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Factorizations related to the reciprocal Pascal matrix

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Abstract: The reciprocal Pascal matrix has entries $\binom{i+j}{j}^{-1}$. Explicit formulæ for its LU-decomposition, the LU-decomposition of its inverse, and some related matrices are obtained. For all results, q-analogues are also presented.

Key words: Pascal matrix, LU-decomposition, q-analogue, Zeilberger's algorithm

1. Introduction

Recently, there has been some interest in the *reciprocal Pascal matrix* M, defined by

$$M_{i,j} = \binom{i+j}{j}^{-1};$$

the indices start here for convenience with $0, 0$, and the matrix is either infinite or has N rows and columns, depending on the context.

Richardson [7] has provided the decomposition $S = GMG$, where the diagonal matrix G has entries $G_{i,i} = \binom{2i}{i}$, and S is the *super Catalan matrix* [2, 4] with entries

$$S_{i,j} = \frac{(2i)!(2j)!}{i!j!(i+j)!}.$$

We want to give an alternative decomposition of M, provided by the LU-decomposition. We will give explicit expressions for L and U, defined by $LU = M$, as well as for L^{-1} and U^{-1}.

Since there is also interest in M^{-1}, in particular in the integrality of its coefficients, we also provide the LU-decomposition $AB = M^{-1}$, and give expressions for A, B, A^{-1}, and B^{-1}.

In the following section, we provide q-analogues of these results.

The paper closes with a list of similar results with two additional parameters, but for the matrix with entries $\binom{i+r+j+s}{j+s}^{-1}$ and $\binom{i+r+j+s}{j+s}$.

We would like to mention that results of the type as presented here are useful to find and prove new expansion formulæ for "Fibonomial sums"; see, for instance, [5].

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2. Identities

The LU-decomposition $M = LU$ is given by

$$L_{i,j} = \frac{i!i!(2j)!}{(i+j)!(i-j)!j!j!}$$

and

$$U_{i,j} = \frac{(-1)^i j!j!i!(i-1)!}{(j+i)!(j-i)!(2i-1)!} \qquad \text{for } i \geq 1.$$

For $i = 0$, the formula is $U_{0,j} = 1$.

The formula that needs to be proved is

$$\sum_{0 \leq k \leq \min\{i,j\}} L_{i,k} U_{k,j} = \binom{i+j}{j}^{-1},$$

which is equivalent to

$$1 + \frac{2i!i!j!j!}{(2i)!(2j)!} \sum_{1 \leq k \leq \min\{i,j\}} (-1)^k \binom{2i}{i+k}\binom{2j}{j+k} = \binom{i+j}{j}^{-1}.$$

The von Szily identity [2, 3, 8] is

$$\frac{(2i)!(2j)!}{i!j!(i+j)!} = \sum_{k \in \mathbb{Z}} (-1)^k \binom{2i}{i+k}\binom{2j}{j+k},$$

and an equivalent form is, by symmetry,

$$\frac{(2i)!(2j)!}{i!j!(i+j)!} = \binom{2i}{i}\binom{2j}{j} + 2\sum_{k \geq 1} (-1)^k \binom{2i}{i+k}\binom{2j}{j+k}.$$

Thus, the identity to be proven is now

$$\binom{i+j}{j} + \frac{i!j!(i+j)!}{(2i)!(2j)!}\left[\frac{(2i)!(2j)!}{i!j!(i+j)!} - \binom{2i}{i}\binom{2j}{j}\right] = 1,$$

which is obviously correct.

The formula for L^{-1} is for $i \geq j \geq 0$:

$$L^{-1}_{i,j} = \frac{(-1)^{i-j} i!i!(i+j-1)!}{(2i-1)!(i-j)!j!j!}.$$

If necessary ($i = j = 0$), this must be interpreted as a limit.

To check this, we consider

$$\sum_k \frac{i!i!(2k)!}{(i+k)!(i-k)!k!k!} \frac{(-1)^{k-j} k!k!(k+j-1)!}{(2k-1)!(k-j)!j!j!}$$
$$= \frac{2i!i!(-1)^j}{j!j!} \sum_{j \leq k \leq i} \frac{k}{(i+k)!(i-k)!} \frac{(-1)^k (k+j-1)!}{(k-j)!}.$$

The sum can be evaluated by computer algebra (or otherwise), and the result is indeed $[\![i=j]\!]$, as desired.

