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On congruences related to central binomial coefficients, harmonic and Lucas numbers

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Abstract: In this paper, using some combinatorial identities, we present new congruences involving central binomial coefficients and harmonic, Catalan, and Fibonacci numbers. For example, for an odd prime p, we have

$$\sum_{k=1}^{(p-1)/2}(-1)^k\binom{2k}{k}H_{k-1} \equiv \frac{2^p}{p}\left(2F_{p+1}-5^{(p-1)/2}-1\right)(\bmod\ p),$$

$$\sum_{k=0}^{(p-1)/2}\frac{H_kC_k}{(-4)^k} \equiv 2\frac{Q_{p+1}}{p}-\frac{2^{p+1}}{p}\left(1+2^{(p+1)/2}\right)(\bmod\ p),$$

and for $\left(\frac{5}{p}\right)=1$,

$$\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{H_{k-1}F_k}{(-4)^k}\equiv\frac{1}{p}\left(F_{2p+1}-F_{p+2}\right)-\frac{2^p}{p}F_{p-1}(\bmod\ p),$$

where $\{F_n\}$ is the Fibonacci sequence and $\{Q_n\}$ is the Pell–Lucas sequence.

Key words: Central binomial coefficients, harmonic numbers, Catalan numbers, Fibonacci numbers, Pell numbers

1. Introduction

The Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ are defined by the recursions for $n>1$,

$$F_{n+1}=F_n+F_{n-1} \text{ and } L_{n+1}=L_n+L_{n-1},$$

where $F_0=0,\ F_1=1$ and $L_0=2,\ L_1=1$, respectively. The Binet formulae are

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta} \quad\text{and}\quad L_n=\alpha^n+\beta^n,$$

where $\alpha,\beta=\left(1\pm\sqrt{5}\right)/2$.

The Pell sequence $\{P_n\}$ and the Pell–Lucas sequence $\{Q_n\}$ are defined by the recursions for $n>1$,

$$P_{n+1}=2P_n+P_{n-1} \text{ and } Q_{n+1}=2Q_n+Q_{n-1},$$

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where $P_0 = 0,\ P_1 = 1$ and $Q_0 = Q_1 = 2$, respectively. The Binet formulae are

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \text{ and } Q_n = \gamma^n + \delta^n,$$

where $\gamma, \delta = \left(1 \pm \sqrt{2}\right)$.

Harmonic numbers H_n are defined as for $n > 0$,

$$H_n = \sum_{k=1}^{n} \frac{1}{k},$$

where $H_0 = 0$. The first few harmonic numbers are $1,\ \frac{3}{2},\ \frac{11}{6},\ \frac{25}{12}, \cdots$.

The Catalan numbers are defined by, for $n > 0$,

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}.$$

For a prime p and an integer $a \not\equiv 0 (\text{mod } p)$, we take $q_p(a)$ to denote the Fermat quotient $\left(a^{p-1} - 1\right)/p$. For an odd prime p and an integer a, $\left(\frac{a}{p}\right)$ denotes the Legendre symbol defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p|\, a, \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

Let $\mathbb{Z}$ be the set of integers, and for an odd prime p, let $\mathbb{Z}_p$ denote the set of those rational numbers whose denominator is not divisible by p. For an integer D, $\sqrt{D} \in \mathbb{Z}_p$ if $\left(\frac{D}{p}\right) = 1$ and $\sqrt{D} \notin \mathbb{Z}_p$ if $\left(\frac{D}{p}\right) = -1$ in [11].

Sun [8] gives the following result:

$$\sum_{k=0}^{p-1} \frac{C_k}{m^k} \equiv \frac{m-4}{2}\left(1 - \left(\frac{m(m-4)}{p}\right)\right) (\text{mod } p),$$

where m is any integer not divisible by p.

There are many types of congruences containing binomial coefficients and Fibonacci, Lucas, Pell, and Pell–Lucas numbers. For example, Sun [9] gives the results

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{F_{2k}}{16^k}\binom{2k}{k} &\equiv (-1)^{(p-1)/2 + \lfloor p/5 \rfloor} (\text{mod } p^2), \\ \sum_{k=0}^{(p-1)/2} \frac{F_{2k+1}}{16^k}\binom{2k}{k} &\equiv (-1)^{(p-1)/2 + \lfloor p/5 \rfloor} \frac{5 + \left(\frac{p}{5}\right)}{4} (\text{mod } p^2), \end{aligned}$$

where p is a prime number different from both 2 and 5.

