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Numerical approach for solving space fractional order diffusion equations using shifted Chebyshev polynomials of the fourth kind

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Abstract: In this paper, a new approach for solving space fractional order diffusion equations is proposed. The fractional derivative in this problem is in the Caputo sense. This approach is based on shifted Chebyshev polynomials of the fourth kind with the collocation method. The finite difference method is used to reduce the equations obtained by our approach for a system of algebraic equations that can be efficiently solved. Numerical results obtained with our approach are presented and compared with the results obtained by other numerical methods. The numerical results show the efficiency of the proposed approach.

Key words: Space fractional order diffusion equation, Caputo derivative, Chebyshev collocation method, finite difference method, Chebyshev polynomials of the fourth kind; Euler approximation

1. Introduction

Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of many materials and processes [12, 14]. This is the main advantage of the fractional order derivatives in comparison with the classical integer order models, in which such effects are in fact neglected [19, 26]. The advantages of the fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks and in many other fields [19, 23, 25]. It should be mentioned that there are also applications from the viewpoint of many fields such as physics, chemistry, engineering, finance, and other sciences that have been developed in the last few decades [12, 24, 25, 27, 28, 35, 43]. A large number of authors studied fractional differential equations, such as [1, 6, 7, 32, 39].

Over the past four decades the appeal of spectral methods for applications such as computational fluid dynamics has expanded, commensurate with the erosion of most of the obstacles to their wider application [5, 8, 44]. They have been applied successfully to numerical simulations in many fields. They have gained wide popularity in automatic computations for a wide class of physical problems in fluid and heat flow[8, 9, 14].

In recent decades, the Chebyshev polynomials have become some of the most useful polynomials that are suitable in numerical analysis including polynomial approximation, integral and differential equations, spectral methods for partial differential equations, and fractional order differential equations (see [9, 12, 20, 33, 36, 42]).

The fractional order diffusion equation is considered one of the attractive concepts in the initial boundary

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value problems in fractionals. It has been used in modeling turbulent flow [10], groundwater contaminant transport [4], chaotic dynamics of classical conservative systems [28, 45], applications in biology [16], and applications in other fields [5, 12, 21, 33]. Later there appeared several numerical methods for solving fractional diffusion equations, such as those in [2, 3, 21, 22] where the authors used the finite difference method. Cui [11] and Geo and Sun [17] used the compact finite difference scheme. On the other hand, Sweilam et al. [37, 38, 40, 41] used the Chebyshev collocation method with finite difference method for solving fractional diffusion equations. Saaddmant and Dehghan [29–31] used an operational matrix. Moreover, there are many methods for solving fractional order diffusion equations [13, 15, 18, 21, 22, 34, 37].

This paper has essentially been motivated by the enormous numbers of very interesting and novel applications for fractional diffusion equations. We have used shifted Chebyshev polynomials of the fourth kind and we recall some of their important properties and analytical forms. We have used these polynomials to approximate the numerical solutions of fractional diffusion equations with the assistance of the Chebyshev collocation method together with the finite difference method to convert the system of equations into algebraic equations so that they can be solved.

The structure of this paper is as follows: the next section recalls some classical facts from calculus that form the basis of our purpose in this work. In Section 3, we introduce the fundamental concepts, definitions, and some properties of Chebyshev polynomials of the fourth kind that are necessary in what follows. In Section 4, we introduce the main theorem of our technique for solving space fractional order diffusion equations subject to homogeneous and nonhomogeneous boundary conditions using shifted Chebyshev polynomials of the fourth kind. The numerical scheme is explained in Section 5. In Section 6, we present numerical examples to exhibit the accuracy and the efficiency of our proposed method, in which our numerical results are computed using MATLAB. Conclusions are presented in Section 7.

2. Preliminaries

Definition 1. The Caputo fractional derivative operator D^μ of order μ is defined as follows [26]:

$$D^\mu f(x) = \frac{1}{\Gamma(m-\mu)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\mu-m+1}} dt, \quad \mu > 0, \tag{1}$$

where $m-1 < \mu \leq m, \quad m \in N, \quad x > 0$. The linear property of the Caputo fractional derivative is similar to integer order differentiation:

$$D^\mu(\lambda f(x) + \gamma g(x)) = \lambda D^\mu f(x) + \gamma D^\mu g(x), \tag{2}$$

where λ and γ are constants.