The formula for U^{-1} is for $j \ge i \ge 1$

$$U^{-1}_{i,j} = \frac{(-1)^i (j+i-1)!(2j)!}{(j-i)!j!(j-1)!i!i!}$$

and for $i=0$

$$U^{-1}_{0,j} = \frac{(2j)!}{j!j!}.$$

The fact that $\sum_k U_{i,k}U^{-1}_{k,j} = [\![i=j]\!]$ can also be done by computer algebra. Since there are a few cases to be distinguished, it is omitted here.

The LU-decomposition $AB = M^{-1}$ depends on the dimension N and is given by

$$A_{i,j} = \frac{(-1)^{i-j}(N-j-1)!j!(N+i-1)!}{i!(N-i-1)!(N+j-1)!(i-j)!},$$

$$B_{i,j} = \frac{(-1)^{j+N-1}(N+j-1)!}{j!(j-i)!(N-j-1)!i!}.$$

Since M^{-1} does not have "nice" entries, we rather provide formulæ for A^{-1} and B^{-1} and prove the identity $B^{-1}A^{-1} = M$ instead. The results are:

$$A^{-1}_{i,j} = \frac{(N-j-1)!j!(N+i-1)!}{i!(N-i-1)!(N+j-1)!(i-j)!},$$

$$B^{-1}_{i,j} = \frac{(-1)^{j+N-1}(N-1-i)!j!i!}{(j-i)!(N+i-1)!}.$$

First we prove that these are indeed the inverses. We consider

$$\begin{aligned}
&\sum_k \frac{(-1)^{i-k}(N-k-1)!k!(N+i-1)!}{i!(N-i-1)!(N+k-1)!(i-k)!}\frac{(N-j-1)!j!(N+k-1)!}{k!(N-k-1)!(N+j-1)!(k-j)!}\\
&\quad= (-1)^i \frac{(N+i-1)!(N-j-1)!j!}{(N-i-1)!(N+j-1)!i!}\sum_{j\le k\le i}\frac{(-1)^k}{(i-k)!(k-j)!}\\
&\quad= \frac{(N+i-1)!(N-j-1)!j!}{(N-i-1)!(N+j-1)!i!(i-j)!}\sum_{j\le k\le i}(-1)^{i-k}\binom{i-j}{i-k}\\
&\quad= \frac{(N+i-1)!(N-j-1)!j!}{(N-i-1)!(N+j-1)!i!(i-j)!}[\![i=j]\!] = [\![i=j]\!],
\end{aligned}$$

which proves $AA^{-1}=I$. Similarly

$$\sum_k \frac{(-1)^{k+N-1}(N-1-i)!k!i!}{(k-i)!(N+i-1)!}\frac{(-1)^{j+N-1}(N+j-1)!}{j!(j-k)!(N-j-1)!k!}$$
$$= (-1)^j \frac{(N-1-i)!i!(N+j-1)!}{(N+i-1)!j!(N-j-1)!}\sum_k \frac{(-1)^k}{(k-i)!(j-k)!}$$
$$= \frac{(N-1-i)!i!(N+j-1)!}{(N+i-1)!j!(N-j-1)!(j-i)!}\sum_k (-1)^{j-k}\binom{j-i}{j-k}$$
$$= \frac{(N-1-i)!i!(N+j-1)!}{(N+i-1)!j!(N-j-1)!(j-i)!}[\![i=j]\!] = [\![i=j]\!],$$

which proves $B^{-1}B=I$.