Sun [10] also shows that for a prime $p > 5$

$$\begin{aligned} \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^3\binom{2k}{k}} &\equiv -2B_{p-3} (\text{mod } p), \\ \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^2}{(2k+1)\,16^k} &\equiv -2q_p(2) - pq_p(2)^2 + \frac{5}{12}p^2 B_{p-3} (\text{mod } p^3), \end{aligned}$$

where B_n is the nth Bernoulli number.

Mao and Sun [4] establish that for a prime $p > 3$,

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} H_k &\equiv \frac{1}{3}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) (\text{mod } p), \\ \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} H_{2k} &\equiv \frac{7}{12}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) (\text{mod } p), \end{aligned}$$

where $B_n(x)$ denotes the Bernoulli polynomial of degree n.

Let p be a fixed prime bigger than 3. Define

$$q(x) := \frac{x^p - (x-1)^p - 1}{p} \quad \text{and} \quad G_n(x) := \sum_{k=1}^{p-1} \frac{x^k}{k^n},$$

where x is a variable.

From [1], it is known that

$$G_2(1) \equiv 0 (\text{mod } p), G_1(1) \equiv 0 (\text{mod } p^2) \ \textit{and} \ G_2(-1) \equiv 0 (\text{mod } p)$$

as well as

$$\begin{aligned} G_2(x) &\equiv G_2(1-x) + x^p G_2(1-1/x) (\text{mod } p), \\ q(x)^2 &\equiv -2x^p G_2(x) - 2(1-x^p) G_2(1-x) (\text{mod } p), \\ q(x) &\equiv -G_1(x) (\text{mod } p). \end{aligned} \tag{1.1}$$

Sun [7] obtains the following congruences: for an odd prime p and $G_n(x) \in \mathbb{Z}_p[x]$,

$$G_2(x) \equiv \frac{1}{p}\left(\frac{1+(x-1)^p - x^p}{p} - \sum_{i=1}^{p-1} \frac{(1-x)^i - 1}{i}\right) + p\sum_{r=2}^{p-1} \frac{x^r}{r^2} \sum_{s=1}^{r-1} \frac{1}{s} (\text{mod } p^2),$$

and for a prime $p > 3$

$$npG_{n+1}(x) \equiv (-1)^n x^p G_n(1/x) - G_n(x) (\text{mod } p^2).$$

Pan and Sun [5] obtain that for a prime $p > 5$,

$$\sum_{k=1}^{p-1} \frac{L_k}{k^2} \equiv 0 (\text{mod } p),$$

and for a prime p different from both 2 and 5,

$$\sum_{k=0}^{p^a-1}(-1)^k\binom{2k}{k}\equiv\left(\frac{p^a}{5}\right)\left(1-2F_{p^a-\left(\frac{p^a}{5}\right)}\right)\pmod{p^3},$$

where a is a positive integer.

2. Some congruences involving harmonic, Catalan, and Fibonacci numbers

In this section, we consider the congruences related to some special numbers. For this, first we will give some auxiliary lemmas:

Lemma 1 *For $n>1$ and $x\in\mathbb{R}$,*

$$\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{(-x)^k}{k}H_{k-1}=\frac{(1-x)^n-(-x)^n-1}{n}H_{n-1}-\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k(n-k)}.$$

Proof From the binomial theorem, we write

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k(n-k)} &= \sum_{k=1}^{n-1}\frac{1}{k(n-k)}\sum_{j=1}^{k}\binom{k}{j}(-x)^j\\ &= \sum_{k=1}^{n-1}\frac{1}{n-k}\sum_{j=1}^{k}\binom{k-1}{j-1}\frac{(-x)^j}{j}=\sum_{j=1}^{n-1}\frac{(-x)^j}{j}\sum_{k=j}^{n-1}\binom{k-1}{j-1}\frac{1}{n-k},\end{aligned}$$

which, by the result

$$\sum_{k=j}^{n-1}\binom{k-1}{j-1}\frac{1}{n-k}=\binom{n-1}{j-1}(H_{n-1}-H_{j-1})$$