For the Caputo derivative we can obtain the following result:

$$D^\mu k = 0, \quad k \textit{ is a constant}, \tag{3}$$

.

$$D^\mu x^n = \begin{cases} 0, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n < \lceil \mu \rceil, \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\mu)} x^{(n-\mu)}, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n \geq \lceil \mu \rceil. \end{cases} \tag{4}$$

The function $\lceil \mu \rceil$ is used to denote the smallest integer greater than or equal to μ. Also, $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Recall that for $\mu \in N$ the Caputo differential operator coincides with the usual differential operator of integer order. For more details about fractional derivative definitions, theorems, and their properties see [14, 26]. The main objective of this work is to provide a solid foundation that may later be used for the construction of an efficient and reliable numerical method for solving space fractional order diffusion equations using shifted Chebyshev polynomials of the fourth kind. Consider the one-dimensional space fractional order diffusion equation of the form

$$\frac{\partial u(x,t)}{\partial t} = p(x)\,\frac{\partial^{\mu} u(x,t)}{\partial x^{\mu}} + q(x,t) \tag{5}$$

on a finite domain $0 < x < L$, $0 < t \leq T$ with the parameter μ referring to the fractional order of the spatial derivative with $1 < \mu \leq 2$. The function $q(x,t)$ is the source term. We also assume an initial condition:

$$u(x,0) = f(x), \quad 0 < x < L, \tag{6}$$

and the boundary conditions:

$$u(0,t) = y_0(t), \quad 0 < t \leq T \tag{7}$$

$$u(L,t) = y_1(t), \quad 0 < t \leq T. \tag{8}$$

In the case of $\mu = 2$, Eq. (5) is the classical second order diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = p(x)\,\frac{\partial^{2} u(x,t)}{\partial x^{2}} + q(x,t). \tag{9}$$

3. Some properties of Chebyshev polynomials of the fourth kind

3.1. Chebyshev polynomials of the fourth kind

Definition 2. The Chebyshev polynomials $W_n(x)$ of the fourth kind are orthogonal polynomials of degree n in x defined on $[-1,1]$ (see Mason and Handscomb [20]):

$$W_k(x) = \frac{\sin(k+\frac{1}{2})\theta}{\sin\frac{\theta}{2}},$$

where $x = \cos\theta$ and $\theta \in [0,\pi]$. They can be obtained explicitly using the Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$, for the special case $\beta = -\alpha = \frac{1}{2}$. These are given by:

$$W_k(x) = \frac{2^{2k}}{\binom{2k}{k}}\, P_k^{(\frac{1}{2},-\frac{1}{2})}(x), \tag{10}$$

where

$$P_k^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+k+1)}{\Gamma(k+1)\Gamma(\alpha+\beta+k+1)} \sum_{m=0}^{k} \binom{k}{m} \frac{\Gamma(\alpha+\beta+k+m+1)}{\Gamma(\alpha+m+1)} \left(\frac{x-1}{2}\right)^{m}. \tag{11}$$

These polynomials $W_n(x)$ are orthogonal on $[-1,1]$ with respect to the inner product:

$$\langle W_n(x), W_m(x)\rangle = \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\, W_n(x)\, W_m(x)\, dx = \begin{cases} 0, & n \neq m, \\ \pi, & n = m, \end{cases} \tag{12}$$

where $\sqrt{\dfrac{1-x}{1+x}}$ is the weight function corresponding to $W_n(x)$. Moreover, the polynomials $W_n(x)$ may be generated by using the recurrence relations

$$W_{n+1}(x) = 2x\,W_n(x) - W_{n-1}(x), \quad n = 1, 2, \ldots,$$

with starting values

$$W_0(x) = 1, \quad W_1(x) = 2x + 1, \quad W_2(x) = 4x^2 + 2x - 1.$$

There is an important relationship between Chebyshev polynomials of the second kind and those of the fourth kind, given as follows:

$$W_n(x) = U_{2n}\left(\frac{1+x}{2}\right)^{1/2}. \tag{13}$$

From the analytical form of the Chebyshev polynomials of the second kind given in [40] together with Eq. (13), the analytical form of the Chebyshev polynomials of the fourth kind $W_n(x)$ of degree n can be expressed as follows:

$$W_n(x) = \sum_{i=0}^{n} (-1)^i (2)^{n-i} \frac{\Gamma(2n-i+1)}{\Gamma(i+1)\Gamma(2n-2i+1)} (1+x)^{n-i} \qquad n \in \mathbb{Z}^+. \tag{14}$$