Now we compute an entry in $B^{-1}A^{-1}$:

$$\sum_k \frac{(-1)^{k+N-1}(N-1-i)!k!i!}{(k-i)!(N+i-1)!}\frac{(N-j-1)!j!(N+k-1)!}{k!(N-k-1)!(N+j-1)!(k-j)!}$$
$$= (-1)^{N-1}\frac{(N-1-i)!i!(N-j-1)!j!}{(N+i-1)!(N+j-1)!}\sum_k \frac{(-1)^k(N+k-1)!}{(k-i)!(N-k-1)!(k-j)!}$$
$$= (-1)^{N-1}\frac{i!j!(N-j-1)!}{(N+i-1)!}\sum_k (-1)^k\binom{N-1-i}{N-1-k}\binom{N+k-1}{N-1+j}$$
$$= \frac{i!j!(N-j-1)!}{(N+i-1)!}\sum_k \binom{i-1-k}{N-1-k}\binom{N+k-1}{N-1+j}$$
$$= \frac{i!j!(N-j-1)!}{(N+i-1)!}\sum_k \binom{i-1-k}{i-N}\binom{N+k-1}{N-1+j}$$
$$= \frac{i!j!(N-j-1)!}{(N+i-1)!}\binom{i-1+N}{i+j}$$
$$= \frac{i!j!}{(i+j)!} = M_{i,j},$$

as claimed.

Now we use the form $M^{-1}=AB$ and write the (i,j) entry:

$$\sum_k \frac{(N-k-1)!k!(N+i-1)!}{i!(N-i-1)!(N+k-1)!(i-k)!}\frac{(-1)^{j+N-1}(N+j-1)!}{j!(j-k)!(N-j-1)!k!}$$
$$= \frac{(N+i-1)!(N+j-1)!}{i!(N-i-1)!j!(N-j-1)!}\sum_k \frac{(N-k-1)!}{(N+k-1)!(i-k)!}\frac{(-1)^{j+N-1}}{(j-k)!}$$
$$= \binom{N-1}{i}\binom{N+j-1}{j}\sum_{0\le k\le \min\{i,j\}} (-1)^{j+N-1}\binom{N+i-1}{i-k}\binom{N-k-1}{j-k}.$$

From this representation, it is clear that this is an integer. This was a question that was addressed in the affirmative in [7].

3. q-analogues

In this section we present q-analogues. Define $(q)_n := (1-q)(1-q^2)\dots(1-q^n)$, and

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{(q)_n}{(q)_k (q)_{n-k}};$$

these definitions are standard, see [1]. Then we have the following results for the matrix M with entries $\begin{bmatrix} i+j \\ j \end{bmatrix}^{-1}$.

$$L_{i,j} = \frac{(q)_i (q)_i (q)_{2j}}{(q)_{i+j}(q)_{i-j}(q)_j (q)_j},$$

$$U_{i,j} = \frac{(-1)^i q^{i(3i-1)/2}(1+q^i)(q)_j (q)_j (q)_i (q)_i}{(q)_{i+j}(q)_{j-i}(q)_{2i}} \quad \text{for } i \ge 1, \quad U_{0,j} = 1,$$

$$L^{-1}_{i,j} = \frac{q^{i(i-1)/2+j(j+1)/2-ij}(-1)^{i-j}(q)_i (q)_i (q)_{i+j-1}}{(q)_{2i-1}(q)_{i-j}} \quad \text{for } j < i, \quad L^{-1}_{i,i} = 1,$$

$$U^{-1}_{i,j} = \frac{(-1)^i q^{-j^2-ji+i(i+1)/2}(q)_{j+i-1}(q)_{2j}(q)_i (q)_i}{(q)_{j-i}(q)_j (q)_{j-1}} \quad \text{for } j > i,$$

$$U^{-1}_{i,i} = \frac{(-1)^i q^{i(3i+1)/2}(q)_{2i}(q)_{2i}}{(q)_i (q)_i (q)_i (q)_i (1+q^i)} \quad \text{for } i \ge 1, \quad U^{-1}_{0,0} = 1,$$

$$A_{i,j} = \frac{(-1)^{i-j} q^{(i+j+3)(i-j)/2+N(j-i)}(q)_{N-j-1}(q)_j (q)_{N+i-1}}{(q)_{N-i-1}(q)_i (q)_{N+j-1}(q)_{i-j}},$$

$$B_{i,j} = \frac{(-1)^{j+N-1} q^{i^2+j(j+3)/2-Nj-N(N-1)/2}(q)_{N+j-1}}{(q)_j (q)_{j-i}(q)_{N-j-1}(q)_i},$$

$$A^{-1}_{i,j} = \frac{q^{(i-j)(i-N+1)}(q)_{N-j-1}(q)_{N+i-1}(q)_j}{(q)_{N-i-1}(q)_{N+j-1}(q)_i (q)_{i-j}},$$

$$B^{-1}_{i,j} = \frac{(-1)^{j+1+N} q^{-j(j+1)/2-ij+N(N-1)/2+(N-1)i}(q)_{N-1-i}(q)_j (q)_i}{(q)_{j-i}(q)_{N+i-1}}.$$