given in [6], equals

$$\begin{aligned}&\sum_{j=1}^{n-1}\frac{(-x)^j}{j}\binom{n-1}{j-1}(H_{n-1}-H_{j-1})=\sum_{j=1}^{n-1}\frac{(-x)^j}{n}\binom{n}{j}(H_{n-1}-H_{j-1})\\ &=\frac{H_{n-1}}{n}\sum_{j=1}^{n-1}\binom{n}{j}(-x)^j-\sum_{j=1}^{n-1}\frac{(-x)^j}{n}\binom{n}{j}H_{j-1}\\ &=\frac{(1-x)^n-(-x)^n-1}{n}H_{n-1}-\frac{1}{n}\sum_{j=1}^{n-1}\binom{n}{j}(-x)^jH_{j-1},\end{aligned}$$

as claimed. □

Lemma 2 *For $n>1$ and $x\in\mathbb{R}$,*

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(-x)^{k+1}}{k+1}H_k$$
$$=-\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(-x)^{k+1}}{(k+1)^2}-(1-x)^n\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{x^{k+1}}{(k+1)^2(1-x)^{k+1}}.$$

Proof By Lemma 1, we have

$$\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{(-x)^k}{k}H_{k-1}$$
$$=\frac{(1-x)^n-(-x)^n-1}{n}H_{n-1}-\frac{1}{n}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}-\frac{1}{n}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{n-k}$$
$$=\frac{(1-x)^n-(-x)^n-1}{n}H_{n-1}-\frac{1}{n}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}$$
$$-\frac{(1-x)^n}{n}\sum_{k=1}^{n-1}\frac{(1-x)^{-k}-1}{k}+\frac{1-(1-x)^n}{n}H_{n-1}$$
$$=-\frac{(-x)^n}{n}H_{n-1}-\frac{1}{n}\sum_{k=1}^{n-1}\frac{(1-x)^k-1}{k}-\frac{(1-x)^n}{n}\sum_{k=1}^{n-1}\frac{(1-x)^{-k}-1}{k},$$

which, by the result

$$\sum_{k=1}^{n-1}\frac{x^k-1}{k}=n\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{(x-1)^k}{k^2}-\frac{x^n-(x-1)^n-1}{n}$$

given in [5], equals

$$-\frac{(-x)^n}{n}H_{n-1}-\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{(-x)^k}{k^2}+\frac{(1-x)^n-(-x)^n-1}{n^2}$$
$$-(1-x)^n\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\left(\frac{x}{1-x}\right)^k}{k^2}+\frac{(1-x)^n}{n^2}\left((1-x)^{-n}-x^n(1-x)^{-n}-1\right)$$
$$=-\frac{(-x)^n}{n}H_{n-1}-\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{(-x)^k}{k^2}-\frac{(-x)^n}{n^2}-(1-x)^n\sum_{k=1}^{n-1}\binom{n-1}{k-1}\frac{\left(\frac{x}{1-x}\right)^k}{k^2}-\frac{x^n}{n^2}$$
$$=-\frac{(-x)^n}{n}H_{n-1}-\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{(-x)^k}{k^2}-(1-x)^n\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{\left(\frac{x}{1-x}\right)^k}{k^2}$$
$$=-\frac{(-x)^n}{n}H_{n-1}-\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(-x)^{k+1}}{(k+1)^2}-(1-x)^n\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{x^{k+1}}{(k+1)^2(1-x)^{k+1}},$$

as claimed. □

Lemma 3 *Let* p *be an odd prime. For* $x \in \mathbb{Z}_p$ *,*

$$\sum_{k=1}^{(p-1)/2} \frac{x^k}{k} \equiv \frac{2}{p} - \frac{(\sqrt{x}+1)^p - (\sqrt{x}-1)^p}{p} (\text{mod } p).$$

Proof Observe that

$$\sum_{k=1}^{(p-1)/2} \frac{x^k}{k} = \sum_{k=1}^{p-1} \left(1+(-1)^k\right) \frac{(\sqrt{x})^k}{k} = \sum_{k=1}^{p-1} \frac{(\sqrt{x})^k}{k} + \sum_{k=1}^{p-1} \frac{(-\sqrt{x})^k}{k}.$$

