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# The Moore–Penrose inverse of differences and products of projectors in a ring with involution

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**Abstract:** In this paper, we study the Moore–Penrose inverses of differences and products of projectors in a ring with involution. Some necessary and sufficient conditions for the existence of the Moore–Penrose inverse are given. Moreover, the expressions of the Moore–Penrose inverses of differences and products of projectors are presented.

**Key words:** Moore–Penrose inverses, normal elements, involutions, projectors

## 1. Introduction

Throughout this paper, $R$ is a unital $*$-ring, that is a ring with unity 1 and an involution $a \mapsto a^*$ satisfying that $(a^*)^* = a$, $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$. Recall that an element $a \in R$ is said to have a Moore–Penrose inverse (abbr. MP-inverse) if there exists $b \in R$ such that the following equations hold [11]:

$$aba = a,\ bab = b,\ (ab)^* = ab,\ (ba)^* = ba.$$

Any $b$ that satisfies the equations above is called a MP-inverse of $a$. The MP-inverse of $a \in R$ is unique if it exists and is denoted by $a^\dagger$. By $R^\dagger$ we denote the set of all MP-invertible elements in $R$.

MP-inverse of differences and products of projectors in various sets attracts wide attention from many scholars. For instance, Cheng and Tian [1] studied the MP-inverses of $pq$ and $p-q$, where $p$, $q$ are projectors in complex matrices. Li [10] investigated how to express MP-inverses of product $pq$ and differences $p-q$ and $pq-qp$, for two given projectors $p$ and $q$ in a $C^*$-algebra. Later, Deng and Wei [3] derived some formulae for the MP-inverse of the differences and the products of projectors in a Hilbert space. Recently, Zhang et al. [12] obtained the equivalences for the existences of differences and products of projectors in a $*$-reducing ring. More results on MP-inverses can be found in [7, 8, 11].

Motivated by [9], we investigate the equivalences for the existences of the MP-inverse of differences and products of projectors in a ring with involution. Moreover, the expressions of the MP-inverse of differences and products of projectors are presented. Some well-known results in $C^*$-algebras are extended.

Note that neither dimensional analysis nor special decomposition in Hilbert spaces and $C^*$-algebras can be used in rings. The results in this paper are proved by a purely ring theoretical method.

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## 2. Some lemmas

In 1992, Harte and Mbekhta [5] showed an excellent result in $C^*$-algebras, i.e. if $a$ is MP-invertible, then $a^*c = ca^*$ and $ac = ca$ imply $a^\dagger c = ca^\dagger$. In 2013, Drazin [4] extended this result to a $*$-semigroup case in Lemma 2.1 below.

**Lemma 2.1** [4, Corollary 2.7] *Let $S$ be any $*$-semigroup, let $a_1, a_2, d \in S$, and suppose that $a_1$ and $a_2$ each have Moore–Penrose inverses $a_1^\dagger$, $a_2^\dagger$, respectively. Then, for any $d \in S$, $da_1 = a_2 d$ and $da_1^* = a_2^* d$ together imply $a_2^\dagger d = da_1^\dagger$.*

The following result in $C^*$-algebras was considered by Koliha [6]. For the convenience of the reader, we give its proof in a ring.

**Lemma 2.2** *Let $a, b \in R^\dagger$ with $ab = ba$ and $a^*b = ba^*$. Then $ab \in R^\dagger$ and $(ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger$.*

**Proof** It follows from Lemma 2.1 that $a^\dagger b = ba^\dagger$ and $b^\dagger a = ab^\dagger$. As $b^*a = ab^*$ and $b^*a^* = a^*b^*$, then $b^*a^\dagger = a^\dagger b^*$, which together with $ba^\dagger = a^\dagger b$ implies $a^\dagger b^\dagger = b^\dagger a^\dagger$. Note that $aa^\dagger$ commutes with $b$ and $b^\dagger$. Also, $bb^\dagger$ commutes with $a$ and $a^\dagger$. Hence, $b^\dagger a^\dagger$ satisfies four equations of Penrose. Indeed, we have:

(i) $(abb^\dagger a^\dagger)^* = (aba^\dagger b^\dagger)^* = (aa^\dagger bb^\dagger)^* = bb^\dagger aa^\dagger = aa^\dagger bb^\dagger = aba^\dagger b^\dagger = abb^\dagger a^\dagger$.

