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# Problems in matricially derived solid Banach sequence spaces

**Peter D. JOHNSON**[1], **Faruk POLAT**[2,*]
[1]Auburn University, Department of Mathematics and Statistics, Auburn, Alabama, USA
[2]Çankırı Karatekin University, Faculty of Science, Department of Mathematics, Çankırı, Turkey

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**Abstract:** Let $\mathbb{F}^{\mathbb{N}}$ denote the vector space of all scalar sequences. If $A$ is an infinite matrix with nonnegative entries and $\lambda$ is a solid subspace of $\mathbb{F}^{\mathbb{N}}$, then $sol - A^{-1}(\lambda) = \{x \in \mathbb{F}^{\mathbb{N}} : A|x| \in \lambda\}$ is also a solid subspace of $\mathbb{F}^{\mathbb{N}}$ that, under certain conditions on $A$ and $\lambda$, inherits a solid topological vector space topology from any such topology on $\lambda$. Letting $\Lambda_0 = \lambda$ and $\Lambda_m = sol - A^{-1}(\Lambda_{m-1})$ for $m > 0$, we derive an infinite sequence $\Lambda_0, \Lambda_1, \Lambda_2, ...$ of solid subspaces of $\mathbb{F}^{\mathbb{N}}$ from the inputs $A$ and $\lambda$. For $A$ and $\lambda$ confined to certain classes, we ask many questions about this derived sequence and answer a few.

**Key words:** Solid sequence space, Toeplitz matrix, projective limit, solid topology

## 1. Introduction

Throughout the paper the scalar field $\mathbb{F}$ will be either $\mathbb{R}$, the real numbers or $\mathbb{C}$, the complex numbers, and $\mathbb{N} = \{1, 2, ...\}$.

Although the only spaces appearing in this paper will be spaces of sequences of scalars, in hopes of inspiring generalizations to a wider context, we give some basic definitions and properties concerning Riesz spaces. For more details on Riesz spaces, see the classical manuscripts [1, 2, 9].

A real vector space $E$, endowed with a partial order $\leq$ in $E^2$, is called a *Riesz space* or a *vector lattice* if $\sup\{x, y\}$ and $\inf\{x, y\}$ exist in $E$ for all $x, y \in E$, and $x \leq y$ implies $\alpha x + z \leq \alpha y + z$ for all $z \in E$ and $0 \leq \alpha \in \mathbb{R}$. We define the modulus or absolute value of an element $x$ in $E$ by the formula $|x| := \sup\{x, -x\}$.

Typical examples of Riesz spaces are provided by function spaces. For example, $C(\Omega)$, all continuous real valued functions on a topological space $\Omega$ and $B(K)$, all bounded real valued functions on a set $K$ are Riesz spaces under pointwise ordering.

If $E$ is a vector lattice, then the set $E^+ = \{x \in E : x \geq 0\}$ is called the *positive cone* of $E$.

The *solid* hull of an element $a$ in a vector lattice E is given by $S(a) = \{b \in E : |b| \leq |a|\}$. A vector subspace $S$ in a vector lattice $E$ is said to be *solid* or an *order ideal* if it follows from $|u| \leq |v|$ in $E$ and $v \in S$ that $u \in S$. In the sequel, we will use the term *solid* in preference to *order ideal.*

A norm $\|.\|$ on a vector lattice $E$ is said to be a *lattice norm* or *solid norm* if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for each $x, y \in E$.

*Correspondence: faruk.polat@gmail.com
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A vector lattice endowed with a solid norm is known as a *normed vector lattice*. If a normed vector lattice $E$ is also norm complete, then it is called a *Banach lattice*. It should be obvious that in a normed vector lattice $E$, $\|x\| = \| \, |x| \, \|$ holds for all $x \in E$.

The space of all scalar valued sequences will be denoted by $\mathbb{F}^{\mathbb{N}}$. The subspace of $\mathbb{F}^{\mathbb{N}}$ consisting of sequences with only finitely many nonzero entries will be denoted by $c_{00}$, whether $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. All operations on sequences will be coordinate-wise. If $x = (x_n) \in \mathbb{F}^{\mathbb{N}}$, then we write $|x| = (|x_n|)$. Let $x = (x_n)$ and $y = (y_n)$ be elements of $\mathbb{R}^{\mathbb{N}}$; $x \leq y$ means that $x_n \leq y_n$ for each $n \in \mathbb{N}$. It is clear that $(\mathbb{R}^{\mathbb{N}}, \leq)$ is a vector lattice, and the vector lattice definition of $|x|$, $x \in \mathbb{R}^{\mathbb{N}}$ agrees with the definition given here, $|x| = (|x_n|)$. $\mathbb{C}^{\mathbb{N}} = \mathbb{R}^{\mathbb{N}} + i\mathbb{R}^{\mathbb{N}}$ is the space of complex valued sequences with the usual addition and multiplication. $\mathbb{C}^{\mathbb{N}}$ can be partially ordered coordinate-wise, i.e. $(z_n) = (x_n + iy_n) \leq (t_n) = (a_n + ib_n)$ in $\mathbb{C}^{\mathbb{N}}$ whenever $(x_n) \leq (a_n)$ and $(y_n) \leq (b_n)$ in $\mathbb{R}^{\mathbb{N}}$ for each $n \in \mathbb{N}$. Then $\mathbb{C}^{\mathbb{N}}$ is a Riesz space and, for $(z_n) = (x_n + iy_n)$, the element $|(z_n)|$ is given by $|(z_n)| = |(x_n)| + i|(y_n)|$. For complex Riesz spaces, see [10].

The sequence of zeros $(0, 0, 0, ...)$ will be denoted by $\underline{0}$.

For $0 < p < \infty$, we denote

$$l_p = \{(x_n) \in \mathbb{F}^{\mathbb{N}} : \ \sum_{n=1}^{\infty} |x_n|^p < \infty\}.$$

The usual "norm" or distance from $\underline{0}$ in $l_p$ is defined by

$$||x||_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}},$$

and for each $x \in l_p$; $\|.\|_p$ is really a norm for $p \geq 1$.