3.2. Shifted Chebyshev polynomials of the fourth kind

We may define Chebyshev polynomials appropriate to any given finite range $[a, b]$ of x by making this range correspond to the range $[-1, 1]$ of new variables under the linear transformation:

$$s = \frac{2x - (a+b)}{(b-a)}. \tag{15}$$

The Chebyshev polynomials of the fourth kind appropriate to $[a, b]$ are thus $W_n(s)$, where s is given by Eq. (15) (see [20]). Moreover, since the range $[0, 1]$ is quite often more convenient to be used than the range $[-1, 1]$, the range $[-1, 1]$ was adjusted to the range $[0, 1]$ for convenience, and this corresponds to the use of the shifted Chebyshev polynomials of all kinds. However, in this work, we focus on the shifted Chebyshev polynomials of the fourth kind $W_n^*(x)$ of degree n in x on $[0, 1]$ given by:

$$W_n^*(x) = W_n(2x - 1).$$

These polynomials are orthogonal on the support interval $[0, 1]$ as the following inner product:

$$\langle W_n^*(x),\, W_m^*(x)\,\rangle = \int_0^1 \sqrt{\frac{1-x}{x}}\; W_n^*(x)\, W_m^*(x)\, dx = \begin{cases} 0, & n \neq m, \\ \dfrac{\pi}{2}, & n = m, \end{cases} \tag{16}$$

where $\sqrt{\dfrac{1-x}{x}}$ is weight function corresponding to $W_n^*(x)$ and normalized by the requirement that $W_n^*(1) = 1$. Also, $W_n^*(x)$ may be generated by using the recurrence relations

$$W_{n+1}^*(x) = 2(2x-1)\,W_n^*(x) - W_{n-1}^*(x), \quad n = 1, 2, \ldots,$$

with starting values:

$$W_0^*(x) = 1, \quad W_1^*(x) = 4x - 1, \quad W_2^*(x) = 16x^2 - 12x + 1.$$

The analytical form of the shifted Chebyshev polynomials of the fourth kind $W_n^*(x)$ of degree n in x is given by

$$W_n^*(x) = \sum_{i=0}^{n} (-1)^i (2)^{2n-2i} \frac{\Gamma(2n-i+1)}{\Gamma(i+1)\Gamma(2n-2i+1)}\, x^{n-i}, \qquad n \in \mathbb{Z}^+. \tag{17}$$

It is clear that the analytical form given in Eq. (17) can be obtained by the analytical form of Chebyshev polynomials of the second kind given in [40] and the relationship between $W_n^*(x)$ and $U_n(x)$, which can be generated by using the following relation:

$$U_{2n}(x) = W_n^*(x^2).$$

In a spectral method, on the assumption that it is possible to expand a given function $g(x)$ in a (suitably convergent) series based on a system $\phi_n(x)$ of polynomials orthogonal over the interval $[a,b]$, $\phi_n(x)$ being of exact degree n, we may write

$$g(x) = \sum_{n=0}^{\infty} c_n \phi_n(x), \quad x \in [a,b], \tag{18}$$

and it follows, by taking inner products with $\phi_i(x)$, that

$$\langle g(x), \phi_i(x) \rangle = \sum_{n=0}^{\infty} c_n \langle \phi_n(x), \phi_i(x) \rangle.$$

Now the function $g(x)$, square integrable in $[0,1]$, is represented by an infinite expansion of the shifted Chebyshev polynomials of the fourth kind as follows:

$$g(x) = \sum_{i=0}^{\infty} c_i \; W_i^*(x), \tag{19}$$

where c_i are constants. One then proceeds to estimate as many as possible of the coefficients c_i, thus approximating $g(x)$ by a finite sum of $(m+1)$-terms such as:

$$g_m(x) = \sum_{i=0}^{m} c_i \; W_i^*(x), \tag{20}$$

where the coefficients c_i, $(i = 0, 1, \ldots, m.)$ are given by

$$c_i = \frac{1}{\pi} \int_{-1}^{1} g\left(\frac{x+1}{2}\right) \sqrt{\frac{1-x}{1+x}}\; W_i(x) dx, \tag{21}$$

or

$$c_i = \frac{2}{\pi} \int_{0}^{1} g(x) \sqrt{\frac{1-x}{x}}\; W_i^*(x) dx. \tag{22}$$

4. Main results

In this section, we will give a derivation of the approximate formula of the fractional derivate of the function $g_m(x)$ given in (20) using shifted Cheybshev polynomials of the fourth kind and properties of the Caputo derivative.