Note that for $q \to 1$, we get the previous formulæ. We do not display all the proofs here, since Zeilberger's algorithm (aka WZ-theory) [6] proves all these results (which were obtained by guessing), using a computer algebra system (such as, e.g., Maple). However, as suggested by a referee, in the next section, we provide how a typical proof is obtained with a computer.

Remark. Richardson's decomposition $S = GMG$ still holds when all binomial coefficients are replaced by the corresponding Gaussian q-binomial coefficients.

4. A sample proof

We deal here with

$$\sum_{j\le k\le i} A_{i,k}A^{-1}_{k,j} = \sum_{j\le k\le i} \frac{(-1)^{i-k}q^{(i+k+3)(i-k)/2+N(k-i)}(q)_{N-k-1}(q)_k(q)_{N+i-1}}{(q)_{N-i-1}(q)_i(q)_{N+k-1}(q)_{i-k}}$$
$$\times \frac{q^{(k-j)(k-N+1)}(q)_{N-j-1}(q)_{N+k-1}(q)_j}{(q)_{N-k-1}(q)_{N+j-1}(q)_k(q)_{k-j}}$$
$$= \frac{q^{i(i+3)/2-j+N(j-i)}(q)_{N+i-1}(q)_{N-j-1}(q)_j}{(q)_{N-i-1}(q)_i(q)_{N+j-1}(q)_{i-j}} \sum_{j\le k\le i} \frac{(-1)^{i-k}q^{k(k-1)/2-kj}(q)_{i-j}}{(q)_{i-k}(q)_{k-j}}.$$

Now Zeilberger's algorithm provides the formula

$$\frac{(-1)^{i-k}q^{k(k-1)/2-kj}(q)_{i-j}}{(q)_{i-k}(q)_{k-j}} = \frac{(-1)^{i-k}q^{k(k+1)/2-j-kj}(q)_{i-j-1}}{(q)_{i-1-k}} - \frac{(-1)^{i-(k-1)}q^{k(k-1)/2-j-(k-1)j}(q)_{i-j-1}}{(q)_{i-k}},$$

so the sum over k is telescoping, with the result

$$\sum_{j\le k\le \ell} \frac{(-1)^{i-k}q^{k(k-1)/2-kj}(q)_{i-j}}{(q)_{i-k}(q)_{k-j}} = \frac{(-1)^{i-\ell}q^{\ell(\ell+1)/2-j-\ell j}(q)_{i-j-1}(1-q^{i-\ell})}{(q)_{i-\ell}}.$$

For $j<i$ and $\ell=i$, this evaluates to 0. For $j=i$, we have directly

$$\sum_{i\le k\le i} \frac{(-1)^{i-k}q^{k(k-1)/2-ki}}{(q)_{i-k}(q)_{k-i}} = q^{i(i-1)/2-i^2} = q^{-i(i+1)/2}.$$

Therefore

$$\sum_{i\le k\le i} A_{i,k}A^{-1}_{k,j} = \frac{q^{i(i+3)/2-i}(q)_{N+i-1}(q)_{N-i-1}(q)_i}{(q)_{N-i-1}(q)_i(q)_{N+i-1}} q^{-i(i+1)/2} = 1,$$

as desired.

5. A two parameter extension

It is even possible to extend the results by replacing $i\to i+r$ and $j\to j+s$, for $r,s\ge 0$. In other words, the matrix now has entries $\binom{i+r+j+s}{j+s}^{-1}$.

