{i+1})-\sum_{k=0}^{3}P_k(\theta)u(x_{i-1+k})\Big| \\ &+\sum_{k=0}^{3}\Big|P_k(\theta)\big(u(x_{i-1+k})-U_{i-1+k}\big)\Big| \\ \le& \Omega + M\|E\|.\end{aligned}$$

Consequently, combining the above inequality with (5.16) and using Theorem 5.5, we obtain the desired estimate. □

Lemma 5.7 *Let B be the $(N-1)\times(N-1)$ matrix of the form*

$$B=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

and let $A=-\tilde{\alpha}B+I-\tilde{\beta}B^T$ with I being the identity matrix. Let $\tilde{\alpha}+\tilde{\beta}=1$, $\tilde{\alpha},\tilde{\beta}\ge 0$. Then

$$\|A^{-1}\|_2 \le O\big((N-1)^2\big).$$

Proof see [30]. □

Theorem 5.8 *Let L_N be the $(N-1)\times(N-1)$ tridiagonal matrix (4.2), and let $\varepsilon = O(h_m^p)$, where p is a positive real number. Then*

$$\|L_N^{-1}\|_2 \le \begin{cases} O(h_m^{-p}), & 0<p<1, \\ O(h_m^{3p-4}), & p\ge 1. \end{cases}$$

Proof We will follow the procedure as done in [7]. Let $\tilde{\alpha}=-r_i^-/r_i^c$ and $\tilde{\beta}=-r_i^+/r_i^c$. Then, using (3.8), it is easy to show that $\tilde{\alpha}$ and $\tilde{\beta}$ satisfy the assumptions of Lemma 5.7 ($\tilde{\alpha}+\tilde{\beta}=1, \tilde{\alpha},\tilde{\beta}\ge 0$). Hence, the tridiagonal matrix L_N defined by (4.2) can be written as

$$L_N = \mathrm{diag}\Big(\hat{r}_1^c,\hat{r}_2^c,\cdots,\hat{r}_{N-1}^c\Big)\Big(-\tilde{\alpha}B+I-\tilde{\beta}B^T\Big) = \mathrm{diag}\Big(\hat{r}_1^c,\hat{r}_2^c,\cdots,\hat{r}_{N-1}^c\Big)A,$$

where B and I are the matrices defined in Lemma 5.7, and $A = \big(-\tilde{\alpha}B + I - \tilde{\beta}B^T\big)$. Therefore, using Lemma 5.7, we have

$$\|L_N^{-1}\|_2 = \frac{1}{\hat{r}_m^c}\|A^{-1}\|_2 \leq \frac{1}{\hat{r}_m^c}\,O\big((N-1)^2\big), \qquad \hat{r}_m^c = \max_{1\leq i\leq N-1}|\hat{r}_i^c|. \tag{5.17}$$

From (3.8) and (3.9), it follows that

$$\hat{r}_m^c = \frac{O(\varepsilon^4) + O(\varepsilon^3 h_m) + O(\varepsilon^2 h_m^2) + O(\varepsilon h_m^3) + O(h_m^4)}{O(\varepsilon^3 h_m^2)}.$$

Using (5.17) and the above equation with the assumption $\varepsilon = O(h_m^p)$ gives

$$\|L_N^{-1}\|_2 \leq \begin{cases} \dfrac{O(h_m^{3p})}{O(h_m^{4p})} = O(h_m^{-p}), & 0 < p < 1, \\[2ex] \dfrac{O(h_m^{3p})}{O(h_m^{4})} = O(h_m^{3p-4}), & p \geq 1. \end{cases}$$

□

6. Numerical results

In this section, we present the numerical results obtained by the FMOCI method described in Section 3. In order to demonstrate the accuracy of the present method to solve nonlinear singularly perturbed differential equations with small shifts, we consider four examples including both cases, when the boundary layer occurs on the left as well as on the right side of the interval. The computational results are listed with the maximum pointwise errors and orders of convergence for different values of δ, η, and ε. The solutions of the considered examples are plotted for different values of δ and η to illustrate the effect of delay and advance parameters on the boundary layer behavior of the solution.