By (1.1), we can write

$$\begin{aligned} \sum_{k=1}^{(p-1)/2} \frac{x^k}{k} &\equiv -\frac{(\sqrt{x})^p - (\sqrt{x}-1)^p - 1}{p} - \frac{(-\sqrt{x})^p - (-\sqrt{x}-1)^p - 1}{p} \\ &= \frac{2}{p} - \frac{(\sqrt{x}+1)^p - (\sqrt{x}-1)\,,^p}{p} (\text{mod } p) \end{aligned}$$

which is as desired. □

Lemma 4 *Let* $p > 3$ *be a prime. For* $x \in \mathbb{Z}_p$ *,*

$$\sum_{k=1}^{(p-3)/2} \frac{x^k}{2k+1} \equiv \frac{1}{2p\sqrt{x}} \left((\sqrt{x}+1)^p + (\sqrt{x}-1)^p - 2\left(\sqrt{x}\right)^p \right) - 1 (\text{mod } p).$$

Proof Observe that

$$\sum_{k=1}^{(p-3)/2} \frac{x^k}{2k+1} = \frac{1}{\sqrt{x}} \left(\sum_{k=1}^{p-1} \frac{(\sqrt{x})^k}{k} - \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{x^k}{k} \right) - 1.$$

With the help of Lemma 3 and (1.1), we write

$$\begin{aligned} \sum_{k=1}^{(p-3)/2} \frac{x^k}{2k+1} &\equiv \frac{1}{\sqrt{x}} \left(\frac{1 - (\sqrt{x})^p + (\sqrt{x}-1)^p}{p} - \frac{1}{p} + \frac{(\sqrt{x}+1)^p - (\sqrt{x}-1)^p}{2p} \right) - 1 \\ &= \frac{1}{2p\sqrt{x}} \left((\sqrt{x}+1)^p + (\sqrt{x}-1)^p - 2\left(\sqrt{x}\right)^p \right) - 1 (\text{mod } p). \end{aligned}$$

Thus, we have the proof of Lemma 4. □

Now we can state our first main theorem.

Theorem 1 *Let* p *be an odd prime. For* $x \in \mathbb{Z}_p$ *,*

$$\begin{aligned} \sum_{k=1}^{(p-1)/2} \binom{2k}{k} H_{k-1} x^k &\equiv \frac{1}{p\sqrt{1-4x}} \left(\left(\sqrt{1-4x}+1\right)^{p+1} - \left(\sqrt{1-4x}-1\right)^{p+1} \right) \\ &\quad - \frac{2^p}{p} \left(\left(\sqrt{1-4x}\right)^{p-1} + 1 \right) (\text{mod} p). \end{aligned}$$

Proof By Lemma 1, we write

$$\frac{2}{p-1}\sum_{k=1}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}(-4x)^k H_{k-1} = 2\frac{(1-4x)^{(p-1)/2}-1}{p-1}H_{(p-3)/2} - 2\sum_{k=1}^{(p-3)/2}\frac{(1-4x)^k-1}{k(p-2k-1)}.$$

With the help of the congruences $\frac{1}{p-k}\equiv-\frac{1}{k}\pmod p$ and $\frac{1}{k(p-1-2k)}\equiv-\frac{1}{k(2k+1)}\pmod p$ for $k=1,2,...,(p-3)/2$, we have

$$\sum_{k=1}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}(-4x)^k H_{k-1}\equiv\left((1-4x)^{(p-1)/2}-1\right)H_{(p-3)/2}-\sum_{k=1}^{(p-3)/2}\frac{(1-4x)^k-1}{k(2k+1)}\pmod p.$$

Note that

$$\binom{2k}{k}=\binom{-1/2}{k}(-4)^k\equiv\binom{(p-1)/2}{k}(-4)^k\pmod p,$$

and

$$H_{(p-3)/2}\equiv 2-2q_p(2)\pmod p$$

by [3]. Thus, we get

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}\binom{2k}{k}x^k H_{k-1} \qquad (2.1)\\ &\equiv 2\left((1-4x)^{(p-1)/2}-1\right)(1-q_p(2))-\sum_{k=1}^{(p-3)/2}\frac{(1-4x)^k-1}{k(2k+1)}\\ &=2\left((1-4x)^{(p-1)/2}-1\right)(1-q_p(2))-\sum_{k=1}^{(p-3)/2}\frac{(1-4x)^k-1}{k}\\ &+2\sum_{k=1}^{(p-3)/2}\frac{(1-4x)^k-1}{2k+1}\pmod p.\end{aligned}$$