Theorem 1 Let $g_m(x)$ be an approximated function in terms of shifted Chebyshev polynomials of the fourth kind as given in (20) and suppose $\mu > 0$. Then we obtain:

$$D^{\mu}(g_m(x)) = \sum_{i=\lceil\mu\rceil}^{m} \sum_{k=0}^{i-\lceil\mu\rceil} c_i \; V_{i,k}^{(\mu)} \; x^{i-k-\mu}, \tag{23}$$

where

$$V_{i,k}^{(\mu)} = (-1)^k \; 2^{(2i-2k)} \; \frac{\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+1)\Gamma(i-k+1-\mu)}. \tag{24}$$

Proof Using the definition of the approximated function $g_m(x)$ given in Eq. (20) and the Caputo fractional differentiation properties given in Eq. (2) we obtain:

$$D^{\mu}(g_m(x)) = \sum_{i=0}^{m} c_i \; D^{\mu}(W_i^*(x)). \tag{25}$$

The properties of linearity of the Caputo derivative together with Eqs. (3) and (4) are used to claim that:

$$D^{\mu}(W_i^*(x)) = 0, \quad i = 0, 1, ..., \lceil\mu\rceil - 1, \quad \mu > 0. \tag{26}$$

Also, we obtain:

$$D^{\mu}(W_i^*(x)) = \sum_{k=0}^{i} (-1)^k \; 2^{(2i-2k)} \; \frac{\Gamma(2i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+1)} \; D^{\mu} x^{i-k}. \tag{27}$$

Eq. (27) can be rewritten with the aid of using Eqs. (3) and (4) as follows:

$$D^{\mu}(W_i^*(x)) = \sum_{k=0}^{i-\lceil\mu\rceil} (-1)^k \; 2^{(2i-2k)} \; \frac{\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+1)\Gamma(i-k+1-\mu)} \; x^{i-k-\mu}. \tag{28}$$

By combinations Eqs. (25), (26), and (28) we obtain:

$$D^{\mu}(g_m(x)) = \sum_{i=\lceil\mu\rceil}^{m} \sum_{k=0}^{i-\lceil\mu\rceil} c_i \; (-1)^k \; 2^{(2i-2k)} \frac{\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+1)\Gamma(i-k+1-\mu)} x^{i-k-\mu}, \tag{29}$$

and Eq. (29) can be rewritten in the following form:

$$D^{\mu}(g_m(x)) = \sum_{i=\lceil\mu\rceil}^{m} \sum_{k=0}^{i-\lceil\mu\rceil} c_i \; V_{i,k}^{(\mu)} \; x^{i-k-\mu}, \tag{30}$$

where

$$V_{i,k}^{(\mu)} = (-1)^k \; 2^{(2i-2k)} \; \frac{\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+1)\Gamma(i-k+1-\mu)}. \tag{31}$$

□

Simple test: *Consider* $g(x) = x^2$ *with* $m = 3$ *and* $\mu = 1.5$. *Using Eq.* (4) *we obtain:*

$$D^{1.5}x^2 = \frac{\Gamma(2+1)}{\Gamma(2+1-1.5)}x^{(2-1.5)} = \frac{\Gamma(3)}{\Gamma(1.5)}x^{\frac{1}{2}} = \frac{2}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}}.$$

Then, using the proposed method given in Theorem 1 together with Eq. (21), *we obtain:*

$$\begin{aligned} \mathcal{D}^{1.5}x^2 &= \sum_{i=2}^{3}\sum_{k=0}^{i-2} c_i\ V_{i,k}^{1.5}\ x^{(i-k-1.5)} \\ &= c_2\ V_{2,0}^{\frac{3}{2}}\ x^{\frac{3}{2}} + c_3\ V_{3,0}^{\frac{3}{2}}\ x^{\frac{3}{2}} + c_3\ V_{3,1}^{\frac{3}{2}}\ x^{\frac{1}{2}}, \end{aligned} \tag{32}$$

where

$$V_{2,0}^{\frac{3}{2}} = \frac{32}{\Gamma(\frac{3}{2})}, \qquad V_{3,0}^{\frac{3}{2}} = \frac{128}{\Gamma(\frac{3}{2})}, \qquad V_{3,1}^{\frac{3}{2}} = \frac{-32}{\Gamma(\frac{3}{2})}.$$

The constants c_2 *and* c_3 *are computed by using Eq.* (21) *or* (22) *and then by substituting in Eq.* (32) *we get*:

$$D^{1.5}x^2 = \frac{2}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}}.$$

5. Numerical scheme

Consider the fractional order diffusion equation of the type given in Eq. (5) with initial condition as in Eq. (6) and boundary conditions given in Eqs. (7) and (8) respectively. In order to use the Chebyshev collocation method and finite difference method, let us approximate $u(x,t)$ as follows [20]:

$$u_m(x,t) = \sum_{i=0}^{m} u_i(t)\ W_i^*(x). \tag{33}$$

From Eqs. (5) and (23) and Theorem 1 we can claim:

$$\sum_{i=0}^{m} \frac{du_i(t)}{dt}\ W_i^*(x) = p(x) \sum_{i=\lceil\mu\rceil}^{m} \sum_{k=0}^{i-\lceil\mu\rceil} u_i(t)\ V_{i,k}^{(\mu)}\ x^{i-k-\mu} + q(x,t). \tag{34}$$