If $\lambda$ is a solid subspace of $\mathbb{F}^{\mathbb{N}}$, a topology on $\lambda$ with which $\lambda$ becomes a t.v.s. is a *solid topology* if it has a basis of neighborhoods at the origin consisting of solid sets.

The spaces $l_p$, $1 \leq p < \infty$, $l_\infty$, the space of bounded sequences, and $c_0$, the space of null sequences, are solid subspaces of $\mathbb{F}^{\mathbb{N}}$, and their usual norms, $||.||_p$ on $l_p$, the sup norm on $l_\infty$ and $c_0$, are solid norms. The space of convergent sequences, $c$, is not solid.

Let $A = [a_{ij} : \ i, j \geq 1] = [a_{ij}]$ be an infinite matrix with nonnegative entries and no zero columns. The domain of $A$, denoted by $dom(A)$, is

$$dom(A) = \{x \in \mathbb{F}^{\mathbb{N}} : \ \sum_{j=1}^{\infty} a_{ij} x_j \ converges \ for \ each \ i \in \mathbb{N}\}.$$

For $x \in dom(A)$, the sequence $Ax$, the $A$-transform of $x$, is given by $(Ax)_i = \sum\limits_{j=1}^{\infty} a_{ij} x_j$ for each $i \in \mathbb{N}$. If $\lambda \subseteq dom(A)$, then $A\lambda = \{Ax : \ x \in \lambda\}$.

If $\lambda \subseteq \mathbb{F}^{\mathbb{N}}$, then $A^{-1}(\lambda) = \{x \in dom(A) : \ Ax \in \lambda\}$.

If $A = [a_{ij}]$ is a lower triangular matrix (i.e. $a_{ij} = 0$, for $i < j$) with nonnegative entries and positive entries on the main diagonal (i.e. $a_{ii} > 0$, for $i \in \mathbb{N}$), then $dom(A) = \mathbb{F}^{\mathbb{N}}$. The assumption of nonzero diagonal entries implies that $A$ has a matricial inverse $A^{-1}$. This inverse $A^{-1}$ is also lower triangular. $A^{-1}$ will fail to have

all nonnegative entries unless $A$ is diagonal. For more details on infinite matrices, see the book [3].
The following definition inspired by [8] was introduced in [6].

**Definition 1** *If $\lambda \subseteq \mathbb{F}^{\mathbb{N}}$ and $A$ is an infinite matrix, with nonnegative entries,*

$$sol - A^{-1}(\lambda) = \{x \in \mathbb{F}^{\mathbb{N}} : \ |x| \in A^{-1}(\lambda)\} = \{x \in \mathbb{F}^{\mathbb{N}} : \ |x| \in dom(A) \ and \ A|x| \in \lambda\}.$$

The next result, given in [6], justifies the name "$sol - A^{-1}(\lambda)$".

**Proposition 2** *Let $A$ be an infinite matrix with nonnegative entries and $\lambda$ be a solid subspace of $\mathbb{F}^{\mathbb{N}}$. Then we have:*
*(a) $sol - A^{-1}(\lambda)$ is solid;*
*(b) $sol - A^{-1}(\lambda) \subseteq A^{-1}(\lambda)$;*
*(c) $sol - A^{-1}(\lambda)$ is the largest solid set of sequences contained in $A^{-1}(\lambda)$;*
*(d) $sol - A^{-1}(\lambda)$ is a subspace of $\mathbb{F}^{\mathbb{N}}$.*

From now on, we shortly write t.v.s. and l.c.t.v.s. for topological vector space and locally convex topological vector space, respectively.

If $\tau$ is a solid t.v.s. topology on $\lambda$, then it naturally induces a solid t.v.s. topology on $sol - A^{-1}(\lambda)$. Suppose $\lambda$ is a solid subspace of $\mathbb{F}^{\mathbb{N}}$ with solid topology $\tau$, and $\mathcal{U}$ is a neighborhood base at the origin in $(\lambda, \tau)$ consisting of solid sets. It is shown in [6] that the sets

$$sol - A^{-1}(U) = \{x \in sol - A^{-1}(\lambda) : \ A|x| \in U\}, \quad (U \in \mathcal{U})$$

constitute a neighborhood base at the origin for a solid t.v.s. topology $sol - A^{-1}(\tau)$ on $sol - A^{-1}(\lambda)$. Furthermore, if the topology on $\lambda$ is Hausdorff and $A$ has no zero columns, then the induced topology on $sol - A^{-1}(\lambda)$ is Hausdorff. As mentioned earlier, all our matrices will have no zero columns.

In addition, when $\lambda$ and $A$ are given, we will require that the columns of $A$ are in $\lambda$. This implies that $c_{00} \subseteq sol - A^{-1}(\lambda)$.

Clearly, if $\lambda$ is equipped with a solid norm $\|.\|$, then the topology induced on $sol - A^{-1}(\lambda)$ is induced by the solid norm $x \to \| \ A|x| \ \|$. A similar comment holds for quasinorms, pseudonorms, and seminorms.

We say that $(\lambda, \tau)$ is $AK$ if and only if for each $x \in \lambda$, the projections $P_n(x) = (x_1, \ldots, x_n, 0, 0, \ldots)$ converge to $x$ in $(\lambda, \tau)$, as $n \to \infty$.
In [6] and [5], it is shown that $(sol - A^{-1}(\lambda), sol - A^{-1}(\tau))$ inherits the following properties from the solid space $(\lambda, \tau)$.

**Proposition 3** *If $\lambda$ is a solid subspace of $\mathbb{F}^{\mathbb{N}}$ with a solid Hausdorff t.v.s. topology $\tau$, and $A$ is an infinite matrix with nonnegative entries, with every column in $\lambda \setminus \{\underline{0}\}$, then:*
*(a) if $(\lambda, \tau)$ is $AK$, then so is $(sol - A^{-1}(\lambda), sol - A^{-1}(\tau))$ ; and*
*(b) if $(\lambda, \tau)$ is complete, then so is $(sol - A^{-1}(\lambda), sol - A^{-1}(\tau))$, provided either*
*(i) $(\lambda, \tau)$ is $AK$ or (ii) $A$ is lower triangular.*

In fact, $(sol - A^{-1}(\lambda), sol - A^{-1}(\tau))$ inherits the completeness of $(\lambda, \tau)$ under much more general conditions than (i) or (ii) above. (See [6]; in fact, no example is known in which $(\lambda, \tau)$ is complete but $(sol - A^{-1}(\lambda), sol - A^{-1}(\tau))$

is not.) However, a discussion of this thorny question, the completeness of $(sol - A^{-1}(\lambda), sol - A^{-1}(\tau))$, would distract from our main purpose.