(ii) $(b^\dagger a^\dagger ab)^* = (b^\dagger ba^\dagger a)^* = a^\dagger ab^\dagger b = b^\dagger a^\dagger ab$.

(iii) $abb^\dagger a^\dagger ab = aa^\dagger bb^\dagger ab = aa^\dagger bb^\dagger ba = aa^\dagger ba = aa^\dagger ab = ab$.

(iv) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger ba^\dagger ab^\dagger a^\dagger = b^\dagger ba^\dagger aa^\dagger b^\dagger = b^\dagger ba^\dagger b^\dagger = b^\dagger a^\dagger$.

Therefore, $ab \in R^\dagger$ and $(ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger$. □

Penrose [11, p. 408] presented the MP-inverse of $A+B$, where $A$ and $B$ are complex matrices such that $A^*B = 0$ and $AB^* = 0$. His formula indeed holds in a ring with involution.

**Lemma 2.3** *Let $a, b \in R^\dagger$ such that $a^*b = ab^* = 0$. Then $(a+b)^\dagger = a^\dagger + b^\dagger$.*

## 3. Main results

We say that an element $p$ is a projector if $p^2 = p = p^*$. Throughout this paper, the elements $p$, $q$ are projectors from the ring $R$.

**Theorem 3.1** *Let $a, b \in R^\dagger$ with $a^*p = pa^*$ and $b^*p = pb^*$. Then $ap + b(1-p) \in R^\dagger$ and $(ap + b(1-p))^\dagger = a^\dagger p + b^\dagger (1-p)$.*

**Proof** As $a^*p = pa^*$, then $ap = pa$ since $p$ is a projector. Similarly, $bp = pb$. We have $(ap)^* b(1-p) = 0$. Indeed, $(ap)^* b(1-p) = pa^*(1-p)b = a^*p(1-p)b = 0$. Also, $ap(b(1-p))^* = 0$. By Lemma 2.2, it follows that $(ap)^\dagger = a^\dagger p$ and $(b(1-p))^\dagger = b^\dagger(1-p)$. In view of Lemma 2.3, we obtain $ap + b(1-p) \in R^\dagger$ and $(ap + b(1-p))^\dagger = a^\dagger p + b^\dagger(1-p)$. □

Recall from [8] that an element $a \in R$ is $*$-cancelable if $a^*ax = 0$ implies $ax = 0$ and $xaa^* = 0$ implies $xa = 0$. A ring $R$ is called a $*$-reducing ring if all elements in $R$ are $*$-cancelable. We get the following result, under the condition of $*$-cancelabilities of some elements, rather than $*$-reducing rings in [12].

**Proposition 3.2** *Let $p(1-q)$ and $q(1-p)$ be $*$-cancelable. Then the following conditions are equivalent:*
(1) $1-pq \in R^{\dagger}$, (2) $1-pqp \in R^{\dagger}$, (3) $p-pqp \in R^{\dagger}$, (4) $p-pq \in R^{\dagger}$, (5) $p-qp \in R^{\dagger}$,
(6) $1-qp \in R^{\dagger}$, (7) $1-qpq \in R^{\dagger}$, (8) $q-qpq \in R^{\dagger}$, (9) $q-qp \in R^{\dagger}$, (10) $q-pq \in R^{\dagger}$.

**Proof** (1) $\Leftrightarrow$ (6) Note that $a \in R^{\dagger}$ if and only if $a^* \in R^{\dagger}$. Hence, it is sufficient to prove (1)–(5).

(1) $\Leftrightarrow$ (2) By [12, Theorem 4].

(2) $\Rightarrow$ (3) Noting $p-pqp = p(1-pqp) = (1-pqp)p$, it is an immediate result of Lemma 2.2.

(3) $\Rightarrow$ (2) Since $1-pqp = p(p-pqp)+1-p$ and $(p-pqp)^* = p-pqp$, it follows from Theorem 3.1 that $1-pqp \in R^{\dagger}$.

(3) $\Leftrightarrow$ (4) Note that $a \in R^{\dagger} \Leftrightarrow aa^* \in R^{\dagger}$ and $a$ is $*$-cancelable by [8, Theorem 5.4]. As $p(1-q)(p(1-q))^* = p-pqp \in R^{\dagger}$ and $p-pq$ is $*$-cancelable, the result follows.