Now we collocate Eq. (34) at $(m+1-\lceil\mu\rceil)$ points x_p as follows:

$$\sum_{i=0}^{m} \frac{du_i(t)}{dt}\ W_i^*(x_p) = p(x_p) \sum_{i=\lceil\mu\rceil}^{m} \sum_{k=0}^{i-\lceil\mu\rceil} u_i(t)\ V_{i,k}^{(\mu)}\ x_p^{i-k-\mu} + q(x_p,t). \tag{35}$$

We use the roots of shifted Chebyshev polynomials of the fourth kind $W^*_{m+1-\lceil\mu\rceil}(x)$ to suitable collocation points. Also, substitute Eqs. (22) and (33) in the initial to get the constants (u_i) in the initial case at $(t=0)$. Moreover, substitute Eq. (33) in the boundary conditions to get $\lceil\mu\rceil$ equations. For example, by substituting Eq. (33) in Eqs. (7) and (8) respectively, in the case of $0 < x < 1$ we obtain:

$$\sum_{i=0}^{m} (-1)^{(i)}\ u_i(t) = y_0(t), \quad \sum_{i=0}^{m} (2i+1)\ u_i(t) = y_1(t). \tag{36}$$

Notice that if the boundary conditions are zeros by Dirichlet conditions Eq. (36) can be rewritten as:

$$\sum_{i=0}^{m} (-1)^{(i)}\ u_i(t) = 0, \quad \sum_{i=0}^{m} (2i+1)\ u_i(t) = 0. \tag{37}$$

Eq. (35) together with $\lceil \mu \rceil$ equations of the boundary conditions (36) give $(m+1)$ ordinary differential equations, which can be solved numerically using the finite difference method (Euler method) to get the unknown coefficients u_i, $i = 0, 1, ..., m$.

6. Numerical experiments and comparison

Example 1 Consider *Eq.* (5) with $\mu = 1.8$:

$$\frac{\partial u(x,t)}{\partial t} = p(x)\, \frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + q(x,t), \quad 0 < x < 1, \quad t > 0,$$

with the diffusion coefficient

$$p(x) = \Gamma(1.2)\ x^{1.8},$$

the source function

$$q(x,t) = 3x^2\ (2x-1)\ e^{-t},$$

with the initial condition

$$u(x,0) = x^2(1-x),$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t > 0.$$

The exact solution of this problem is given by*:*

$$u(x,t) = x^2(1-x)e^{-t}.$$

Let us consider $m = 3$; then we have*:*

$$u_3(x,t) = \sum_{i=0}^{3}\ u_i(t)\ W_i^*(x). \tag{38}$$

Using *Eq.* (35)*,* we claim*:*

$$\sum_{i=0}^{3} \frac{du_i(t)}{dt}\ W_i^*(x_p) = p(x_p) \sum_{i=2}^{3} \sum_{k=0}^{i-2} u_i(t)\ V_{i,k}^{(1.8)}\ x_p^{i-k-1.8} + q(x_p,t), \quad p = 0, 1, \tag{39}$$

where x_p are the roots of the shifted Chebyshev polynomial of the fourth kind $W_2^*(x)$. Using Eqs. (37) and (39) we obtain the following system of ordinary differential equations*:*

$$\dot{u_0}(t) + G_1\ \dot{u_1}(t) + G_2\ \dot{u_3}(t) = H_1\ u_2(t) + H_2\ u_3(t) + q(x_0,t), \tag{40}$$

$$\dot{u_0}(t) + G_{11}\ \dot{u_1}(t) + G_{22}\ \dot{u_3}(t) = H_{11}\ u_2(t) + H_{22}\ u_3(t) + q(x_1,t), \tag{41}$$

$$u_0(t) - u_1(t) + u_2(t) - u_3(t) = 0, \tag{42}$$

$$u_0(t) + 3u_1(t) + 5u_2(t) + 7u_3(t) = 0, \tag{43}$$

where

$$G_1 = W_1^*(x_0), \quad G_2 = W_3^*(x_0), \quad G_{11} = W_1^*(x_1), \quad G_{22} = W_3^*(x_1),$$

$$H_1 = p(x_0,t)\ V_{2,0}^{(1.8)}\ x_0^{2-1.8}, \quad H_2 = p(x_0,t)\ [V_{3,0}^{(1.8)}\ x_0^{3-1.8} + V_{3,1}^{(1.8)}\ x_0^{2-1.8}],$$

$$H_{11} = p(x_1,t)\ V_{2,0}^{(1.8)}\ x_1^{2-1.8}, \quad H_{22} = p(x_1,t)\ [V_{3,0}^{(1.8)}\ x_1^{3-1.8} + V_{3,1}^{(1.8)}\ x_1^{2-1.8}].$$

