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Some upper bounds on the dimension of the Schur multiplier of a pair of nilpotent Lie algebras

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Abstract: Let (L, N) be a pair of Lie algebras where N is an ideal of the finite dimensional nilpotent Lie algebra L. Some upper bounds on the dimension of the Schur multiplier of (L, N) are obtained without considering the existence of a complement for N. These results are applied to derive a new bound on the dimension of the Schur multiplier of a nilpotent Lie algebra.

Key words: Pair of Lie algebras, Schur multiplier, nilpotent Lie algebra

1. Introduction

Throughout this paper, we denote by (L, N) a pair of Lie algebras where N is an ideal of the Lie algebra L. The Schur multiplier of the pair (L, N) is defined to be the abelian Lie algebra $\mathcal{M}(L, N)$, whose principal feature is the following natural exact sequence of Lie algebras:

$$\begin{aligned} H_3(L) \to H_3(L/N) \to \mathcal{M}(L, N) \to H_2(L) \to H_2(L/N) \\ \to N/[N, L] \to H_1(L) \to H_1(L/N) \to 0, \end{aligned} \tag{1}$$

where $H_i(-)$ is the i-th Chevalley–Eilenberg homology group of a Lie algebra. From the homotopical point of view, $\mathcal{M}(L, N)$ is the second relative homology of (L, N), see [3, 4] for more details and a brief introduction. Taking $N = L$ we find that $\mathcal{M}(L, N) = H_2(L)$, which is called the Schur multiplier of L and denoted by $\mathcal{M}(L)$.

Determining bounds on the dimension of the Schur multiplier of a (nilpotent) Lie algebra was a hot topic in recent decades. Nilpotent Lie algebras have been widely discussed in the literature in order to be classified by their multipliers; however, there are many other interesting open problems on the dimension of the homology groups of nilpotent Lie algebras; see [1, 2, 5, 6, 8] for instance.

Most of the bounds that have been obtained on the dimension of the Schur multiplier of the pair (L, N) are just generalizations of a previously known bound on the dimension of the Schur multiplier of L. In the most discussed case, authors have considered that the ideal N is complemented in L. Thus, the morphisms $H_i(L) \twoheadrightarrow H_i(L/N)$ split for any i, and $\mathcal{M}(L, N)$ is a complement of $H_2(L/N)$ in $H_2(L)$. Therefore, if $L \cong F/R$ and $N \cong S/R$ are arbitrary free presentations of L and N respectively, then by Hopf's formula we have

$$\mathcal{M}(L) = (R \cap [F, F])/[R, F] \quad , \quad \mathcal{M}(L/N) = (S \cap [F, F])/[F, S].$$

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This dedicates the free presentation $(R \cap [S,F])/[R,F]$ for $\mathcal{M}(L,N)$ that applies to determine the bounds; see [9, 12] for instance.

By assuming that N admits a complement in L, the following theorem was proved in [12]. We use different tools to eliminate this limitation and give a similar bound that can widely extend some results of [11, 12].

Theorem A. *Let L be a finite dimensional nilpotent Lie algebra and N an ideal of L. Then*

$$\dim(\mathcal{M}(L,N)) \leq \dim\left(\mathcal{M}\left(\frac{L}{[L,N]}, \frac{N}{[L,N]}\right)\right) + \dim([L,N])(d(\frac{L}{Z(L,N)}) - 1),$$

where $d(X)$ is the minimal number of generators of a Lie algebra X and $Z(L,N) = \{n \in N \mid [l,n] = 0, \text{ for all } l \in L\} = Z(L) \cap N$.

It was shown in [13] that if L is a nilpotent Lie algebra then $\dim(\mathcal{M}(L)) + \dim(L^2) \leq \dim(L)d(L)$. The following theorem can be a generalization of this bound on the dimension of $\mathcal{M}(L,N)$.

Theorem B. *Let L be a finite dimensional nilpotent Lie algebra with an ideal N. Then*

$$\dim(\mathcal{M}(L,N)) + \dim([L,N]) \leq \dim(N)(d(N) + d(L/N)).$$

We finally give the following theorem, which can be used to obtain a new bound for the Schur multiplier of a nilpotent Lie algebra.

Theorem C. *Let L be a finite dimensional nilpotent Lie algebra and N be an ideal of L which is not central. Then*

$$\dim(\mathcal{M}(L,N)) \leq d(L)(\dim(N) - 1) - \dim\left(\mathcal{M}\left(\frac{L}{[L,N]}, \frac{N}{[L,N]}\right)\right).$$