(4) $\Leftrightarrow$ (5) As $(p-pq)^* = p-qp$ and $a \in R^{\dagger} \Leftrightarrow a^* \in R^{\dagger}$, then $p-pq \in R^{\dagger} \Leftrightarrow p-qp \in R^{\dagger}$. $\square$

Recall that an element $a \in R$ is normal if $aa^* = a^*a$. Further, if a normal element $a$ is MP-invertible, then $aa^{\dagger} = a^{\dagger}a$ by Lemma 2.2.

In 2004, Koliha et al. [9] showed that $p-q$ is nonsingular if and only if $1-pq$ and $p+q-pq$ are both nonsingular, for projectors $p$, $q$ in complex matrices. It is natural to consider whether the same property can be inherited to the MP-inverse in a ring with involution. The following result illustrates its possibility.

**Theorem 3.3** *Let $p-q$, $p(1-q)$ and $q(1-p)$ be $*$-cancelable. Then the following conditions are equivalent:*

(1) $p-q \in R^{\dagger}$,

(2) $1-pq \in R^{\dagger}$,

(3) $p+q-pq \in R^{\dagger}$.

**Proof** (1) $\Rightarrow$ (2) Note that $p-q$ is normal. It follows from Lemma 2.2 that $((p-q)^2)^{\dagger} = ((p-q)^{\dagger})^2$. As $p(p-q)^2 = (p-q)^2p = p-pqp$, then $1-pqp = (p-q)^2p+1-p$ and hence $1-pqp \in R^{\dagger}$ according to Theorem 3.1. So, $1-pq \in R^{\dagger}$ by [12, Theorem 4].

(2) $\Rightarrow$ (1) By [12, Theorem 4], we know that $1-pq \in R^{\dagger}$ implies $1-pqp \in R^{\dagger}$. Let $\overline{p} = 1-p$ and $\overline{q} = 1-q$. Note that $p(1-q)$ is $*$-cancelable. We have $1-pq \in R^{\dagger} \Rightarrow p-pq = \overline{q}-\overline{p}\,\overline{q} \in R^{\dagger}$ by (1) $\Rightarrow$ (4) in Proposition 3.2. Also, as $\overline{q}(1-\overline{p}) = p(1-q)$ is $*$-cancelable, then $\overline{q}-\overline{p}\,\overline{q} \in R^{\dagger}$ implies $1-\overline{q}\,\overline{p} \in R^{\dagger}$ by (10) $\Rightarrow$ (6) in Proposition 3.2, which means $1-\overline{p}\,\overline{q} \in R^{\dagger}$ since $a \in R^{\dagger} \Leftrightarrow a^* \in R^{\dagger}$. Again, applying [12, Theorem 4], it follows that $1-\overline{p}\,\overline{q}\,\overline{p} \in R^{\dagger}$.

Setting $a = 1-pqp$ and $b = 1-\overline{p}\,\overline{q}\,\overline{p}$, then $a^*p = pa^*$ and $b^*p = pb^*$. Since $(p-q)^2 = ap+b(1-p)$, we obtain $(p-q)^2 = (p-q)(p-q)^* \in R^{\dagger}$ by Theorem 3.1 and hence $p-q \in R^{\dagger}$ from [8, Theorem 5.4].

(1) $\Leftrightarrow$ (3) In (1) $\Leftrightarrow$ (2), replacing $p$, $q$ by $1-p$, $1-q$, respectively. $\square$

Next, we mainly consider the representations of the MP-inverse by the aforementioned results.

**Theorem 3.4** *Let $p-q \in R^{\dagger}$. Define $F$, $G$, and $H$ as*

$$F = p(p-q)^{\dagger},\ G = (p-q)^{\dagger}p,\ H = (p-q)(p-q)^{\dagger}.$$

*Then we have:*

(1) $F^2 = F = (p-q)^{\dagger}(1-q)$,

(2) $G^2 = G = (1-q)(p-q)^{\dagger}$,

(3) $H^2 = H = H^*$.