From Lemma 4, it is clearly known that

$$\begin{aligned}\sum_{k=1}^{(p-3)/2}\frac{(1-4x)^k-1}{2k+1} &\equiv \frac{1}{2p\sqrt{1-4x}}\\ &\times\left(\left(\sqrt{1-4x}+1\right)^p+\left(\sqrt{1-4x}-1\right)^p-2\left(\sqrt{1-4x}\right)^p\right)-q_p(2)\pmod p.\end{aligned}$$

In this way, substituting this congruence in (2.1), we find

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2}\binom{2k}{k}x^kH_{k-1}\\
\equiv\ & 2\left((1-4x)^{(p-1)/2}-1\right)(1-q_p(2))-\sum_{k=1}^{(p-3)/2}\frac{(1-4x)^k-1}{k}\\
&+\frac{1}{p\sqrt{1-4x}}\left(\left(\sqrt{1-4x}+1\right)^p+\left(\sqrt{1-4x}-1\right)^p-2\left(\sqrt{1-4x}\right)^p\right)\\
&-2q_p(2)\,(\mathrm{mod}\ p).
\end{aligned}$$

By Lemma 3 and $H_{(p-3)/2}\equiv 2-2q_p(2)\,(\mathrm{mod}\ p)$, we have

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2}\binom{2k}{k}x^kH_{k-1}\equiv 2(1-q_p(2))\left((1-4x)^{(p-1)/2}-1\right)\\
&\quad-\frac{1}{p}\left(\left(\sqrt{1-4x}-1\right)^p-\left(\sqrt{1-4x}+1\right)^p+2^p\right)-2\left((1-4x)^{(p-1)/2}-1\right)\\
&\quad+\frac{1}{p\sqrt{1-4x}}\left(\left(\sqrt{1-4x}+1\right)^p+\left(\sqrt{1-4x}-1\right)^p-2\left(\sqrt{1-4x}\right)^p\right)-2q_p(2)\\
&= -2q_p(2)(1-4x)^{(p-1)/2}-\frac{2}{p}(1-4x)^{(p-1)/2}-\frac{2^p}{p}\\
&\qquad+\frac{1}{p\sqrt{1-4x}}\left(\left(\sqrt{1-4x}+1\right)^{p+1}-\left(\sqrt{1-4x}-1\right)^{p+1}\right)\\
&= \frac{1}{p\sqrt{1-4x}}\left(\left(\sqrt{1-4x}+1\right)^{p+1}-\left(\sqrt{1-4x}-1\right)^{p+1}\right)\\
&\qquad-\frac{2^p}{p}\left(\left(\sqrt{1-4x}\right)^{p-1}+1\right)(\mathrm{mod}\ p),
\end{aligned}$$

as claimed. □

Now we present two consequences of Theorem 1.

Corollary 1 *Let p be an odd prime. Then*

$$\sum_{k=1}^{(p-1)/2}(-1)^k\binom{2k}{k}H_{k-1}\equiv\frac{2^p}{p}\left(2F_{p+1}-5^{(p-1)/2}-1\right)(\mathrm{mod}\ p), \tag{2.2}$$

$$\begin{aligned}
\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{H_{k-1}}{(-4)^k}&\equiv 2\frac{P_{p+1}}{p}-\frac{2^p}{p}\left(1+2^{(p-1)/2}\right)(\mathrm{mod}\ p),\\
\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{H_{k-1}}{8^k}&\equiv\frac{1}{p}\left(\frac{P_{p+1}}{2^{(p-3)/2}}-2^p-2^{(p+1)/2}\right)(\mathrm{mod}\ p),
\end{aligned} \tag{2.3}$$

$$\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{H_{k-1}}{9^k} \equiv \frac{2}{p}\left(\frac{2}{3}\right)^{p-1}\left(\frac{2}{3}F_{2p+2}-3^{p-1}-5^{(p-1)/2}\right) (\mathrm{mod}\ p),$$
$$\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{H_{k-1}}{(-16)^k} \equiv \frac{F_{3p+3}}{p2^p}-\frac{2^p}{p}-\frac{2}{p}5^{(p-1)/2}(\mathrm{mod}\ p).$$