Given $(\lambda, \tau)$ and $A$, satisfying the hypothesis of Proposition 3, set $\Lambda_0 = \lambda$ and $\Lambda_m = sol - A^{-1}(\Lambda_{m-1})$, $m \in \mathbb{N}$. (Even if not every column of $A$ is in $\Lambda_{m-1}$, we can still form $\Lambda_m$, and give it a topology $(sol{-}A^{-1})^m(\tau)$. If, say, the $j^{th}$ column of $A$ is not in $\Lambda_{m-1}$, then $x \in \Lambda_m$ implies $x_j = 0$.) This gives us an infinite sequence $\Lambda_0, \Lambda_1, \Lambda_2, \ldots$ of solid sequence spaces each with a solid topology.

Since $x \to A|x|$ maps $\Lambda_m$ into $\Lambda_{m-1}$, and $\Lambda_{m-1}$ is solid, it follows that $x \to Ax$, that is, multiplication by $A$ maps $\Lambda_m$ into $\Lambda_{m-1}$, and the map is clearly continuous, by the way the topology on $\Lambda_m = sol - A^{-1}(\Lambda_{m-1})$ is derived from that on $\Lambda_{m-1}$ . Therefore, we have the projective limit of the $\Lambda_m$ with respect to the maps $A : \Lambda_m \to \Lambda_{m-1}$, $m \in \mathbb{N}$, defined by

$$X = \{(x^{(m)}) \in \prod_{m=0}^{\infty} \Lambda_m : \ for\ m > 0,\ Ax^{(m)} = x^{(m-1)}\}.$$

In specific cases, even the most fundamental questions about the sequence $\Lambda_0, \Lambda_1, \Lambda_2, \ldots$, and $X$ are not so easy to answer:

1) Is $\Lambda_m$ nontrivial ($\{\underline{0}\} \subsetneq \Lambda_m$ ) for every $m > 0$ ?

2) Related to 1): is $c_{00} \subseteq \Lambda_m$ for every $m > 0$?

3) Is $X$ nontrivial ? That is, does $X$ contain any sequence $(x^{(0)}, x^{(1)}, ...)$ other than $(\underline{0}, \underline{0}, ...)$?

In these introductory studies we think it wise to confine $A$ to be lower triangular and invertible. Under this restriction it is easy to see that for each $m > 0$, $\Lambda_m = \{x \in \mathbb{F}^{\mathbb{N}} \ : \ A^m|x| \in \lambda\}$, and if $x^{(m)} \in \Lambda_m$, $m = 0, 1, 2, \ldots$, then $(x^{(0)}, x^{(1)}, \ldots) \in X$ if and only if $x^{(m)} = (A^{-1})^m x^{(0)} = A^{-m}x^{(0)}$ for each $m > 0$. It follows that in these circumstances (when $A$ is lower triangular and invertible ), $X$ is in one-to-one correspondence with a mysterious subspace of $\lambda$,

$$\begin{aligned} \lambda_X &= \{x \in \lambda : \ for\ all\ m \geq 1,\ A^{-m}x \in \Lambda_m\} \\ &= \{x \in \lambda : \ A^m|A^{-m}x| \in \lambda\}. \end{aligned}$$

The one-to-one correspondence between $X$ and $\lambda_X$ is simply: $(x^{(0)}, x^{(1)}, ...) \in X$ corresponds to $x^{(0)} \in \lambda_X$. This is very interesting and raises a lot of questions, such as: is the topology on $X$ induced by the product topology on $\prod_{m=0}^{\infty} \Lambda_m$ the same as the topology on $X$ induced by its identification with $\lambda_X$, which bears the relative topology induced upon it by $\tau$, the topology on $\lambda = \Lambda_0$? Certainly the product topology on $X$ is no weaker than the topology on it as $\lambda_X$, and the two are the same, trivially, when $X$ is trivial $(\lambda_X = \{\underline{0}\})$.

From previous work [7] we know that the product topology on $X$ can be strictly finer than the relative topology induced by $\tau$ on $\lambda_X \cong X$. We wonder if they can ever be the same when $X$ is nontrivial. In that previous work we concentrated on matrices of Cesàro type,

$$C = C(1, a_2, a_3, ...) = \begin{bmatrix} 1 & 0 & 0 & . & . & . & . \\ a_2 & a_2 & 0 & . & . & . & . \\ a_3 & a_3 & a_3 & 0 & . & . & . \\ a_4 & a_4 & a_4 & a_4 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix},$$

and $\lambda$ satisfying $C\lambda \subseteq \lambda$. (The Cesàro matrix is the Cesàro type matrix with $a_n = \frac{1}{n}$, $n \in \mathbb{N}$; for that $C$ we have $Cl_p \subseteq l_p$ for all $p \in (1, \infty)$, by Hardy's inequality [4].)

In [7], we show that for all such $C$ and $\lambda$, $\Lambda_0 \subseteq \Lambda_1 \subseteq ...$, whence $c_{00} \subseteq \Lambda_m$ for all $m$, and also that $X$ is nontrivial. However, even in the special case where $C$ is the Cesàro matrix and $\lambda = l_p$ for some $p \in (1, \infty)$, there are straightforward questions about the (relative) product topology on $X$ that we are not able to answer; for instance, is this topology on $X$ given by a norm?

Here we concentrate on Toeplitz type matrices,

$$T = T(1, a_2, a_3, \ldots) = \begin{bmatrix} 1 & 0 & 0 & . & . & . & . \\ a_2 & 1 & 0 & . & . & . & . \\ a_3 & a_2 & 1 & 0 & . & . & . \\ a_4 & a_3 & a_2 & 1 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}.$$

As always, we require $\lambda$ to be solid with a solid topology $\tau$, and $(1, a_2, a_3, ...) \in \lambda$, which guarantees $c_{00} \subseteq \Lambda_1 = sol{-}T^{-1}(\lambda)$, if, as is usually the case, $\lambda$ is closed under the right shift, $(x_1, x_2, \ldots) \to (0, x_1, x_2, \ldots)$.