Now we use the finite difference method to solve the system of (40)–(43) with the following notations: $T = T_{final}$, $0 < t_j \le T$ and we suppose $\Delta t = T/N$, $t_j = j\Delta t$, for $j = 0, 1, ..., N$. Also, we define

$$u_i(t_n) = u_i^n, \quad q_i(t_n) = q_i^n.$$

Then the system in Eqs. (40)–(43) is discretized in time and has the following form:

$$\frac{u_0^n - u_0^{n-1}}{\Delta t} + G_1\ \frac{u_1^n - u_1^{n-1}}{\Delta t} + G_2\ \frac{u_3^n - u_3^{n-1}}{\Delta t} = H_1\ u_2^n + H_2\ u_3^n + q_0^n, \tag{44}$$

$$\frac{u_0^n - u_0^{n-1}}{\Delta t} + G_{11}\ \frac{u_1^n - u_1^{n-1}}{\Delta t} + G_{22}\ \frac{u_3^n - u_3^{n-1}}{\Delta t} = H_{11}\ u_2^n + H_{22}\ u_3^n + q_1^n, \tag{45}$$

$$u_0^n - u_1^n + u_2^n - u_3^n = 0, \tag{46}$$

$$u_0^n + 3u_1^n + 5u_2^n + 7u_3^n = 0. \tag{47}$$

The above system of Eqs. (44)–(47) can be rewritten in the following matrix form:

$$AU^n = BU^{n-1} + \Delta t q^n, \quad or \quad U^n = A^{-1}BU^{n-1} + \Delta t A^{-1} q^n, \tag{48}$$

where

$$A = \begin{pmatrix} 1 & G_1 & -\Delta t H_1 & G_2 - \Delta t H_2 \\ 1 & G_{11} & -\Delta t H_{11} & G_{22} - \Delta t H_{22} \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 5 & 7 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & G_1 & 0 & G_2 \\ 1 & G_{11} & 0 & G_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$U^n = (u_0^n,\ u_1^n,\ u_2^n,\ u_3^n)^T$, and $q^n = (q_0^n,\ q_1^n,\ 0,\ 0,)^T$. In order to obtain the initial solution U^0 of Eq. (48) we use the initial condition of the problem, $u(x,0)$, combined with Eq. (21) or (22). Moreover, the approximation solution in Eq. (38) is obtained by substituting the analytical form series of the shifted Chebyshev polynomials of the fourth kind $W_i^*(x)$, $(i = 0, 1, 2, 3)$ as well as the coefficients $(u_i\ , (i = 0, 1, 2, 3))$. Which are computed in Eq. (48).

The following tables compute the absolute error between the exact and the approximate solutions. In order to assess the accuracy of the solutions obtained by our approach, in Table 1, we have compared the obtained results with the results given in [40] at $T = 1$. Moreover, in Table 2, we have compared the results of the presented method with the previously published data given in [18], [31], and [40] for the value of $T = 2$. From Tables 1 and 2 we can deduce that the presented method is more accurate than other methods. The computational cost (CPU time) for the proposed method also seems to be faster than other methods because we just need a few terms (i.e. $m = 3$) to obtain a highly accurate solution. The exact and the numerical solutions at $m = 5$, $T = 1$ and $m = 7$, $T = 2$ are plotted in Figures 1 and 2, respectively.

Table 1. The absolute error of our method and the method given in [40] for Example 1, at $T = 1$.

x	Method of [40] with $m = 3$	Present method with $m = 3$
0	0	0
0.1	5.46×10^{-6}	5.32×10^{-10}
0.2	8.51×10^{-6}	6.94×10^{-10}
0.3	9.60×10^{-6}	5.79×10^{-10}
0.4	9.18×10^{-6}	2.82×10^{-10}
0.5	7.69×10^{-6}	1.00×10^{-10}
0.6	5.60×10^{-6}	4.76×10^{-10}
0.7	3.33×10^{-6}	7.48×10^{-10}
0.8	1.34×10^{-6}	8.22×10^{-10}
0.9	8.39×10^{-6}	6.05×10^{-11}
1	0	0

Table 2. The absolute error of our method and methods given in [18], [31], and [40] for Example 1, at $T = 2$.