**Proof** (1) We first prove $F = (p-q)^{\dagger}(1-q)$.

As $(p-q)^* = p-q$ and $p-q \in R^{\dagger}$, then $(p-q)^2 \in R^{\dagger}$ by Lemma 2.2. Moreover, $((p-q)^2)^{\dagger} = ((p-q)^{\dagger})^2$. Also, $(p-q)(p-q)^{\dagger} = (p-q)^{\dagger}(p-q)$. From $p(p-q)^2 = (p-q)^2 p$ and $p((p-q)^2)^* = ((p-q)^2)^* p$, we have $p((p-q)^{\dagger})^2 = ((p-q)^{\dagger})^2 p$ using Lemma 2.1.

Hence,

$$\begin{aligned}
(p-q)^{\dagger}(1-q) &= ((p-q)^{\dagger})^2(p-q)(1-q) = ((p-q)^{\dagger})^2 p(1-q) \\
&= ((p-q)^{\dagger})^2 p(p-q) = p((p-q)^{\dagger})^2(p-q) \\
&= p(p-q)^{\dagger} \\
&= F.
\end{aligned}$$

We now show $F^2 = F$. Since $p(p-q)^{\dagger} = (p-q)^{\dagger}(1-q)$, one can get

$$\begin{aligned}
F^2 &= (p-q)^{\dagger}(1-q)p(p-q)^{\dagger} \\
&= (p-q)^{\dagger}(1-q)(p-q)(p-q)^{\dagger} \\
&= p(p-q)^{\dagger}(p-q)(p-q)^{\dagger} \\
&= p(p-q)^{\dagger} \\
&= F.
\end{aligned}$$

(2) By $F^* = G$.

(3) It is trivial. □

Under the same symbol in Theorem 3.4, more relations among $F$, $G$, and $H$ are given in the following result.

**Corollary 3.5** *Let $p-q \in R^{\dagger}$. Then*

(1) $q(p-q)^{\dagger} = (p-q)^{\dagger}(1-p)$,

(2) $qH = Hq$,

(3) $G(1-q) = (1-q)F$.

**Proof** (1) can be obtained by a similar proof of Theorem 3.4(1).

(2) Taking involution on (1), it follows that $(1-p)(p-q)^{\dagger} = (p-q)^{\dagger}q$ and hence

$$\begin{aligned}
qH &= q(p-q)(p-q)^{\dagger} = q(p-1)(p-q)^{\dagger} \\
&= -q(p-q)^{\dagger}q = -(p-q)^{\dagger}(1-p)q \\
&= -(p-q)^{\dagger}(q-p)q \\
&= Hq.
\end{aligned}$$

(3) We have

$$\begin{aligned} G(1-q) &= (p-q)^\dagger(p-q)(1-q) = (p-q)^\dagger p(p-q) \\ &= (1-q)(p-q)^\dagger(p-q) \\ &= (1-q)F. \end{aligned}$$

□

Keeping in mind the relations in Theorem 3.4 and Corollary 3.5, we give the following equalities, where $\overline{a}$ denotes $1-a$.

**Corollary 3.6** *Let $p-q \in R^\dagger$. Then:*

(1) $Fp = pG = pH = Hp$,

(2) $qHq = qH = Hq = HqH$,

(3) $\overline{q}\overline{F} = \overline{G}\overline{q} = \overline{q}\overline{F}\overline{q}$,

(4) $(p-q)^\dagger = F + G - H$.

In general, $p-q \in R^\dagger$ can not imply $p+q \in R^\dagger$. Foe example, take $R = \mathbb{Z}$ and $1 = p = q \in R$, then $p-q = 0 \in R^\dagger$, but $p+q = 2 \notin R^\dagger$ since 2 is not invertible.

The next theorem presents the necessary and sufficient conditions for the existence of $(p+q)^\dagger$.

**Theorem 3.7** *Let 2 be invertible in $R$. Then the following conditions are equivalent*:

(1) $pH = p$,

(2) $(p+q)H = (p+q)$,

(3) $p+q \in R^\dagger$ and $(p+q)^\dagger = (p-q)^\dagger(p+q)(p-q)^\dagger$.

**Proof**

(1) ⇒ (2) If $pH = p$, then $qH = q$ by the symmetry of $p$ and $q$. Hence, $(p+q)H = (p+q)$.