Example 6.1 Consider the following nonlinear singularly perturbed boundary value problem with small delay

$$\varepsilon u''(x) + u'(x) = u^2(x-\delta) - \Big(2(x-\delta) + \frac{1 - e^{-\frac{(x-\delta)}{\varepsilon}}}{e^{\frac{-1}{\varepsilon}} - 1}\Big)^2 + 2,$$
$$u(x) = 0, \quad -\delta \leq x \leq 0, \qquad u(1) = 1,$$

Example 6.2 Consider the problem with small positive shift

$$\varepsilon u''(x) + \frac{1}{x+1}u'(x) = u^3(x) - 0.25\,u(x+\eta) + \frac{1}{4(x+\eta+1)},$$
$$u(0) = 0, \qquad u(x) = 0, \quad 1 \leq x \leq 1+\eta,$$

Example 6.3 Consider the problem with mixed type of small shift

$$\varepsilon u''(x) + u'(x) = -u^2(x-\delta) + u(x+\eta) + \Big(\frac{e^{-\frac{(x-\delta)}{\varepsilon}} - 1}{e^{\frac{-1}{\varepsilon}} - 1}\Big)^2 - \frac{e^{-\frac{(x+\eta)}{\varepsilon}} - 1}{e^{\frac{-1}{\varepsilon}} - 1},$$
$$u(x) = 0, \quad -\delta \leq x \leq 0, \qquad u(x) = 1, \quad 1 \leq x \leq 1+\eta,$$

Example 6.4 Consider the following problem with right-hand side boundary layer

$$\varepsilon u''(x) - u'(x) = e^{-u} - e^{x-1} - u(x-\delta) + u(x+\eta),$$
$$u(x) = 1, \quad -\delta \leq x \leq 0, \qquad u(x) = 0, \quad 1 \leq x \leq 1+\eta,$$

The exact solutions of the above problems are not known and so the maximum pointwise error is evaluated using the double mesh principle [3],

$$E_\varepsilon^N = \max_{0\leq j\leq N} |U_j^N - U_j^{2N}|,$$

Furthermore, the ε-uniform maximum pointwise error E^N and the corresponding ε-uniform order of convergence p^N are computed by

$$E^N = \max_{\varepsilon} E_\varepsilon^N \qquad \text{and} \qquad p^N = \log_2\left(\frac{E^N}{E^{2N}}\right).$$

Table 1. Maximum errors E_ε^N, E^N, and ε-uniform orders of convergence p^N for Example 6.1 with $\delta = 0.5\varepsilon$.

ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-2}	3.99E−04	5.67E−05	6.16E−06	6.78E−07	7.29E−08
10^{-4}	5.35E−04	6.63E−05	7.23E−06	7.10E−07	5.30E−08
10^{-6}	5.34E−04	6.63E−05	7.31E−06	7.46E−07	7.17E−08
10^{-8}	5.34E−04	6.63E−05	7.31E−06	7.46E−07	7.19E−08
10^{-10}	5.34E−04	6.63E−05	7.31E−06	7.46E−07	7.19E−08
E^N	5.35E−04	6.63E−05	7.31E−06	7.46E−07	7.29E−08
p^N	3.01	3.18	3.29	3.36	

Table 2. Maximum errors E_ε^N, E^N, and ε-uniform orders of convergence p^N for Example 6.1 with $\delta = 0.03$.

ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-2}	3.64E−08	5.82E−09	1.32E−09	1.88E−10	1.62E−11
2^{-3}	6.71E−07	9.97E−08	2.14E−08	3.03E−09	2.73E−10
2^{-4}	4.74E−06	5.25E−07	2.63E−08	1.14E−08	2.72E−09
2^{-5}	9.58E−05	3.14E−06	1.06E−07	1.55E−08	6.04E−10
2^{-6}	1.55E−03	5.79E−05	1.99E−06	6.47E−08	2.37E−09
2^{-7}	1.52E−03	1.99E−04	2.06E−05	2.01E−06	2.64E−07
E^N	1.55E−03	1.99E−04	2.06E−05	2.01E−06	2.64E−07
p^N	2.96	3.27	3.36	2.93	

For different values of ε, δ, and η, the maximum pointwise error E_ε^N has been computed and presented in Tables 1–8. The last two rows in each table show the ε-uniform maximum pointwise errors E^N and the ε-uniform orders of convergence p^N. From these tables, we conclude that the proposed method is ε-uniformly convergent of third order, and for all values of ε the maximum pointwise error E^N decreases rapidly with increasing N. In addition, Tables 2 and 8 demonstrate that the proposed method is effective in both cases, when shift parameters are of $O(\varepsilon)$ or $o(\varepsilon)$.

Table 3. Maximum errors E_ε^N, E^N, and ε-uniform orders of convergence p^N for Example 6.2 with $\eta = 0.5\varepsilon$.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-2}	4.50E−04	5.46E−05	6.15E−06	6.33E−07	4.16E−08
10^{-4}	4.93E−04	6.05E−05	7.08E−06	6.82E−07	9.13E−08
10^{-6}	4.93E−04	6.05E−05	7.07E−06	7.93E−07	9.57E−08
10^{-8}	4.93E−04	6.05E−05	7.07E−06	7.92E−07	9.56E−08
10^{-10}	4.93E−04	6.05E−05	7.07E−06	7.92E−07	9.56E−08
E^N	4.93E−04	6.05E−05	7.08E−06	7.93E−07	9.57E−08
p^N	3.03	3.09	3.16	3.05	

Table 4. Maximum errors E_ε^N, E^N, and ε-uniform orders of convergence p^N for Example 6.2.

ε	η	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-1}	ε	2.26E−07	1.53E−07	1.61E−08	6.31E−09	3.44E−10
	6ε	1.71E−07	1.79E−07	1.86E−08	7.64E−09	4.07E−10
	10ε	8.30E−08	5.30E−09	3.34E−10	2.10E−11	1.40E−12
10^{-2}	ε	4.50E−04	5.43E−05	6.16E−06	6.39E−07	4.14E−08
	6ε	4.29E−04	5.18E−05	5.92E−06	6.11E−07	3.88E−08
	10ε	4.12E−04	4.96E−05	5.72E−06	5.93E−07	3.76E−08
10^{-3}	ε	4.90E−04	6.03E−05	7.02E−06	7.30E−07	7.45E−08
	6ε	4.84E−04	5.86E−05	6.99E−06	7.35E−07	7.54E−08
	10ε	4.81E−04	5.87E−05	6.73E−06	7.43E−07	6.86E−08
E^N		4.90E−04	6.03E−05	7.02E−06	7.43E−07	7.54E−08
p^N		3.02	3.10	3.26	3.30	

Table 5. Maximum errors E_ε^N, E^N, and ε-uniform orders of convergence p^N for Example 6.3 with $\delta = \eta = 2.5\varepsilon$.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-2}	1.60E−04	1.98E−05	2.44E−06	2.63E−07	2.35E−08
10^{-4}	1.13E−04	1.35E−05	1.49E−06	1.51E−07	1.45E−08
10^{-6}	1.12E−04	1.34E−05	1.47E−06	1.49E−07	1.43E−08
10^{-8}	1.12E−04	1.34E−05	1.47E−06	1.49E−07	1.43E−08
10^{-10}	1.12E−04	1.34E−05	1.47E−06	1.49E−07	1.43E−08
E^N	1.60E−04	1.98E−05	2.44E−06	2.63E−07	2.35E−08
p^N	3.016	3.02	3.21	3.49	

Table 6. Maximum errors E_ε^N, E^N, and ε-uniform orders of convergence p^N for Example 6.3 with $\eta = \delta$.