x	In [18], $(m = 7)$	In [31], $(m = 5)$	In [40], $(m = 3)$	Present method, $(m = 3)$
0	3×10^{-6}	0	0	0
0.1	4.18×10^{-6}	4.47×10^{-6}	3.33×10^{-6}	3.44×10^{-10}
0.2	5.45×10^{-6}	2.78×10^{-7}	5.65×10^{-6}	4.04×10^{-10}
0.3	6.18×10^{-6}	5.81×10^{-6}	7.05×10^{-6}	2.55×10^{-10}
0.4	6.49×10^{-6}	1.02×10^{-5}	7.64×10^{-6}	2.14×10^{-11}
0.5	6.39×10^{-6}	1.17×10^{-5}	7.52×10^{-6}	3.49×10^{-10}
0.6	5.95×10^{-6}	1.08×10^{-5}	6.80×10^{-6}	6.49×10^{-10}
0.7	5.32×10^{-6}	8.54×10^{-6}	5.59×10^{-6}	8.41×10^{-10}
0.8	4.59×10^{-6}	6.06×10^{-6}	3.98×10^{-6}	8.50×10^{-10}
0.9	3.79×10^{-6}	3.67×10^{-6}	2.08×10^{-6}	5.96×10^{-10}
1	3×10^{-6}	0	0	0

Example 2 Consider *Eq.* (5) with $\mu = 1.8$:

$$\frac{\partial u(x,t)}{\partial t} = p(x)\,\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + q(x,t), \quad 0 < x < 1, \quad t > 0,$$

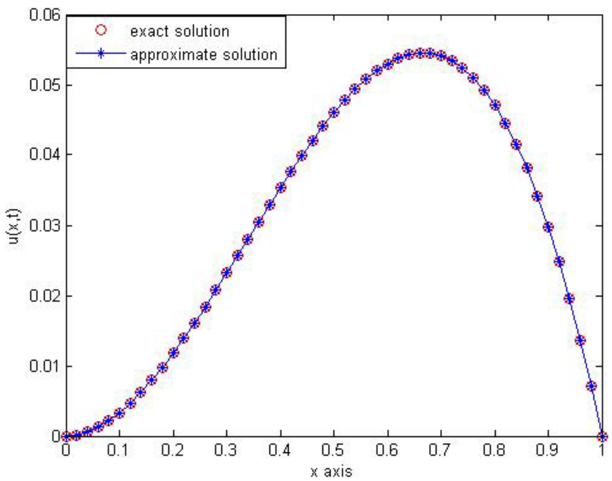

Figure 1. The behavior of the exact solution and approximation solution with $m = 5$ and $T = 1$ for Example 1.

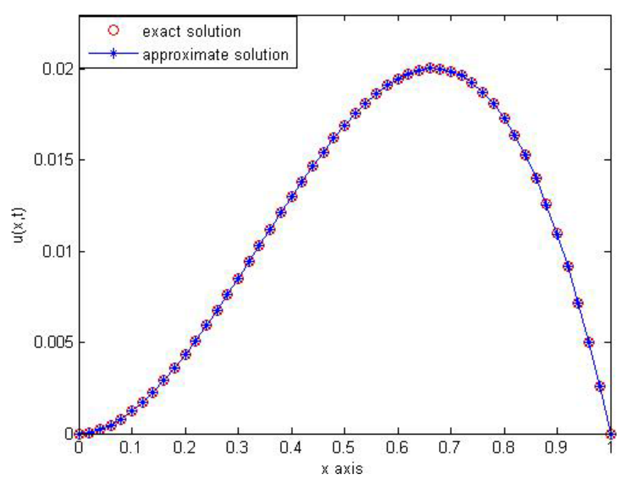

Figure 2. The behavior of the exact solution and approximation solution with $m = 7$ and $T = 2$ for Example 1.

with the diffusion coefficient

$$p(x) = \frac{\Gamma(2.2)}{6}\ x^{2.8},$$

the source function

$$q(x,t) = -(1+x)\ x^3\ e^{-t},$$

with the initial condition

$$u(x,0) = x^3,$$

and the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = e^{-t}, \quad t > 0.$$

The exact solution of this problem is given by:

$$u(x,t) = x^3 e^{-t}.$$

From the results of Table 3 and Figure 3, it is shown that the numerical results obtained by the proposed approach are more accurate than the results given in [40] and [42].

Table 3. Comparison of maximum error of different methods of [40] and [42] and our presented method for Example 2, at $T = 1$.