## 2. Results and examples

Consider an infinite lower triangular Toeplitz matrix,

$$T = \begin{bmatrix} 1 & 0 & 0 & . & . & . & . \\ a_2 & 1 & 0 & . & . & . & . \\ a_3 & a_2 & 1 & 0 & . & . & . \\ a_4 & a_3 & a_2 & 1 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}, \ (a_k \geq 0 \ \ \forall k \in \mathbb{N}),$$

and let $\lambda \subseteq \mathbb{F}^{\mathbb{N}}$ be a solid sequence space such that each column of $T$ is in $\lambda$. For this matrix $T$ and any sequence $b = (b_1, b_2, b_3, \ldots)$, we have

$$Tb = \begin{bmatrix} 1 & 0 & 0 & . & . & . & . \\ a_2 & 1 & 0 & . & . & . & . \\ a_3 & a_2 & 1 & 0 & . & . & . \\ a_4 & a_3 & a_2 & 1 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} . \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ . \\ . \\ . \\ . \\ . \\ . \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 + a_2 b_1 \\ b_3 + a_2 b_2 + a_3 b_1 \\ b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1 \\ . \\ . \\ b_n + \sum_{j=2}^{n} a_j b_{n-j+1} \\ . \\ . \\ . \end{bmatrix}.$$

Observe that $b \in S(T|b|)$, because the $a_j$ are nonnegative.

We define $\Lambda_0, \Lambda_1, \Lambda_2, ...,$ as in the preceding section, by $\Lambda_0 = \lambda$ and for $k > 0$,

$$\Lambda_k = sol - T^{-1}(\Lambda_{k-1}) = \{x \in \mathbb{F}^{\mathbb{N}} : \ T|x| \in \Lambda_{k-1}\} = \{x \in \mathbb{F}^{\mathbb{N}} : T^k|x| \in \lambda\}.$$

As discussed in the introduction, if $\lambda$ is equipped with a solid Hausdorff t.v.s. topology $\tau$, there will be a solid Hausdorff t.v.s. topology $\tau_1 = sol - T^{-1}(\tau)$ naturally induced on $\Lambda_1$, possibly the coarsest such topology such that multiplication by $T$ is a continuous linear map from $\Lambda_1$ into $\Lambda_0$. Then with $\Lambda_1$ replacing $\lambda = \Lambda_0$ and $\Lambda_2 = sol - T^{-1}(\Lambda_1)$ we obtain a solid t.v.s topology $\tau_2 = sol - T^{-1}(\tau_1)$, and so on through $\Lambda_3, \Lambda_4, .....$ As noted in the introduction, the apparent impediment to all this topologizing -- the possibility that $\Lambda_k$ may fail to contain one or more of $T's$ columns -- is not really an impediment at all. If the $j^{th}$ column of $T$ is not in $\Lambda_k$ then for every $x \in \Lambda_{k+1} = sol - T^{-1}(\Lambda_k)$, $x_j = 0$. Thus, $\Lambda_{k+1}$ can be considered to be a solid space of functions from $\mathbb{N}_k = \{j \in \mathbb{N} : \textit{ the } j^{th} \textit{ column of } T \textit{ is an element of } \Lambda_k\}$ into the scalar field, a perfectly good solid sequence space if $\mathbb{N}_k$ is infinite. If $\mathbb{N}_k = \emptyset$, then $\Lambda_{k+1} = \{\underline{0}\}$.

Since $T$ is lower triangular, by Proposition 3 the properties of being complete, and/or $AK$, will be inherited by $\Lambda_1$ from $\lambda$, and the inheritance proceeds to $\Lambda_2$, $\Lambda_3$, etc.

**Proposition 4** *$(\Lambda_k)_{k\geq 0}$ is a decreasing sequence.*

**Proof** As noted above, because the $a_j$ are nonnegative, $x \in S(T|x|)$ for every $x \in \mathbb{F}^{\mathbb{N}}$. Therefore, if $x \in \Lambda_{k+1}$, i.e. if $T|x| \in \Lambda_k$, then $x \in S(T|x|) \subseteq \Lambda_k$. Thus, $\Lambda_{k+1} \subseteq \Lambda_k$. □

**Corollary 5** *$\Lambda_j = \lambda$ for some $j > 0$ if and only if $\Lambda_j = \lambda$ for all $j > 0$.*

When $\lambda$ is a Banach lattice with a solid norm $\|.\|_\lambda$, then $(\Lambda_k, \tau_k)$ is a Banach lattice with a solid norm $\|.\|_k$, defined by $\|x\|_k = \|\ T^k|x|\ \|_\lambda$. Sometimes $\|.\|_\lambda$ will be denoted $\|.\|_0$, because $\lambda = \Lambda_0$.

**Proposition 6** *$\Lambda_1 = \lambda$ if and only if $T$ multiplies $\lambda$ into $\lambda$. When $\lambda$ is a solid Banach lattice (with a solid norm) and $T$ multiples $\lambda$ into $\lambda$, then multiplication by $T$ is a bounded linear operator on $\lambda$. Furthermore, $\|.\|_0$ and $\|.\|_1$ are equivalent norms on $\lambda$ in this case.*

**Proof** Suppose that $T$ multiplies $\lambda$ into $\lambda$. If $x \in \lambda$ then $|x| \in \lambda$, so $T|x| \in \lambda$. Thus, $\lambda \subseteq \Lambda_1$. By Proposition 4, it follows that $\lambda = \Lambda_1$.

Suppose that $\lambda = \Lambda_1$. If $x \in \lambda$ then $|x| \in \lambda = \Lambda_1$, so $T|x| \in \lambda$. Since $Tx \in S(\ T|x|\ )$, it follows that $Tx \in \lambda$. Since $x \in \lambda$ was arbitrary, it follows that $T$ multiplies $\lambda$ into $\lambda$ .