ε	δ	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-2}	ε	2.38E−04	2.58E−05	3.52E−06	4.10E−07	4.97E−08
	2ε	2.68E−04	2.93E−05	4.11E−06	5.04E−07	5.34E−08
	3ε	3.07E−04	3.84E−05	4.92E−06	5.81E−07	6.12E−08
10^{-4}	ε	1.77E−04	2.15E−05	2.35E−06	2.39E−07	2.36E−08
	2ε	1.77E−04	2.15E−05	2.36E−06	2.41E−07	2.32E−08
	3ε	1.78E−04	2.16E−05	2.37E−06	2.42E−07	2.33E−08
10^{-6}	ε	1.76E−04	2.14E−05	2.34E−06	2.38E−07	2.28E−08
	2ε	1.76E−04	2.14E−05	2.34E−06	2.38E−07	2.28E−08
	3ε	1.76E−04	2.14E−05	2.34E−06	2.38E−07	2.28E−08
E^N		3.07E−04	3.84E−05	4.92E−06	5.81E−07	6.12E−08
p^N		3.00	2.97	3.08	3.25	

Table 7. Maximum errors E_ε^N, E^N, and ε-uniform orders of convergence p^N for Example 6.4 with $\delta = \eta = 0.5\varepsilon$.

ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-2}	2.76E−03	2.99E−04	2.85E−05	2.10E−06	1.46E−07
10^{-3}	3.10E−03	3.47E−04	3.46E−05	2.87E−06	2.20E−07
10^{-4}	3.14E−03	3.57E−04	3.81E−05	3.43E−06	3.26E−07
10^{-5}	3.14E−03	3.58E−04	3.84E−05	3.57E−06	2.59E−07
10^{-6}	3.14E−03	3.58E−04	3.85E−05	3.57E−06	2.48E−07
10^{-7}	3.14E−03	3.58E−04	3.85E−05	3.56E−06	2.55E−07
10^{-8}	3.14E−03	3.58E−04	3.84E−05	3.29E−06	2.73E−07
E^N	3.14E−03	3.58E−04	3.85E−05	3.57E−06	3.26E−07
p^N	3.13	3.22	3.43	3.46	

Table 8. Maximum errors E_ε^N, E^N, and ε-uniform orders of convergence p^N for Example 6.4 for with $\delta = \eta$.

δ, η	ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
0.01	10^{-1}	1.82E−05	2.08E−06	4.59E−07	1.96E−07	2.94E−08
	10^{-2}	2.69E−03	2.62E−04	1.61E−05	9.25E−06	1.12E−06
	10^{-3}	3.12E−03	3.51E−04	3.13E−05	2.75E−06	3.53E−07
0.03	10^{-1}	4.88E−05	1.09E−05	1.78E−06	1.54E−07	1.18E−07
	10^{-2}	2.76E−03	2.21E−04	4.27E−05	9.61E−06	1.11E−06
	10^{-3}	5.80E−03	6.85E−04	7.86E−05	2.96E−06	8.93E−07
E^N		5.80E−03	6.85E−04	7.86E−05	9.61E−06	1.12E−06
p^N		3.10	3.12	3.03	3.12	

Figure 1 shows the corresponding numerical solutions of Examples 6.1 and 6.2 with $\varepsilon = 2^{-5}$ and for different values of δ and η, respectively. Similarly, the numerical solution for Example 6.4 with $\varepsilon = 2^{-5}$ and different values of δ and η are plotted in Figure 2. From these Figures, one can observe the effect of shift parameters δ and η on the solution behavior in both cases (left and right) of boundary layer. In the case of the left boundary layer, as δ increases, the thickness of the boundary layer decreases while it increases when η increases, and the effect of these parameters is reverse in the right case.