(2) ⇒ (1) Note that $H = (p-q)(p-q)^\dagger$ and $p-q$ is normal. We have $(p-q)H = p-q$ and $p+q = (p+q)H = (q-p)H + 2pH = -(p-q) + 2pH$, which implies $2pH = 2p$. Hence, $pH = p$ since 2 is invertible.

(2) ⇒ (3) Let $x = (p-q)^\dagger(p+q)(p-q)^\dagger$. We prove that $x$ is the MP-inverse of $p+q$ by checking four equations of Penrose.

(i) $((p+q)x)^* = (p+q)x$. Indeed,

$$\begin{aligned} (p+q)x &= (p+q)(p-q)^\dagger(p+q)(p-q)^\dagger \\ &= (p-q)^\dagger(1-q+1-p)(p+q)(p-q)^\dagger \\ &= (p-q)^\dagger(p-q)^2(p-q)^\dagger \\ &= (p-q)(p-q)^\dagger. \end{aligned}$$

(ii) $(x(p+q))^* = x(p+q)$. By similar proof of (i), we have $x(p+q) = (p-q)^\dagger(p-q)$.

(iii) Note the relations $pH = Hp$ and $qH = Hq$ in Corollary 3.6. Then

$$\begin{aligned}(p+q)x(p+q) &= (p-q)(p-q)^{\dagger}(p+q)\\ &= H(p+q) = (p+q)H\\ &= p+q.\end{aligned}$$

(iv) It follows that $x(p+q)x = (p-q)^{\dagger}(p+q)(p-q)^{\dagger}(p-q)(p-q)^{\dagger} = x$.

(3) $\Rightarrow$ (2) As $p+q \in R^{\dagger}$ with $(p+q)^{\dagger} = (p-q)^{\dagger}(p+q)(p-q)^{\dagger}$, then

$$\begin{aligned}p+q &= (p+q)(p+q)^{\dagger}(p+q) = (p+q)(p-q)^{\dagger}(p+q)(p-q)^{\dagger}(p+q)\\ &= (p+q)(p-q)^{\dagger}(p-q)^{\dagger}(1-q+1-p)(p+q)\\ &= (p+q)(p-q)^{\dagger}(p-q)^{\dagger}[(1-q)p+(1-p)q]\\ &= (p+q)(p-q)^{\dagger}(p-q)^{\dagger}[(p-q)p+(q-p)q]\\ &= (p+q)(p-q)^{\dagger}(p-q)^{\dagger}(p-q)p-(p+q)(p-q)^{\dagger}(p-q)^{\dagger}(p-q)q\\ &= (p+q)(p-q)^{\dagger}(p-q)(p-q)^{\dagger}p-(p+q)(p-q)^{\dagger}(p-q)(p-q)^{\dagger}q\\ &= (p+q)(p-q)^{\dagger}p-(p+q)(p-q)^{\dagger}q\\ &= (p+q)(p-q)^{\dagger}(p-q)\\ &= (p+q)H.\end{aligned}$$

$\square$

Next we give a new necessary and sufficient condition of the existence of $(p+q)^{\dagger}$.

**Theorem 3.8** *Let $p, q \in R$ with $pq = qp$. Then $p+q \in R^{\dagger}$ if and only if $1+pq \in R^{\dagger}$.*

*In this case, $(p+q)^{\dagger} = (1+pq)^{\dagger}p + q(1-p)$ and $(1+pq)^{\dagger} = (p+q)^{\dagger}p + 1 - p$.*

**Proof** Suppose $p+q \in R^{\dagger}$. As $1+pq = p(p+q)+1-p$, then $(1+pq)^{\dagger} = (p+q)^{\dagger}p+1-p$ by Theorem 3.1.

Conversely, let $x = (1+pq)^{\dagger}p + q(1-p)$. We next show that $x$ is the MP-inverse of $p+q$.

(i) $[(p+q)x]^{*} = (p+q)x$. We have

$$\begin{aligned}(p+q)x &= (p+q)[(1+pq)^{\dagger}p+q(1-p)]\\ &= (1+pq)^{\dagger}p+(1+pq)^{\dagger}pq+q(1-p)\\ &= (1+pq)^{\dagger}(1+pq)p+q(1-p).\end{aligned}$$

Hence, $[(p+q)x]^{*} = (p+q)x$.