Still supposing that $\lambda = \Lambda_1$, we have that the identity injection from $(\lambda, \|.\|_1)$ to $(\lambda, \|.\|_0)$ is onto, and is therefore an open mapping. Consequently, the inverse injection from $(\lambda, \|.\|_0)$ onto $(\lambda, \|.\|_1)$ is continuous, meaning that for some $C > 0$, $\|x\|_1 = \|\ T|x|\ \|_0 \leq C\|x\|_0$ *for all* $x \in \lambda$.

Since we already have $\|x\|_1 \geq \|x\|_0$ for all $x \in \Lambda_1 = \lambda$ (because $x \in S(\ T|x|\ )$), it follows that $\|.\|_0$ and $\|.\|_1$ are equivalent norms on $\lambda = \Lambda_1$.

Since $T : (\Lambda_1, \|.\|_1) = (\lambda, \|.\|_1) \to (\lambda, \|.\|_0)$ is continuous, it follows that $T : (\lambda, \|.\|_0) \to (\lambda, \|.\|_0)$ is continuous. □

Surprisingly, $\lambda = \Lambda_1$ does not imply that $T$ multiplies $\lambda$ *onto* $\lambda$.

**Proposition 7** *Suppose that $T$ multiplies $\lambda$ into $\lambda$. Then $T$ multiplies $\lambda$ onto $\lambda$ if and only if $T^{-1}$ multiplies $\lambda$ into $\lambda$.*

**Proof** Since both $T$ and $T^{-1}$ are invertible lower triangular matrices, multiplication by either takes $\mathbb{F}^{\mathbb{N}}$ one-to-one onto $\mathbb{F}^{\mathbb{N}}$, and these maps are inverses of each other. From this the conclusion follows, not only for solid subspaces $\lambda$ of $\mathbb{F}^{\mathbb{N}}$, but for any subset of $\mathbb{F}^{\mathbb{N}}$. $\square$

**Proposition 8** *Suppose that $T$ multiplies $\lambda$ onto $\lambda$. Let $X \subseteq \prod_{k=0}^{\infty} \Lambda_k$ be the projective limit defined in the introduction, with respect to $T$ and $\lambda$. Then*

$$X = \{(x, T^{-1}x, T^{-2}x, \ldots) \ : \ x \in \lambda\}, \text{ and}$$

*$P : X \to \lambda$, defined by $P(x, T^{-1}x, T^{-2}x, ...) = x$, is a linear isomorphism of $X$ onto $\lambda$. If $\lambda$ is equipped with a solid t.v.s. topology $\tau$ and $X$ is equipped with the relative topology from the product topology on $\prod_{k=0}^{\infty} \Lambda_k$, then $P$ is continuous, as a linear map between topological vector spaces. If $\tau$ is a locally convex complete metric topology, then $P^{-1}$ is continuous as well, so $\lambda$ and $X$ are isomorphic as topological vector spaces.*

**Proof** By Propositions 6 and 7, $\Lambda_k = \lambda$ for all $k$, and, by the proof of Proposition 7, $T^{-1}$ multiplies $\lambda$ onto $\lambda$. Therefore, for any $k > 0$ and $x \in \lambda$, $T^{-k}x = (T^{-1})^k x \in \lambda = \Lambda_k$. Therefore,

$$\begin{aligned} X &= \{(x, T^{-1}x, T^{-2}x, \ldots) \ : \ T^{-k}x \in \Lambda_k \text{ for all } k > 0\} \\ &= \{(x, T^{-1}x, T^{-2}x, \ldots) \ : \ x \in \lambda\}. \end{aligned}$$

If $\lambda$ is equipped with a solid t.v.s. topology $\tau$, then each $\Lambda_k$ inherits a solid t.v.s. topology $\tau_k$ from $\tau$ and the action of $T$. Clearly $P$ is the restriction of projection onto $\Lambda_0$ on $\prod_{k=0}^{\infty} \Lambda_k$, and this is continuous with respect to the product topology on $\prod_{k=0}^{\infty} \Lambda_k$ and $\tau$ on $\Lambda_0 = \lambda$. (This holds whether or not $T$ multiplies $\lambda$ onto $\lambda$.)

Now suppose that $\tau$ is a locally convex, complete metric topology. Then so is the product topology on $\prod_{k=0}^{\infty} \Lambda_k$. Clearly $X$ is a closed subspace of $\prod_{k=0}^{\infty} \Lambda_k$. (This holds for any solid t.v.s. topology $\tau$, and without the assumption that $T$ multiplies $\lambda$ onto $\lambda$; it suffices to note that multiplication by $T$ maps $(\Lambda_k, \tau_k)$ continuously into $(\Lambda_{k-1}, \tau_{k-1})$, for $k > 0$.)

Therefore, $P$ is a one-to-one continuous map from one locally convex complete metric t.v.s. onto another. By the open mapping theorem for such spaces, $P^{-1}$ is continuous. $\square$

**Example 9** Let

$$T = T(1, 1, 0, 0, \ldots) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \ldots & & \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots & \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots & \\ 0 & 0 & 1 & 1 & 0 & 0 & \ldots & \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}.$$

Then

$$T^{-1} = T(1,-1,1,-1,\ldots) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \ldots & & \\ -1 & 1 & 0 & 0 & 0 & 0 & \ldots & \\ 1 & -1 & 1 & 0 & 0 & 0 & \ldots & \\ -1 & 1 & -1 & 1 & 0 & 0 & \ldots & \\ 1 & -1 & 1 & -1 & 1 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}.$$

Clearly $T$ maps many $\lambda$ into themselves, for instance, all $\lambda = l_p$, $0 < p \leq \infty$. For $1 < p \leq \infty$, $(1,0,\frac{1}{2},0,\frac{1}{3},0,\ldots) \in l_p$ while clearly $T^{-1}x \notin l_\infty$, so $T^{-1}x \notin l_p$. By Propositions 6 and 7, if $\lambda = l_p$, $0 < p \leq \infty$, then $\Lambda_1 = sol - T^{-1}(\lambda) = \lambda$, and $T$ multiplies $\lambda$ into $\lambda$, but not onto $\lambda$.