We only give the formulæ in a list:

$$L_{i,j} = \frac{(i+r)!i!(2j+r+s)!}{(i+j+r+s)!(i-j)!(j+r)!j!},$$

$$U_{i,j} = \frac{(-1)^i (j+s)!j!(i+r)!(i+r+s-1)!}{(j+i+r+s)!(j-i)!(2i+r+s-1)!},$$

$$L^{-1}_{i,j} = \frac{(-1)^{i-j}(i+r)!i!(i+j+r+s-1)!}{(2i+r+s-1)!(i-j)!(j+r)!j!},$$

$$U^{-1}_{i,j} = \frac{(-1)^i (i+j+r+s-1)!(2j+r+s)!}{(j-i)!(j+r)!(j+i)!(j+r+s-1)!(i+s)!i!},$$

$$A_{i,j} = \frac{(-1)^{i-j}(N-j-1)!(j+s)!(N+i+r+s-1)!}{(i+s)!(N-i-1)!(N+j+r+s-1)!(i-j)!},$$

$$B_{i,j} = \frac{(-1)^{j+N-1}(N+j+r+s-1)!}{(j+r)!(j-i)!(N-j-1)!(i+s)!},$$

$$A^{-1}_{i,j} = \frac{(N-j-1)!(j+s)!(N+i+r+s-1)!}{(i+s)!(N-i-1)!(N+j+r+s-1)!(i-j)!},$$

$$B^{-1}_{i,j} = \frac{(-1)^{j+1+N}(N-1-i)!(j+s)!(i+r)!}{(j-i)!(N+i+r+s-1)!}.$$

For $\begin{bmatrix} i+r+j+s \\ j+s \end{bmatrix}^{-1}$ we get q-analogues:

$$L_{i,j} = \frac{(q)_{i+r}(q)_i(q)_{2j+r+s}}{(q)_{i+j+r+s}(q)_{i-j}(q)_{j+r}(q)_j},$$

$$U_{i,j} = \frac{(-1)^i q^{i(3i-1)/2}(q)_{j+s}(q)_j(q)_{i+r}(q)_{i+r+s-1}}{(q)_{j+i+r+s}(q)_{j-i}(q)_{2i+r+s-1}},$$

$$L^{-1}_{i,j} = \frac{(-1)^{i-j} q^{i(i-1)/2+j(j+1)/2-ij}(q)_{i+r}(q)_i(q)_{i+j+r+s-1}}{(q)_{2i+r+s-1}(q)_{i-j}(q)_{j+r}(q)_j},$$

$$U^{-1}_{i,j} = \frac{(-1)^i q^{i(i+1)/2-j^2-ij-(r+s)j}(q)_{i+j+r+s-1}(q)_{2j+r+s}}{(q)_{j-i}(q)_{j+r}(q)_{j+i}(q)_{j+r+s-1}(q)_{i+s}(q)_i},$$

$$A_{i,j} = \frac{(-1)^{i-j} q^{i(i+3)/2-j(j+3)/2+N(j-i)}(q)_{N-j-1}(q)_{j+s}(q)_{N+i+r+s-1}}{(q)_{i+s}(q)_{N-i-1}(q)_{N+j+r+s-1}(q)_{i-j}},$$

$$B_{i,j} = \frac{(-1)^{j+N-1} q^{(r+s)(i+1)+i^2+j(j+3)/2-(r+s+j)N-N(N-1)/2}(q)_{N+j+r+s-1}}{(q)_{j+r}(q)_{j-i}(q)_{N-j-1}(q)_{i+s}},$$

$$A^{-1}_{i,j} = \frac{q^{i(i+1)-j-ij+N(j-i)}(q)_{N-j-1}(q)_{j+s}(q)_{N+i+r+s-1}}{(q)_{i+s}(q)_{N-i-1}(q)_{N+j+r+s-1}(q)_{i-j}},$$

$$B^{-1}_{i,j} = \frac{(-1)^{j+1+N} q^{-j(j+1)/2-ij+N(N-1)/2+(N-1)(r+s+i)-(r+s)j}(q)_{N-1-i}(q)_{j+s}(q)_{i+r}}{(q)_{j-i}(q)_{N+i+r+s-1}}.$$

The previous results follow from these by plugging in $r=s=0$ or taking appropriate limits.