(ii) It follows that $[x(p+q)]^{*} = x(p+q)$ since $p$ and $q$ commute.

(iii) $(p+q)x(p+q) = p+q$. Indeed,

$$\begin{aligned}(p+q)x(p+q) &= (p+q)[(1+pq)^\dagger(1+pq)p + q(1-p)]\\ &= (1+pq)^\dagger(1+pq)p + (1+pq)^\dagger(1+pq)pq + q(1-p)\\ &= (1+pq)^\dagger(1+pq)p(1+pq) + q(1-pq)\\ &= p(1+pq) + q(1-pq)\\ &= p+q.\end{aligned}$$

(iv) By a similar way of (3), we get $x(p+q)x = x$.

Thus, $(p+q)^\dagger = (1+pq)^\dagger p + q(1-p)$. □

The next theorem, a main result of this paper, admits proficient skills on $F$, $G$, and $H$, expressing the formulae of the MP-inverse of difference of projectors.

**Theorem 3.9** *Let $p-q \in R^\dagger$. Then:*

(1) $(1-pqp)^\dagger = p((p-q)^\dagger)^2 + (1-p)$,

(2) $(1-pq)^\dagger = p((p-q)^\dagger)^2 - pq(p-q)^\dagger + 1 - p$,

(3) $(p-pqp)^\dagger = p((p-q)^\dagger)^2$,

(4) *If $p-pq$ is $*$-cancellable, then* $(p-pq)^\dagger = (p-q)^\dagger p$,

(5) *If $p-pq$ is $*$-cancellable, then* $(p-qp)^\dagger = p(p-q)^\dagger$.

**Proof** (1) As $1-pqp = p(p-q)^2 + 1 - p$, then $(1-pqp)^\dagger = p((p-q)^\dagger)^2 + 1 - p$ according to Theorem 3.1.

(2) It follows from Theorem 3.3 that $p-q \in R^\dagger$ implies $1-pq \in R^\dagger$. Let $x = p((p-q)^\dagger)^2 - pq(p-q)^\dagger + 1 - p$. We next show that $x$ is the MP-inverse of $1-pq$.

(i) We have

$$\begin{aligned}(1-pq)x &= (1-pq)[p((p-q)^\dagger)^2 - pq(p-q)^\dagger + 1 - p]\\ &= (p-pqp)((p-q)^\dagger)^2 - (1-pq)pq(p-q)^\dagger + (1-pq)(1-p)\\ &= p(p-q)^2((p-q)^\dagger)^2 - (p-pqp)(p-q)^\dagger(1-p) + (1-pq)(1-p)\\ &= p(p-q)(p-q)^\dagger - p(p-q)^2(p-q)^\dagger(1-p) + (1-pq)(1-p)\\ &= p(p-q)(p-q)^\dagger - p(p-q)(1-p) + (1-pq)(1-p)\\ &= p(p-q)(p-q)^\dagger + 1 - p\\ &= pH + 1 - p.\end{aligned}$$

Hence, $((1-pq)x)^* = (1-pq)x$ since $pH = Hp$ and $H^* = H$.

(ii) We get $x(1-pq) = p(p-q)^\dagger p + 1 - p$. Hence, $(x(1-pq))^* = x(1-pq)$.

(iii) $(1-pq)x(1-pq)=1-pq$. Indeed,

$$\begin{aligned}(1-pq)x(1-pq) &= (pH+1-p)(1-pq)=Hp(1-pq)+(1-p)(1-pq)\\ &= Hp(p-pq)+1-p=pH(p-pq)+1-p\\ &= pHp(p-q)+1-p=pH(p-q)+1-p\\ &= p(p-q)+1-p\\ &= 1-pq.\end{aligned}$$

(iv) $x(1-pq)x=1-pq$. Actually, we can obtain this result by a similar proof of (iii).

(3) Since $p-pqp=p(p-q)^2=(p-q)^2p$, we get $(p-pqp)^\dagger=p((p-q)^\dagger)^2$ by Lemma 2.2.

(4) Keeping in mind that $a^\dagger=a^*(aa^*)^\dagger=(a^*a)^\dagger a^*$, we have $(p-pq)^\dagger=(p-qp)p((p-q)^\dagger)^2=(p-q)((p-q)^\dagger)^2p=(p-q)^\dagger p$.