Max error [40]	Max error-CN [42]	Max error-ext CN [42]	Max error of ours with $m = 3$
8.3830×10^{-10}	6.84895×10^{-4}	2.82750×10^{-5}	3.1075×10^{-10}

Example 3 Consider *Eq.* (5) with the diffusion coefficient

$$p(x) = \Gamma(3-\alpha)\ x^{\alpha},$$

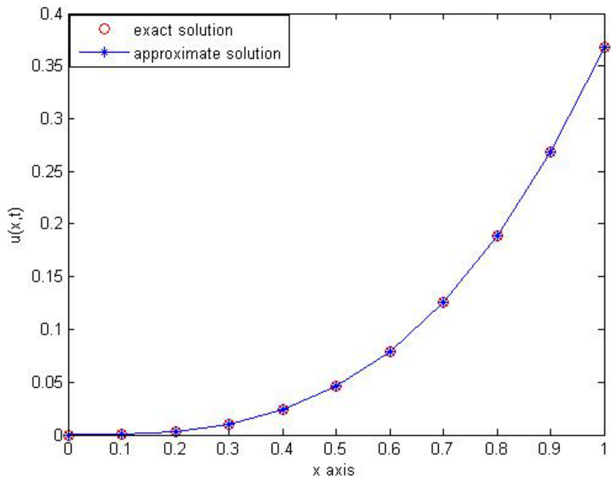

Figure 3. The behavior of the exact solution and approximation solution with $m = 3$ and $T = 1$ for Example 2.

the source function

$$q(x,t) = (x(x-1) - \Gamma(3)\ x^2)\ e^t,$$

with the initial condition

$$u(x,0) = x\ (x-1),$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t > 0.$$

The exact solution of this problem is given by:

$$u(x,t) = x\ (x-1)e^t.$$

From the results of Table 4 and Figure 4, it is shown that the numerical results obtained by the proposed approach for different values of μ are accurate.

Table 4. The absolute error of the presented method at different values of μ for Example 3 and $T = 1$.

x	$\mu = 1.25$	$\mu = 1.5$	$\mu = 1.75$	$\mu = 1.95$
0	0	0	0	0
0.1	1.43×10^{-8}	1.68×10^{-8}	1.86×10^{-8}	1.97×10^{-8}
0.2	1.28×10^{-8}	2.16×10^{-8}	2.78×10^{-8}	3.14×10^{-8}
0.3	1.66×10^{-10}	1.75×10^{-8}	2.95×10^{-8}	3.63×10^{-8}
0.4	1.89×10^{-8}	7.50×10^{-9}	2.58×10^{-8}	3.59×10^{-8}
0.5	3.95×10^{-8}	5.16×10^{-9}	1.85×10^{-8}	3.16×10^{-8}
0.6	5.71×10^{-8}	1.74×10^{-8}	9.79×10^{-9}	2.48×10^{-8}
0.7	6.66×10^{-8}	2.61×10^{-8}	1.57×10^{-9}	1.68×10^{-8}
0.8	6.35×10^{-8}	2.82×10^{-8}	4.14×10^{-9}	9.07×10^{-9}
0.9	4.29×10^{-8}	2.05×10^{-8}	5.33×10^{-9}	3.01×10^{-9}
1	0	0	0	0

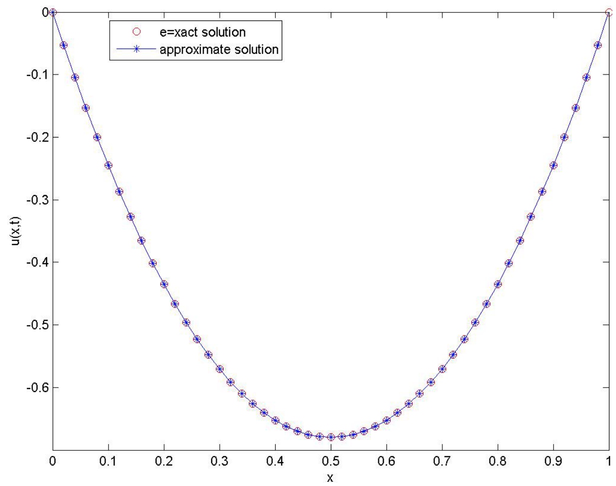

Figure 4. The behavior of the exact solution and approximation solution with $m = 5$ and $T = 2$ for Example 3.

7. Conclusions

In this paper, shifted Chebyshev polynomials of the fourth kind and their properties together with the Chebyshev collocation method are used to reduce the space fractional order diffusion equation for a system of ordinary differential equations. The finite difference method, the Euler method, is used to solve these systems of equations. The fractional derivative is considered in the Caputo sense. The validity and applicability of our presented method is illustrated through the obtained numerical results. These results are compared with results published in some papers. From the numerical results, it is obvious that our proposed method exhibits accuracy and efficiency better than the other methods. All computed results are obtained by using MATLAB.

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