6. Additional results

For completeness, we also deal with the binomial matrix (no reciprocals)

$$\mathscr{M}_{i,j} = \left(\binom{i+r+j+s}{j+s}\right)_{i,j\ge 0}.$$

We get the same type of factorizations and use calligraphic letters to mark the difference. We only cite the results; justifications are in the same style as in the previous instances.

$$\mathscr{L}_{i,j} = \frac{i!(i+r+s)!(j+r)!}{(i-j)!j!(i+r)!(j+r+s)!},$$

$$\mathscr{U}_{i,j} = \frac{(j+r+s)!j!}{(j-i)!(i+r)!(j+s)!},$$

$$\mathscr{L}^{-1}_{i,j} = \frac{(-1)^{i-j}(i+r+s)!i!(j+r)!}{(i-j)!(j+r+s)!(i+r)!j!},$$

$$\mathscr{U}^{-1}_{i,j} = \frac{(-1)^{i-j}(j+r)!(i+s)!}{(j-i)!(i+r+s)!i!},$$

$$\mathscr{A}_{i,j} = \frac{(-1)^{i-j}(N-j-1)!(i+s)!(2j+r+s+1)!}{(i-j)!(N-i-1)!(i+j+r+s+1)!(j+s)!},$$

$$\mathscr{B}_{i,j} = \frac{(-1)^{i-j}(N+i+r+s)!(j+r)!(i+s)!}{(j-i)!(N-j-1)!(2i+r+s)!(i+j+r+s+1)!},$$

$$\mathscr{A}^{-1}_{i,j} = \frac{(N-j-1)!(i+j+r+s)!(i+s)!}{(i-j)!(N-i-1)!(2i+r+s)!(j+s)!},$$

$$\mathscr{B}^{-1}_{i,j} = \frac{(N-i-1)!(2j+r+s+1)!(i+j+r+s)!}{(j-i)!(N+j+r+s)!(i+r)!(j+s)!}.$$

There are also q-analogues for the matrix

$$\mathscr{M}_{i,j} = \left(\begin{bmatrix} i+r+j+s \\ j+s \end{bmatrix}\right)_{i,j\ge 0}.$$

$$\mathscr{L}_{i,j} = \frac{(q)_i(q)_{i+r+s}(q)_{j+r}}{(q)_{i-j}(q)_j(q)_{i+r}(q)_{j+r+s}},$$

$$\mathscr{U}_{i,j} = \frac{q^{i^2+(r+s)i}(q)_{j+r+s}(q)_j}{(q)_{j-i}(q)_{i+r}(q)_{j+s}},$$

$$\mathscr{L}^{-1}_{i,j} = \frac{(-1)^{i-j}q^{i(i-1)/2-ij+j(j+1)/2}(q)_{i+r+s}(q)_i(q)_{j+r}}{(q)_{i-j}(q)_{j+r+s}(q)_{i+r}(q)_j},$$

$$\mathscr{U}^{-1}_{i,j} = \frac{(-1)^{i-j}q^{i(i+1)/2-ij-j(j+1)/2-(r+s)j}(q)_{j+r}(q)_{i+s}}{(q)_{j-i}(q)_{i+r+s}(q)_i},$$

$$\mathscr{A}_{i,j} = \frac{(-1)^{i-j} q^{i(i+3)/2 - j(j+3)/2 + N(j-i)} (q)_{N-j-1} (q)_{i+s} (q)_{2j+r+s+1}}{(q)_{i-j} (q)_{N-i-1} (q)_{i+j+r+s+1} (q)_{j+s}},$$

$$\mathscr{B}_{i,j} = \frac{(-1)^{i-j} q^{(j+1)(j+2)/2 + 3i(i+1)/2 + (r+s)(i+1) - N(j+1+i+r+s)} (q)_{N+i+r+s} (q)_{j+r} (q)_{i+s}}{(q)_{j-i} (q)_{N-j-1} (q)_{2i+r+s} (q)_{i+j+r+s+1}},$$

$$\mathscr{A}^{-1}_{i,j} = \frac{q^{i^2 - (N-1)(i-j) - ij} (q)_{N-j-1} (q)_{i+j+r+s} (q)_{i+s}}{(q)_{i-j} (q)_{N-i-1} (q)_{2i+r+s} (q)_{j+s}},$$

$$\mathscr{B}^{-1}_{i,j} = \frac{q^{(i+j+1+r+s)(N-j-1)} (q)_{N-i-1} (q)_{2j+r+s+1} (q)_{i+j+r+s}}{(q)_{j-i} (q)_{N+j+r+s} (q)_{i+r} (q)_{j+s}}.$$

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