2. Proof of theorems

Let K, N be ideals of a Lie algebra L. The nonabelian exterior product $K \wedge N$ is the Lie algebra generated by the elements $k \wedge n$ with $(k,n) \in K \times N$, subject to the relations

$$\begin{aligned}
&c(k \wedge n) = ck \wedge n = k \wedge cn &&, \quad [k,k'] \wedge n = k \wedge [k',n] - k' \wedge [n,k]\\
&(k+k') \wedge n = k \wedge n + k' \wedge n &&, \quad k \wedge [n,n'] = [n',k] \wedge k - [k,n] \wedge n'\\
&k \wedge (n+n') = k \wedge n + k \wedge n' &&, \quad [(k \wedge n),(k' \wedge n')] = [k,n] \wedge [k',n']\\
&x \wedge x = 0,
\end{aligned}$$

for all $x \in K \cap N$, $k,k' \in K$, $n,n' \in N$ and scalar c. It follows from [4, Theorem 35] that the Schur multiplier of (L,N) can be computed as

$$\mathcal{M}(L,N) \cong \ker(L \wedge N \overset{[-,-]}{\to} L), \tag{2}$$

where $[-,-]$ is the commutator map defined on generators of $L \wedge N$ by $[-,-](l \wedge n) = [l,n]$.
The following theorem plays a key role in our main results.

Theorem 2.1 *Let L be a Lie algebra and N,K be ideals of L such that $K \subseteq N \cap Z(L)$. Then the following sequence is exact:*

$$K \wedge L \to \mathcal{M}(L,N) \to \mathcal{M}(L/K, N/K) \to K \cap [N,L] \to 0.$$

Proof Using the functorial properties of the nonabelian exterior product, the short exact sequence of Lie algebras $0 \to K \to L \overset{\pi}{\to} L/K \to 0$ induces the exact sequence

$$L \wedge K \to L \wedge N \overset{\pi\wedge\pi}{\to} L/K \wedge N/K \to 0. \tag{3}$$

Now, we have the following diagram of Lie algebras

$$\begin{array}{ccccccccc} & & L\wedge K & \longrightarrow & L\wedge N & \xrightarrow{\pi\wedge\pi} & L/K\wedge N/K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle [\ ,\ ]_1} & & \downarrow{\scriptstyle [\ ,\ ]_2} & & \downarrow{\scriptstyle [\ ,\ ]_3} & & \\ 0 & \longrightarrow & ([L,N]\cap K) & \longrightarrow & [L,N] & \xrightarrow{\pi} & [L/K,N/K] & \longrightarrow & 0, \end{array}$$

where the vertical arrows are the commutator maps; see [4]. In this diagram, the right-hand-side square is always commutative. Note that since K is a central ideal of L the commutator map $[-,-]_1$ is equal to the zero morphism and so the left-hand-side square is also commutative. Now the "Snake Lemma" yields that there is the following exact sequence:

$$\ker([\ ,\ ]_1) \to \ker([\ ,\ ]_2) \to \ker([\ ,\ ]_3) \to \mathit{coker}([\ ,\ ]_1) \to 0.$$

The last homomorphism is surjective because $[-,-]_2$ is onto. Finally, the result follows from (2). □

Remark 2.2 *By taking $N = L$ in Theorem 2.1, we can obtain the Ganea sequence in homology of Lie algebras; see [11, Proposition 4.1]. In the case that L splits over N, a similar sequence was obtained in [9].*

Using Theorem 2.1, we obtain the following corollary that generalizes [11, corollary 4.2] and [12, Proposition 2.2].

Corollary 2.3 *Let L be a finite dimensional Lie algebra and N, K be ideals of L such that $K \subseteq N \cap Z(L)$. Then $\dim(\mathcal{M}(L/K,N/K)) \leq \dim(\mathcal{M}(L,N)) + \dim([N,L]\cap K)$; in particular, if N is a central ideal of L then*

$$\dim(\mathcal{M}(L/K,N/K)) \leq \dim(\mathcal{M}(L,N)).$$

Now, we are ready to prove the theorems.

Proof [Proof of Theorem A] The proof is stated on induction on $\dim(L)$. If N is central then $[L,N] = 0$ and there is nothing to prove. Therefore, suppose that $[L,N] \neq 0$ and choose a one-dimensional ideal K of L such that $K \subseteq Z(L) \cap [L,N]$. Thanks to Theorem 2.1 and applying the induction hypothesis, we have

$$\begin{aligned} \dim(\mathcal{M}(L,N)) &\leq \dim(\mathcal{M}(L/K,N/K)) + \dim(K\wedge L) - 1 \\ &\leq \dim\left(\mathcal{M}\left(\frac{L}{[L,N]},\frac{N}{[L,N]}\right)\right) + (\dim([L,N])-1)\times \\ &\quad (d(\frac{L/K}{Z(L/K,N/K)})-1) + \dim(K\wedge L) - 1 \\ &\leq \dim\left(\mathcal{M}\left(\frac{L}{[L,N]},\frac{N}{[L,N]}\right)\right) + (\dim([L,N])-1)(d(\frac{L}{Z(L,N)})-1) + d(L) - 1 \\ &= \dim\left(\mathcal{M}\left(\frac{L}{[L,N]},\frac{N}{[L,N]}\right)\right) + \dim([L,N])(d(\frac{L}{Z(L,N)})-1), \end{aligned}$$

which completes the proof. □

Proof [Proof of Theorem B] Similar to the previous proof, we proceed by induction on the dimension of L. Suppose that the result occurs for any Lie algebra of dimension less than $\dim(L)$. Choose a one-dimensional ideal K such that $K \subseteq N \cap Z(L)$. Since L is a finite dimensional nilpotent Lie algebra, $d(L)$ is equal to $\dim(L/L^2)$ and

$$d(L) \leq \dim(L) - \dim(L^2) + \dim(L^2 \cap N) - \dim(N^2) = d(L/N) + d(N).$$