The next example will show that it can easily happen that $T$ maps $\lambda$ onto $\lambda$. To understand this example, the reader will need to be acquainted with the connection between Toeplitz matrices and formal power series. Suppose $(a_1, a_2, \ldots) = a$ is a sequence of scalars, not necessarily nonnegative, and $a_1$ may be $0$. Let $f(z) = \sum\limits_{n=1}^{\infty} a_n z^{n-1}$, and let

$$T(f) = T(a) = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & \ldots & & \\ a_2 & a_1 & 0 & 0 & 0 & 0 & \ldots & \\ a_3 & a_2 & a_1 & 0 & 0 & 0 & \ldots & \\ a_4 & a_3 & a_2 & a_1 & 0 & 0 & \ldots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}.$$

If $b = (b_n)_{n\geq 1}$ is another sequence of scalars, let $a \star b = (\sum\limits_{k=1}^{n} a_k b_{n-k+1})_{n\geq 1}$, the sequence of coefficients of the formal power series product $(f \centerdot g)(z) = f(z)g(z)$. If $f(z)$ and $g(z)$ represent functions in a neighborhood of $0$, i.e. if the radii of convergence of these series are both positive, then the power series $f(z)g(z)$ represents the product of the functions $f$ and $g$ in that neighborhood of $0$.

The following matrix products are elementary:

$$T(a) \centerdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} = T(a) \centerdot b = a \star b,$$

and $T(a)T(b) = T(a \star b)$.

Therefore, when $f(z)$ and $g(z)$ represent functions in a neighborhood of $0$, $T(f)T(g) = T(f \centerdot g)$.

Therefore, if $f(z)$ has radius of convergence $\geq r > 0$ and $f(z) \neq 0$ for all complex $z$ such that $|z| < r$, then, because $\frac{1}{f}$ is analytic in the open disc of radius $r$ centered at $0$, the function $\frac{1}{f}$ has a power series representation about $0$, in that disc, and $T(\frac{1}{f}) = T(f)^{-1}$. For instance, in Example 9, $T = T(1,1,0,\ldots) = T(1+z)$, so

$$T^{-1} = T(\frac{1}{1+z}) = T(\sum_{n=1}^{\infty}(-z)^{n-1}) = T(1,-1,1,-1,\ldots).$$

**Example 10** Fix $\rho \in (0,\infty]$ and let

$$\begin{aligned}\lambda &= \{(x_n) \in \mathbb{F}^{\mathbb{N}} \ : \ \sum_{n=1}^{\infty} x_n z^{n-1} \ \textit{has radius of convergence} \ \geq \rho\} \\ &= \{(x_n) \in \mathbb{F}^{\mathbb{N}} \ : \ \frac{1}{\limsup_n |x_n|^{\frac{1}{n}}} \geq \rho\}.\end{aligned}$$

$[Convention : \frac{1}{0} = \infty]$. Clearly $\lambda$ is a solid vector subspace of $\mathbb{F}^{\mathbb{N}}$. If $(1, a_2, a_3, \ldots) \in \lambda$, $a_i \geq 0$, $i = 2, 3, \ldots$, and $f(z) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1} \neq 0$ for all $z \in \mathbb{C}$, $|z| < \rho$, then both $f$ and $\frac{1}{f}$ multiply the space of analytic functions on the disc $\{z \in \mathbb{C} \ : \ |z| < \rho\}$ onto itself; therefore, $T = T(f)$ and $T^{-1} = T(\frac{1}{f})$ multiply $\lambda$ onto itself.

For a specific instance, let $\rho = 1$ and $T, T^{-1}$ be the matrices of Example 9, corresponding to $f(z) = 1 + z \neq 0$ for all $z \in \mathbb{C}$ such that $|z| < 1$. But obviously, the possible examples of such $T$ and $T^{-1}$ are an uncountable legion.

If $\lambda$ is as in Example 10, for some $\rho > 0$, $a_i \geq 0$, $i = 2, 3, \ldots$, and $f(z) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1}$ converges for all $|z| < \rho$, but $f$ takes the value 0 in that open disc, then the power series for $\frac{1}{f}$ will have a radius of convergence in the open interval $(0, \rho)$. In such a case, $T(f)$ will multiply $\lambda$ into $\lambda$, and $T(\frac{1}{f}) = T(f)^{-1}$ will not, so $T(f)$ does not multiply $\lambda$ onto $\lambda$. These remarks do not imply the conclusions of Example 9, because the $l_p$ are quite a different case from the spaces $\lambda$ of Example 10.

A sequence space $\lambda$ as in Example 10 bears a natural, solid, metric, locally convex t.v.s. topology called the "topology of uniform convergence on compact subsets of the disc $U(\rho) = \{z \in \mathbb{C} \ : \ |z| < \rho\}$. As the norm suggests, this topology is usually described on the space of analytic functions on $U(\rho)$ associated with $\lambda$, by the association $(x_n) \in \lambda \leftrightarrow f(z) = \sum_{n=1}^{\infty} x_n z^{n-1}$. The space $\lambda$ is complete with this topology. The topology is not normable, but it can be described by a sequence of norms as follows: let $(r_k)_{k\geq 1}$ be any increasing sequence of positive reals, tending to $\rho$, and define $\|.\|_k$ by $\|x\|_k = \sum_{n=1}^{\infty} |x_n| r_k^{n-1}$. It is not completely trivial to see that the topology on $\lambda$ defined by the sequence of norms $\|.\|_1, \|.\|_2, \ldots$ is the same as the topology of uniform convergence on compact subsets of $U(\rho)$ as usually defined on the function space associated with $\lambda$; the conclusion follows by elementary functional analysis from the observations that the sequence-of-norms topology is the finer of the two, and that they are both complete metric l.c.t.v.s. topologies.

An important payoff from the verification of the equivalence of the two topologies is the conclusion that if $x \in \lambda$ and $T(x)$ is the Toeplitz matrix associated with $x$, then not only does $T(x)$ multiply $\lambda$ into $\lambda$, but also, this multiplication is a continuous linear operator on $\lambda$, equipped with its natural topology.

Before looking at our last example, which concerns the Toeplitz matrix $T(1, \frac{1}{2}, \frac{1}{3}, \ldots)$, we resurrect a result from [7], which bears on the preceding discussion and on the questions to be posed in the last section.