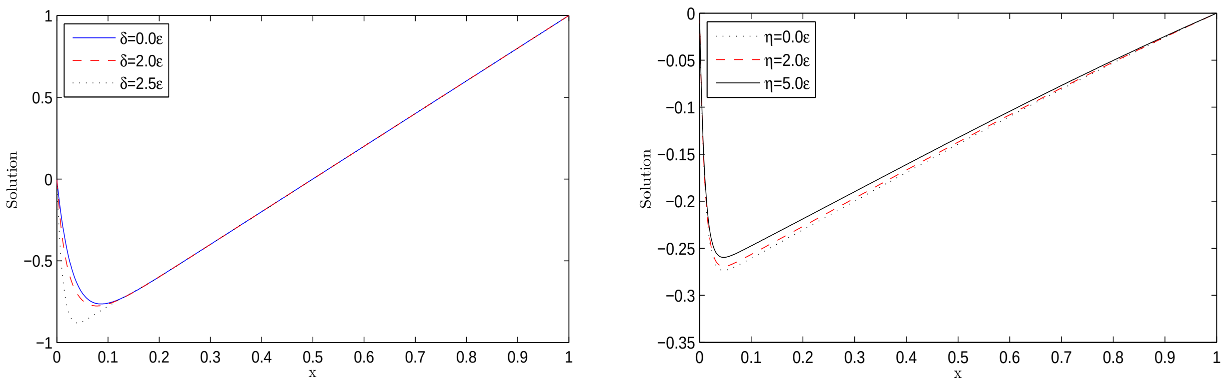

Figure 1. The numerical solution of Example 6.1 (left) and Example 6.2 (right) for $\varepsilon = 2^{-5}$.

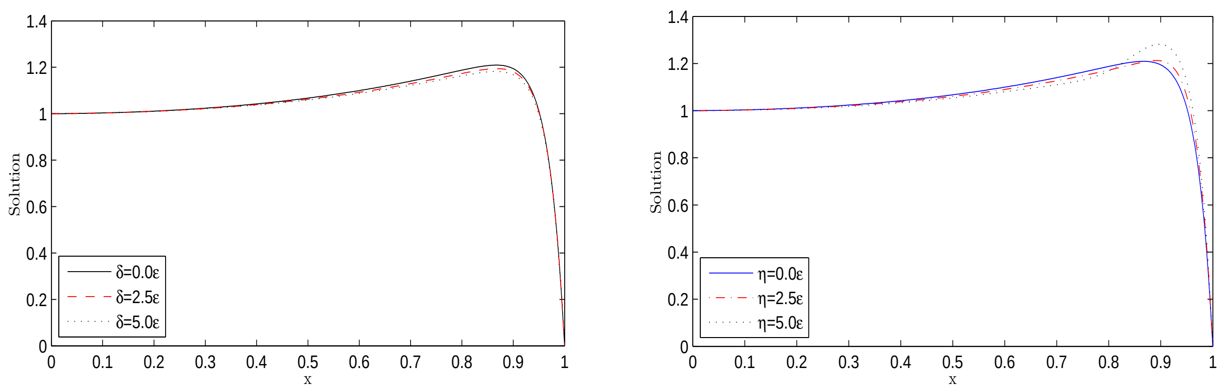

Figure 2. The numerical solution of Example 6.4 for $\varepsilon = 2^{-5}$.

7. Conclusion

An efficient high order uniform method has been developed for solving nonlinear singularly perturbed delay differential equation with small shifts. Both cases, when the boundary layer occurs on the left and on the right side of the interval, are studied. The proposed method is analyzed for convergence and the bound of global error is also discussed. Error analysis is carried out and it has been shown that the present method is ε-uniform convergent with third-order accuracy. The advantages of this method are the higher order of accuracy, the simplicity of implementation, and the strong performance in both cases, when the delay and advance parameters are of $O(\varepsilon)$ or $o(\varepsilon)$.

The effect of small shifts on the layer behavior of the solution has been discussed by considering both cases of boundary layers. It is observed that in the case of the left boundary layer, the thickness of the layer decreases as the delay parameter δ increases while it increases when the advance parameter η increases as shown in Figure 1. In the right boundary layer, the impact of these shifts is the reverse, i.e. as δ increases, the thickness of the boundary layer increases and it decreases when η increases as shown in Figure 2. Moreover, we observe that the effect of delay parameter is more in the case of the left boundary layer in comparison to the right boundary layer case, whereas the advance parameter affects more in the right side boundary layer case.

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