Hence, the sequence (3) implies that

$$\begin{aligned}\dim(L \wedge N) &\leq \dim(K \wedge L) + \dim(L/K \wedge N/K)\\ &\leq d(L) + \dim(N/K)(d(N/K) + d(L/N))\\ &\leq d(L/N) + d(N) + (\dim(N) - 1)(d(N) + d(L/N))\\ &= \dim(N)(d(N) + d(L/N)),\end{aligned}$$

Since $\dim(\mathcal{M}(L,N)) + \dim([L,N]) = \dim(L \wedge N)$ by (2) the proof completes. □

We can use a similar method of Theorem B to prove the following proposition.

Proposition 2.4 *Let L be a finite dimensional nilpotent Lie algebra and N be an ideal of L that is not contained in $Z(L)$. Then*

$$\dim(\mathcal{M}(L,N)) \leq \dim(N)(d(N) + d(L/N) - 1).$$

Proof [Proof of Theorem C] Similarly, the proof is based on induction on $\dim(L)$. Suppose that $\dim(L) > 1$ and choose a one-dimensional ideal K of L such that $K \subseteq Z(L) \cap [L,N]$. Using Theorem 2.1 and applying the induction hypothesis, we have

$$\begin{aligned}\dim(\mathcal{M}(L,N)) &\leq \dim(\mathcal{M}(L/K, N/K)) + \dim(K \wedge L)\\ &\leq d(L/K)(\dim(N/K) - 1) - \dim\left(\mathcal{M}\left(\frac{L}{[L,N]}, \frac{N}{[L,N]}\right)\right) + \dim(K \wedge L)\\ &\leq d(L)(\dim(N) - 2) - \dim\left(\mathcal{M}\left(\frac{L}{[L,N]}, \frac{N}{[L,N]}\right)\right) + d(L)\\ &\leq d(L)(\dim(N) - 1) - \dim\left(\mathcal{M}\left(\frac{L}{[L,N]}, \frac{N}{[L,N]}\right)\right).\end{aligned}$$

Note that since K is a central ideal of L, the Lie actions of K and L on each other are trivial, and so $K \wedge L \cong K \wedge L/L^2$ and

$$\dim(K \wedge L) \leq \dim(L/L^2) = d(L).$$

□

Now we can derive a new bound for the dimension of the Schur multiplier of a nilpotent Lie algebra.

Corollary 2.5 *Let L be a d-generator nilpotent Lie algebra of dimension n. Then*

$$\dim(\mathcal{M}(L)) \leq \frac{1}{2}d(2n - d - 1).$$

Proof If L is an abelian Lie algebra then $d = n$, $\dim(\mathcal{M}(L)) = \frac{1}{2}n(n-1)$ and the statement is obviously true. Hence, suppose that L is not an abelian Lie algebra. Using the fact

$$\dim(\mathcal{M}(L/L^2, L/L^2)) = \dim(\mathcal{M}(L/L^2)) = \frac{1}{2}d(d-1),$$

the desired result follows by taking $N = L$ in Theorem C. □

Note that since $d(2n-d-1) \leq n(n-1)$ for all integers $1 \leq d \leq n$, the upper bound obtained in Corollary 2.5 is sharper than the known bound $\dim(M(L)) \leq \frac{1}{2}n(n-1)$, which is due to Moneyhum [7].

Remark 2.6 *Let* (L, N) *be a pair of Lie algebras such that* N *is of codimension less than two. Since* $H_3(L/N) = 0$ *in the sequence (1), one can deduce that* $\dim(\mathcal{M}(L,N)) \leq \dim(\mathcal{M}(L))$*. Hence any upper bound on the dimension of* $\mathcal{M}(L)$ *can be considered as an upper bound for* $\dim(\mathcal{M}(L,N))$*. In particular, if* N *is an ideal of codimension one, then* $\mathcal{M}(L/N) = H_3(L/N) = 0$*, which immediately implies* $\mathcal{M}(L) \cong \mathcal{M}(L,N)$*. Therefore, any upper and lower bound on* $\mathcal{M}(L)$ *is a bound for* $\mathcal{M}(L,N)$*. The result obtained in [12, Theorem D] is an example of the bound that was previously obtained by Jones (1974) on the dimension of* $\mathcal{M}(L)$*.*

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