If $B = [b_{ij}]$ is an infinite matrix with nonnegative entries, we will permit $B$ to be raised to positive integer powers even if $\infty$ appears as a matrix entry. In the arithmetic performed in taking these products, as usual, $c.\infty = \infty$ for any $c > 0$, $0.\infty = 0$, and any sum of nonnegative terms including $\infty$ is $\infty$. If $\infty$ appears in the $j^{th}$ column of $B^k$ then for $x \in \mathbb{F}^{\mathbb{N}}$, $B^k|x| \in \mathbb{F}^{\mathbb{N}}$ (i.e. the entries of $B^k|x|$ are all finite) only if $x_j = 0$.

**Proposition 11** *Suppose that $B = [b_{ij}]$ is an infinite matrix with nonnegative entries, and $\lambda$ is a solid subspace of $\mathbb{F}^{\mathbb{N}}$ containing $c_{00}$. For $k \geq 1$ let $\Lambda_k = \{x \in \mathbb{F}^{\mathbb{N}} \ : \ B^k|x| \in \lambda\}$, and let*

$$X = \{(x^{(k)})_{k\geq 0} \in \prod_{k=0}^{\infty} \Lambda_k \ : \ for\ all\ k > 0,\ Bx^{(k)} = x^{(k-1)}\}.$$

*(a) If every column of $B$ is finitely nonzero (i.e. in $c_{00}$), then $c_{00} \subseteq \Lambda_k$ for all $k \geq 1$.*
*(b) If $B$ has a matricial inverse, $B^{-1}$, with every column of $B^{-1}$ in $c_{00}$, and $c_{00} \subseteq \Lambda_k$ for all $k = 1, 2, \ldots$, then $X$ is nontrivial.*

**Proof** (a) If $x \in c_{00}$, then $Bx$ is a finite linear combination of columns of $B$; since each of those columns is in $c_{00}$, it follows that $Bx \in c_{00}$.

Thus, every column of $B^2$ is in $c_{00}$; by induction on $k$, every column of $B^k = B.B^{k-1}$ is in $c_{00}$. If $x \in c_{00}$, then $|x| \in c_{00}$, so $B^k|x| \in c_{00} \subseteq \lambda$. Thus, $c_{00} \subseteq \Lambda_k$, $k = 1, 2, \ldots$.

(b) By the argument in (a), if the columns of $B^{-1}$ are finitely nonzero, then so are the columns of $(B^{-1})^k = B^{-k}$, $n = 1, 2, \ldots$ By the argument in (a), $B^{-k}$ has finite entries, in fact columns in $c_{00}$, for all positive integers $k$.

Because $B^{-k}$ has columns in $c_{00}$, for every $x \in c_{00}$, $B^{-k}x \in c_{00} \subseteq \Lambda_k$, by hypothesis. Therefore, for every $x \in c_{00} \subseteq \lambda$, $(x, B^{-1}x, B^{-2}x, \ldots) \in X$. □

**Corollary 12** *(a) Suppose that $B = T(f)$ where $f(z)$ is a polynomial in $z$ with nonnegative real coefficients. Then, for any $\lambda$ as in Example 10 and with $\Lambda_k$ as defined there, $c_{00} \subseteq \Lambda_k$, $k = 1, 2, \ldots$*

*(b) Suppose that $p(z)$ is a polynomial with real coefficients such that*

$$f(z) = \frac{1}{p(z)} = \sum_{j=1}^{k} \frac{c_j}{(a_j - z)^{n_j}}$$

*for some positive real numbers $a_1, a_2, \ldots, a_k, c_1, \ldots, c_k$ and positive integers $n_1, \ldots, n_k$. Let $B = T(f)$, and let $\lambda, \Lambda_k$, $k = 1, 2, \ldots$, and $X$ be as in Example 10. If $c_{00} \subseteq \Lambda_k$ for all $k = 1, 2, \ldots$, then $X$ is nontrivial.*

**Proof** (a) Obviously the columns of $B$ are nonnegative finitely nonzero sequences. The conclusion follows from Proposition 11 (a).

(b) The entries of the Toeplitz matrix $B = T(f)$ are nonnegative, because the coefficients of the power series for $f$ around 0 are nonnegative. The inverse of $B$ is $B^{-1} = T(\frac{1}{f}) = T(p)$, which has all columns in $c_{00}$ because $p(z)$ is a polynomial. The conclusion now follows from Proposition 11 (b). □

**Example 13** Let $T = T(1, \frac{1}{2}, \frac{1}{3}, \ldots) = T(f)$, where $f(z) = 1 + \sum\limits_{n=2}^{\infty} \frac{z^{n-1}}{n}$. Obviously, $f$ is analytic on the open unit disc $\{z \in \mathbb{C} \ : \ |z| < 1\}$, and it is also nonzero there. To see this, observe that for $|z| < 1$,

$$-\ln(1-z) = \int_0^z \frac{dt}{1-t} = \int_0^z \sum_{k=0}^{\infty} t^k dt = \sum_{k=1}^{\infty} \frac{z^k}{k},$$

from which it follows that $f(z) = -\frac{\log(1-z)}{z}$, where log is defined (and analytic) in the right half plane by $\log(w) = \ln|w| + i\arg(w)$, with $\arg(w) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ satisfying $w = |w|e^{i\arg(w)}$. Since $\log w = 0$ only for $w = 1$, if $Re(w) > 0$, it follows that $f(z) \neq 0$ for all $z$ such that $|z| < 1$.

Therefore, if $\lambda$ is as in Example 10, for any $\rho \in (0,1]$, both $T$ and $T^{-1} = T(\frac{1}{f})$ multiply $\lambda$ onto $\lambda$. Therefore, $\Lambda_k = \lambda$ for all $k > 0$, and, by Proposition 8, we know a lot about the projective limit $X$.

But what if $\lambda = l_p$ for some $p \in (1, \infty]$? Here we have many questions and few answers. We will ask the questions in the next section. Here we will put forth the few things that we know. Let $e_j$ denote the sequence with 1 in the $j^{th}$ position.

If a lower triangular one-sided infinite matrix or, indeed, simply a matrix with every row in $c_{00}$ multiplies a Banach lattice of sequences in which the coordinate projections are continuous into another such space, then that multiplication is a bounded linear transformation. This conclusion is straightforward from the closed graph theorem. (Perhaps we should have mentioned this earlier! It provides an alternate proof of part of Proposition 6.)