Proof For the proof of (2.2), taking $x=-1$ in (2.1), we write

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}(-1)^k\binom{2k}{k}H_{k-1} &\equiv \frac{1}{p\sqrt{5}}\left(\left(1+\sqrt{5}\right)^{p+1}-\left(1-\sqrt{5}\right)^{p+1}\right)-\frac{2^p}{p}\left(5^{(p-1)/2}+1\right)\\ &= \frac{2^{p+1}}{p}\left(\frac{\alpha^{p+1}-\beta^{p+1}}{\alpha-\beta}\right)-\frac{2^p}{p}\left(5^{(p-1)/2}+1\right)(\mathrm{mod}\ p).\end{aligned}$$

Using the Binet formula of $\{F_n\}$, the desired result is clearly obtained. Similarly, the other results are given.
□

Corollary 2 *Let p be an odd prime. For $\left(\frac{5}{p}\right)=1$,*

$$\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{H_{k-1}F_k}{(-4)^k} \equiv \frac{1}{p}\left(F_{2p+1}-F_{p+2}\right)-\frac{2^p}{p}F_{p-1}(\mathrm{mod}\ p),$$
$$\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{H_{k-1}L_k}{(-4)^k} \equiv \frac{1}{p}\left(L_{2p+1}+L_{p+2}\right)-\frac{2^p}{p}\left(L_{p-1}+2\right)(\mathrm{mod}\ p),$$

and for $\left(\frac{2}{p}\right)=1$,

$$\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{H_{k-1}P_k}{(-2)^k} \equiv \frac{2^{(p+1)/2}}{p}\left(P_p-1\right)-\frac{2^p}{p}P_{p-1}(\mathrm{mod}\ p), \quad (2.4)$$
$$\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{H_{k-1}Q_k}{(-2)^k} \equiv \frac{2^{(p+1)/2}}{p}\left(Q_p+2\right)-\frac{2^p}{p}\left(Q_{p-1}+2\right)(\mathrm{mod}\ p).$$

Proof We will give the proof of (2.4) as a showcase and leave the others to the readers. Taking $x=-\frac{\gamma}{2}$ in (2.1) and since $\gamma^2=2\gamma+1$, we get

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\binom{2k}{k}H_{k-1}\left(-\frac{\gamma}{2}\right)^k &\equiv \frac{1}{p\sqrt{1+2\gamma}}\left(\left(\sqrt{1+2\gamma}+1\right)^{p+1}-\left(\sqrt{1+2\gamma}-1\right)^{p+1}\right)\\ &\quad -\frac{2^p}{p}\left(\left(\sqrt{1+2\gamma}\right)^{p-1}+1\right)\\ &= \frac{1}{p\gamma}\left((\gamma+1)^{p+1}-(\gamma-1)^{p+1}\right)-\frac{2^p}{p}\left(\gamma^{p-1}+1\right)(\mathrm{mod}\ p).\end{aligned}$$

From the Pell sequence $\{P_n\}$, it is known that $\gamma - 1 = \sqrt{2}$ and $\gamma\sqrt{2} = \gamma + 1$. Thus,

$$\begin{aligned}\sum_{k=1}^{(p-1)/2} \binom{2k}{k} H_{k-1} \left(-\frac{\gamma}{2}\right)^k &\equiv \frac{1}{p\gamma}\left(\left(\gamma\sqrt{2}\right)^{p+1} - \left(\sqrt{2}\right)^{p+1}\right) - \frac{2^p}{p}\left(\gamma^{p-1}+1\right) \\ &= \frac{\left(\sqrt{2}\right)^{p+1}}{p}\left(\gamma^p + \delta\right) - \frac{2^p}{p}\left(\gamma^{p-1}+1\right) \pmod{p}. \qquad (2.5)\end{aligned}$$

Similarly, putting $x = -\frac{\delta}{2}$ in (2.1), and using $\delta - 1 = -\sqrt{2}$, $\delta^2 = 1 + 2\delta$, and $\delta\sqrt{2} = -\delta - 1$, we write

$$\sum_{k=1}^{(p-1)/2} \binom{2k}{k} H_{k-1} \left(-\frac{\delta}{2}\right)^k \equiv \frac{\left(\sqrt{2}\right)^{p+1}}{p}\left(\delta^p + \gamma\right) - \frac{2^p}{p}\left(\delta^{p-1}+1\right) \pmod{p}. \qquad (2.6)$$

From (2.5) and (2.6), the proof is completed. □

We present our second theorem.