(5) Note that $a$ is $*$-cancelable if and only if $a^*$ is $*$-cancelable. It follows from $(a^*)^\dagger=(a^\dagger)^*$ that $(p-qp)^\dagger=p(p-q)^\dagger$. □

**Corollary 3.10** *Let $p-pq$ be $*$-cancelable and let $1-pq\in R^\dagger$. Then $p-q\in R^\dagger$ and*

$$(p-q)^\dagger=(1-pq)^\dagger(p-pq)+(p+q-pq)^\dagger(pq-q).$$

**Proof** From Theorem 3.3, we have $p-q\in R^\dagger\Leftrightarrow 1-pq\in R^\dagger$.

By Theorem 3.9 (2), we have $(p+q-pq)^\dagger=(1-p)((p-q)^\dagger)^2+(1-p)(1-q)(p-q)^\dagger+p$. It is straightforward to check that $(1-pq)^\dagger(p-pq)+(p+q-pq)^\dagger(pq-q)$ satisfies four equations of Penrose. □

The following result is motivated by [2]. Therein, Deng considered the Drazin inverses of difference of idempotent bounded operators on Hilbert spaces.

**Theorem 3.11** *Let $pq-qp$ be $*$-cancelable. Then:*

(1) *$(p-q)^\dagger=p-q$ if and only if $pq=qp$,*

(2) *If 6 is invertible in $R$, then $(p+q)^\dagger=p+q$ if and only if $pq=0$.*

**Proof** (1) If $pq=qp$, it is straightforward to check $(p-q)^\dagger=p-q$.

Conversely, $(p-q)^\dagger=p-q$ implies $(p-q)^3=p-q$, we get $pqp=qpq$ and hence $(pq-qp)^*(pq-qp)=0$. It follows that $pq=qp$ since $pq-qp$ is $*$-cancelable.

(2) Suppose $pq=0$. Then $p^*q=pq^*=0$ since $p$, $q$ are projectors. Then $(p+q)^\dagger=p+q$ by Lemma 2.3.

Conversely, $(p+q)^\dagger=p+q$ concludes $(p+q)^3=p+q$. By direct calculations, it follows that $2pq+2qp+pqp+qpq=0$. (3.1)

Multiplying the equality (3.1) by $p$ on the left yields $2pq+3pqp+pqpq=0$. (3.2)

Multiplying the equality (3.1) by $q$ on the right gives $2pq+3qpq+pqpq=0$. (3.3)

Combining the equalities (3.2) and (3.3), it follows that $pqp=qpq$ since 3 is invertible. As $pq-qp$ is $*$-cancelable, then $pqp=qpq$ implies $pq=qp$. Hence, equality (3.1) can be reduced to $6pq=0$.

Thus, $pq=0$. □

**Theorem 3.12** *Let* $1-p-q \in R^{\dagger}$. *Then:*

(1) $pqp \in R^{\dagger}$ *and* $(pqp)^{\dagger} = p((1-p-q)^{\dagger})^2 = ((1-p-q)^{\dagger})^2 p$,

(2) *If* $pq$ *is* $*$*-cancelable, then* $pq \in R^{\dagger}$ *and* $(pq)^{\dagger} = qp((1-p-q)^{\dagger})^2$.

**Proof** (1) Since $(1-p-q)^* = 1-p-q$, we have $((1-p-q)^2)^{\dagger} = ((1-p-q)^{\dagger})^2$ by Lemma 2.2. As $pqp = p(1-p-q)^2 = (1-p-q)^2 p$, then $pqp \in R^{\dagger}$ from Lemma 2.2 and hence $(pqp)^{\dagger} = p((1-p-q)^{\dagger})^2 = ((1-p-q)^{\dagger})^2 p$.

(2) Note that $1-p-q \in R^{\dagger}$ implies $pqp \in R^{\dagger}$. As $pqp = pq(pq)^*$ and $pq$ is $*$-cancelable, then $pq \in R^{\dagger}$ by [8, Theorem 5.4]. The formula $a^{\dagger} = a^*(aa^*)^{\dagger}$ guarantees that $(pq)^{\dagger} = qp((1-p-q)^{\dagger})^2$. □

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