With this in mind we can see that $T$ does not multiply $l_p$ into $l_p$, $1 < p \leq \infty$. For $p = \infty$ this is easy: $T$ multiplies the sequence of all ones into an unbounded sequence. In the case of $1 < p < \infty$, we observe that

$$\frac{\|T(\sum_{j=1}^{n} e_j)\|_p}{\|\sum_{j=1}^{n} e_j\|_p} \to \infty \ as \ n \to \infty.$$

(Verification of this is left as a pleasurable exercise for the reader.)

Therefore, $T$ does not multiply $l_p$ into itself, for, if it did, that multiplication would have to be a bounded linear operator on $l_p$, by the remark above, and that manifestly cannot be so.

By Propositions 4 and 6, it follows that $\Lambda_1 = sol - T^{-1}(l_p) \subsetneq l_p$, for $1 < p \leq \infty$. Can it be that the sequence $l_p = \Lambda_0 \supseteq \Lambda_1 \supseteq \Lambda_2 \ldots$, shrinks to $\underline{0}$, i.e. that $\bigcap_{k\geq 0} \Lambda_k = \{\underline{0}\}$?

We shall show that this is not the case by showing that $c_{00} \subseteq \Lambda_k$ for all $k$. This is clear for $k = 0, 1$.

To show that $c_{00} \subseteq \Lambda_k$ it suffices to show that $T^k e_j \in l_p$, j=1,2,.... Since $T^k$ is a Toeplitz matrix, $T^k e_j$ is just a shift of $T^k e_1$, so it suffices to show that $T^k e_1 \in l_p$. If this holds for any $p < \infty$ then it holds for $p = \infty$, so we may as well suppose that $p < \infty$.

We will show that $T^k e_1 \in l_p$ by showing, by induction on $k \geq 1$, that

$$(T^k e_1)_n \leq \frac{2^{k-1}(1 + \ln n)^{k-1}}{n + k - 1}, \ n \geq 1.$$

This will do it, since $(\frac{2^{k-1}(1+\ln n)^{k-1}}{n+k-1})_{n\geq 1} \in l_p$ for all $k \geq 1$, as it is quite easy to see.

$$(Te_1)_n = \frac{1}{n} = \frac{2^{1-1}(1+\ln n)^{1-1}}{n+1-1},$$

so the claim holds for $k=1$. Suppose that $k>1$. Note that

$$(T^k e_1)_1 = 1 \leq \frac{2^{k-1}}{k},$$

so we may as well suppose that $n>1$. Let $(T^{k-1}e_1)_n = b_n, n = 1, 2, \ldots$ By induction hypothesis, for all $n \geq 1$, $b_n \leq \frac{2^{k-2}(1+\ln n)^{k-2}}{n+k-2}$. Then

$$\begin{aligned}
(T^k e_1)_n &= (TT^{k-1}e_1)_n = \sum_{j=1}^{n} \frac{1}{j} b_{n-j+1} \\
&\leq \sum_{j=1}^{n} \frac{1}{j} \frac{2^{k-2}(1+\ln(n-j+1))^{k-2}}{n-j+k-1} \\
&\leq \frac{2^{k-2}(1+\ln n)^{k-2}}{n+k-1} \sum_{j=1}^{n} (\frac{1}{j} + \frac{1}{n-j+k-1}) \\
&\leq \frac{2^{k-2}(1+\ln n)^{k-2}}{n+k-1} \sum_{j=1}^{n} (\frac{1}{j} + \frac{1}{n-j+1}) \\
&\leq \frac{2^{k-2}(1+\ln(n))^{k-2}}{n+k-1} 2(1+\ln n) = \frac{2^{k-1}(1+\ln(n))^{k-1}}{n+k-1}.
\end{aligned}$$

## 3. Questions

Throughout this section $T = T(1, a_2, a_3, \ldots)$, $a_j \geq 0$, $j \geq 2$, and $\lambda$ will be a solid vector subspace of $\mathbb{F}^{\mathbb{N}}$ containing $c_{00}$ and the columns of $T$; $\Lambda_k$, $k = 0, 1, 2, \ldots$, and $X$ will be as defined earlier, with reference to $T$ and $\lambda$.

1. If $T$ multiplies $\lambda$ onto $\lambda$, then by Propositions 4, 6, and 8, $\Lambda_k = \lambda$ for all $k$ and $X = \{(x, T^{-1}x, T^{-2}x, \ldots) : x \in \lambda\}$. If $T$ multiplies $\lambda$ into $\lambda$, but not onto, then $\Lambda_k = \lambda$ for all $k$, but

$$\begin{aligned}
\lambda_X &= \{x \in \lambda : T^{-k}x \in \Lambda_k,\ k = 1, 2, \ldots\} \\
&= \{x \in \lambda : (x, T^{-1}x, T^{-2}x, \ldots) \in X\}
\end{aligned}$$

is a proper subspace of $\lambda$.

Is it possible that $\lambda_X = \{\underline{0}\}$ if $\Lambda_k = \lambda$ for all $k$?

Is it possible that $\lambda_X$ could be a solid subspace of $\lambda$ other than $\{\underline{0}\}$ or $\lambda$ itself, whether $\Lambda_k = \lambda$ for all $k$, or not?

2. Are there examples of $\lambda$ and $T$ such that $\Lambda_k = \{\underline{0}\}$ for some $k > 1$ ? (Because the columns of $T$ are in $\lambda$, $c_{00} \subseteq \Lambda_1$.)

Are there examples of $\lambda$ and $T$ such that $\Lambda_k \neq \{\underline{0}\}$ for all $k$, but $\bigcap_{k\geq 0} \Lambda_k = \{\underline{0}\}$?

Are there examples of $\lambda$ and $T$ such that $\Lambda_1 \subsetneq \lambda$, $\Lambda_k \neq \{\underline{0}\}$ for all $k$, and $\bigcap_{k \geq 0} \Lambda_k = \Lambda_m$ for some $m$?

3. We are especially interested in all the questions above when $T = T(1, \frac{1}{2}, \frac{1}{3}, \ldots)$ and $\lambda = l_p$, $1 < p < \infty$.

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