Theorem 2 *Let p be an odd prime. For $x \in \mathbb{Z}_p$,*

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2} H_k C_k x^{k+1} \qquad (2.7)\\ &\equiv \frac{2^p\left((1-4x)^{(p+1)/2}+1\right) - \left(\sqrt{1-4x}+1\right)^{p+1} - \left(\sqrt{1-4x}-1\right)^{p+1}}{2p} \pmod{p}.\end{aligned}$$

Proof By Lemma 2, we have

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2} \binom{\frac{p-1}{2}}{k} \frac{(-4x)^{k+1}}{k+1} H_k \\ &= -\sum_{k=0}^{(p-1)/2} \binom{\frac{p-1}{2}}{k} \frac{(-4x)^{k+1}}{(k+1)^2} - (1-4x)^{(p+1)/2} \sum_{k=0}^{(p-1)/2} \binom{\frac{p-1}{2}}{k} \frac{(4x)^{k+1}}{(k+1)^2(1-4x)^{k+1}}.\end{aligned}$$

From $\binom{(p-1)/2}{k}(-4)^k \equiv \binom{2k}{k} \pmod{p}$, we write

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2} \binom{2k}{k} \frac{x^{k+1}}{k+1} H_k \\ &\equiv -\sum_{k=0}^{(p-1)/2} \binom{2k}{k} \frac{x^{k+1}}{(k+1)^2} - (1-4x)^{(p+1)/2} \sum_{k=0}^{(p-1)/2} \binom{2k}{k} \frac{(-x)^{k+1}}{(k+1)^2(1-4x)^{k+1}} \pmod{p},\end{aligned}$$

and then

$$\sum_{k=0}^{(p-1)/2} H_k C_k x^{k+1} \equiv -\sum_{k=0}^{(p-1)/2} \frac{C_k}{k+1} x^{k+1} - (1-4x)^{(p+1)/2} \sum_{k=0}^{(p-1)/2} \frac{C_k}{k+1} \left(\frac{x}{4x-1}\right)^{k+1} \pmod{p}.$$

Replacing x by $\frac{x}{4x-1}$ in

$$\sum_{k=0}^{(p-1)/2} \frac{C_k}{k+1} x^{k+1} \equiv \frac{\left(\sqrt{1-4x}+1\right)^p - \left(\sqrt{1-4x}-1\right)^p - 2^p}{2p} - (1-4x)^{(p+1)/2} + 1 (\text{mod } p)$$

given in [2], we have

$$\begin{aligned}
\sum_{k=0}^{(p-1)/2} H_k C_k x^{k+1} &\equiv -(1-4x)^{(p+1)/2} \\
&\times \left(\frac{\left(\sqrt{1-\frac{4x}{4x-1}}+1\right)^p - \left(\sqrt{1-\frac{4x}{4x-1}}-1\right)^p - 2^p}{2p} - \left(1-\frac{4x}{4x-1}\right)^{(p+1)/2} + 1 \right) \\
&- \frac{\left(\sqrt{1-4x}+1\right)^p - \left(\sqrt{1-4x}-1\right)^p - 2^p}{2p} + (1-4x)^{(p+1)/2} - 1 \\
&= \frac{\left(\sqrt{1-4x}-1\right)^p - \left(\sqrt{1-4x}+1\right)^p + 2^p}{2p} + (1-4x)^{(p+1)/2} - 1 \\
&- \frac{(1-4x)^{(p+1)/2}}{(1-4x)^{p/2}} \frac{\left(\sqrt{1-4x}+1\right)^p - \left(1-\sqrt{1-4x}\right)^p - 2^p \left(\sqrt{1-4x}\right)^p}{2p} + 1 - (1-4x)^{(p+1)/2} \\
&= \frac{\left(\sqrt{1-4x}-1\right)^p - \left(\sqrt{1-4x}+1\right)^p + 2^p}{2p} \\
&- \sqrt{1-4x} \frac{\left(\sqrt{1-4x}+1\right)^p - \left(1-\sqrt{1-4x}\right)^p}{2p} + \frac{2^{p-1}}{p} (1-4x)^{(p+1)/2} \\
&= \frac{2^p \left((1-4x)^{(p+1)/2} + 1\right) - \left(\sqrt{1-4x}+1\right)^{p+1} - \left(\sqrt{1-4x}-1\right)^{p+1}}{2p} (\text{mod } p),
\end{aligned}$$

as desired. □

Now we give some applications of the above results.

Corollary 3 *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{H_k C_k}{4^k} \equiv 4 q_p(2) (\text{mod } p),$$

$$\sum_{k=0}^{(p-1)/2} \frac{H_k C_k}{8^k} \equiv \frac{1}{p} \left(2^{p+2} + 2^{(p+3)/2} - \frac{Q_{p+1}}{2^{(p-3)/2}} \right) (\text{mod } p),$$

$$\sum_{k=0}^{(p-1)/2} \frac{H_k C_k}{9^k} \equiv \frac{1}{p} \left(\frac{2}{3}\right)^{p-1} \left(3^{p+1} + 5^{(p+1)/2} - 2 L_{2p+2} \right) (\text{mod } p),$$

$$\sum_{k=0}^{(p-1)/2} \frac{H_k C_k}{(-4)^k} \equiv 2 \frac{Q_{p+1}}{p} - \frac{2^{p+1}}{p} \left(1 + 2^{(p+1)/2} \right) (\text{mod } p),$$

$$\sum_{k=0}^{(p-1)/2} \frac{H_k C_k}{(-16)^k} \equiv \frac{4}{p}\left(\frac{L_{3p+3}}{2^p} - 2^{p+1} - 5^{(p+1)/2}\right) (\text{mod } p),$$

$$\sum_{k=0}^{(p-1)/2} (-1)^k H_k C_k \equiv \frac{2^{p-1}}{p}\left(2L_{p+1} - 5^{(p+1)/2} - 1\right) (\text{mod } p).$$

Proof Using the Binet formulae of $\{L_n\}$ and $\{Q_n\}$, the results are obtained from (2.7). □

Corollary 4 *Let p be an odd prime. For $\left(\frac{5}{p}\right) = 1$,*

$$\sum_{k=0}^{(p-1)/2} \frac{H_k C_k F_{k+1}}{(-4)^k} \equiv \frac{2}{p}\left(F_{2p+2} - (2^p + 1) F_{p+1}\right) (\text{mod } p), \tag{2.8}$$

$$\sum_{k=0}^{(p-1)/2} \frac{H_k C_k L_{k+1}}{(-4)^k} \equiv \frac{2}{p}\left(L_{2p+2} + (1 - 2^p) L_{p+1} - 2^{p+1}\right) (\text{mod } p),$$

and for $\left(\frac{2}{p}\right) = 1$,

$$\sum_{k=0}^{(p-1)/2} \frac{H_k C_k P_{k+1}}{(-2)^k} \equiv \frac{2^{(p+1)/2} - 2^p}{p} P_{p+1} (\text{mod } p),$$

$$\sum_{k=0}^{(p-1)/2} \frac{H_k C_k Q_{k+1}}{(-2)^k} \equiv \frac{2^{(p+1)/2} - 2^p}{p} (Q_{p+1} + 2) (\text{mod } p).$$

Proof We give only a proof for the congruence (2.8) as a showcase. The others could be similarly proved. By taking $x = -\frac{\alpha}{4}$ in (2.7), since $\alpha^2 = \alpha + 1$ and $\alpha + \beta = 1$, we write

$$\begin{aligned}
&\sum_{k=0}^{(p-1)/2} H_k C_k \left(-\frac{\alpha}{4}\right)^{k+1} \\
&\equiv \frac{2^p\left(1 + \left(\sqrt{1+\alpha}\right)^{p+1}\right) - \left(\sqrt{1+\alpha} + 1\right)^{p+1} - \left(\sqrt{1+\alpha} - 1\right)^{p+1}}{2p} \\
&= \frac{2^p\left(1 + \alpha^{p+1}\right) - \alpha^{2(p+1)} - \beta^{p+1}}{2p} (\text{mod } p).
\end{aligned} \tag{2.9}$$

Similarly, taking $x = -\frac{\beta}{4}$ in (2.7), since $\beta^2 = \beta + 1$ and $\alpha + \beta = 1$, we write

$$\sum_{k=0}^{(p-1)/2} H_k C_k \left(-\frac{\beta}{4}\right)^{k+1} \equiv \frac{2^p\left(1 + \beta^{p+1}\right) - \beta^{2(p+1)} - \alpha^{p+1}}{2p} (\text{mod } p). \tag{2.10}$$

From (2.9) and (2.10), the claim is seen. □

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