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The relation between rough Wijsman convergence and asymptotic cones

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Abstract: In this paper, we explore the effect of the asymptotic cone of the limit set of a sequence that is rough Wijsman convergent.

Key words: Rough convergence, Wijsman convergence, asymptotic cone

1. Introduction

As is well known, one of the fundamental concepts in mathematical analysis is the concept of the limit of a sequence. In this context, we often study sufficiently small neighborhoods of a point in a topological space, and investigate the behavior of a sequence approaching that point. In recent years, many authors have explored sequences diverging to infinity, by using much larger neighborhoods of a point. This idea is a useful way in both differential geometry and geometric group theory. In addition, it has led to the emergence of asymptotic cones. The notion of an asymptotic cone was first introduced by Steinitz [10]. He also gave the asymptotic properties of unbounded convex sets. Later, asymptotic cones were called "horizon cones" by some authors. In 1940, the theory of asymptotic cones was developed by Stoker [11]. In 1953, Fenchel [4] introduced the concept of the convergence of a sequence of rays by using the distance between two rays. Moreover, he proved that such a distance is a metric on the space of rays. In 1966, an alternative definition of the asymptotic cone was given by Wijsman [13] via normalized sequences.

As for the notion of rough convergence of a sequence, it was first introduced by Phu [7] in a finite-dimensional normed space as follows. Let r be a nonnegative real number. A sequence $\{x_n\}$ is said to be *r-convergent* to x, denoted by $x_n \xrightarrow{r} x$, provided that

$$\forall \varepsilon > 0 \quad \exists N(\varepsilon) \in \mathbb{N} \quad : \quad n \geq N(\varepsilon) \quad \Rightarrow \quad \|x_n - x\| < r + \varepsilon.$$

With the help of this definition, Phu [7] showed that a sequence that is not convergent in the usual sense might be convergent to a point, with a certain degree of roughness. Then he proved analogous results for a sequence in an infinite-dimensional space [8]. In 2008, Aytar [2] investigated the relations between the core and the r-limit set of a real sequence. In [5], Listan-Garcia and Rambla-Barreno gave some results analogous to those of Phu, which are given by using strict convexity and uniform convexity, by means of uniform rotundity in every direction (URED). The condition URED is strictly weaker than the uniform convexity property. In [6], the same

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authors stated two new geometric properties by using the rough convergence in Banach spaces. They showed that the rough limit set is closely related to Chebyshev centers. They also gave some classical properties such as Kalton's M property. In [12], the authors gave an extension of rough convergence, by using the notion of an ideal. They also stated some basic results related to the rough ideal limit set. Recently, Dündar and Çakan [3] introduced the concept of rough convergence for a double sequence.

In this paper, we explore the effect of the asymptotic cone of the limit set of a sequence that is rough Wijsman convergent. To this end, we introduce the concept of a rough asymptotic cone, and investigate the properties of such a cone.

2. Preliminaries

First, we recall the notions of a ray and a cone in $\mathbb{R}^m$. A ray is a closed half-line emanating from the origin. If $x \neq 0$, a ray from origin 0 through x is denoted by (x). A subset X of $\mathbb{R}^m$ is called a *cone* if 0 is in X and $x \in X$ implies $\lambda x \in X$ for every nonnegative real scalar λ. The particular cones consisting of a nonzero vector x and all its multiples λx $(\lambda \geq 0)$ are rays. A cone that contains at least one nonzero vector is therefore just the union of the rays that it contains. Throughout the paper, we will be interested in the cones on $\mathbb{R}^m$ except for the cone $X = \{0\}$.

Since cones may be thought of as sets of rays, it is desirable to introduce a topology on these rays, by using the topology on $\mathbb{R}^m$. This might be done by defining the angle

$$\theta(x, y) = \arccos \frac{\langle x, y\rangle}{\|x\| \, \|y\|}, 0 \leq \theta \leq \pi$$

as a metric on $\mathbb{R}^m - \{0\}$. This angle depends only on the rays (x) and (y) to which x and y belong. It may be thought of as the angle between the two rays. The proof that this angle is indeed a metric for the rays, in particular that it satisfies the triangle inequality, is not obvious. An equivalent metric is

$$[x, y] = \sqrt{2 - \frac{2\langle x, y\rangle}{\|x\| \, \|y\|}}.$$

This new metric is the chord distance between the two points $\frac{x}{\|x\|}$ and $\frac{y}{\|y\|}$ on the unit sphere. That is, $[x, y] = \rho\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right)$, where ρ is the Euclidean metric on $\mathbb{R}^m$ and $[x, y]$ depends only on the rays (x) and (y). The geometric description shows that these two metrics are topologically equivalent to each other [4].

The concept of the convergence of a sequence of rays is given as follows: a sequence $\{(x_n)\}$ is said to be *convergent* to a ray (x) if $[x_n, x] \to 0$ as $n \to \infty$, and we denote this case by $(x_n) \to (x)$. We will also use the notation $\rho\left(\frac{x_n}{\|x_n\|}, \frac{x}{\|x\|}\right) \to 0,$ as $n \to \infty$ [4].

Now we give the definitions of open cone and closed cone in $\mathbb{R}^m$. Let X be a cone in $\mathbb{R}^m$. A ray (x) is called a *limit ray* of a cone X if there is a sequence of rays of the cone that are different from (x) and that converge to (x). A *closed cone* is a cone that contains all its limit rays. A cone is closed in this sense if and only if it is closed in the usual topology of $\mathbb{R}^m$. A cone is *open* if and only if the complementary set of rays is a closed cone [4].

In the next section, we will explore the effect of the rough asymptotic cone of the limit set of a sequence that is rough Wijsman convergent. Thus, we will briefly introduce the concept of rough Wijsman convergence.

Throughout this paper, we assume that X and X_n are any subsets of $\mathbb{R}^m$ for each n. As usual, the distance from a point x to a nonempty set X is defined by

$$d(x,X) = \inf\{\|x-y\| : y \in X\},$$

where $\|.\|$ represents the Euclidean norm.

Given $r > 0$, we say that a sequence $\{X_n\}$ is *rough Wijsman convergent* to a set X if

$$d(x,X_n) \xrightarrow{r} d(x,X) \text{ for all } x \in \mathbb{R}^m.$$

In this case, we write $X_n \xrightarrow{r} X$ as $n \to \infty$.

Now we consider the concept of an asymptotic cone. *The asymptotic cone* of the set X is defined by

$$\begin{aligned} A(X) &= \{(x) : (x) = \lim(x_n),\ x_n \in X,\ \|x_n\| \to \infty\} \\ &= \left\{ax : a \geq 0,\ x = \lim \frac{x_n}{\|x_n\|},\ x_n \in X,\ \|x_n\| \to \infty\right\}. \end{aligned} \tag{2.1}$$

It is clear that if a set X is bounded, then its asymptotic cone is an empty set [13].

If the set X is closed and convex, then the asymptotic cone of this set is called a *recession cone*.

An alternative definition of the asymptotic cone is given by the following:

The asymptotic cone of the set X is defined by

$$A(X) = \begin{cases} \{x \in \mathbb{R}^m : \exists x_n \in X,\ \lambda_n \searrow 0,\ \lambda_n x_n \to x\} & ,\ X \neq \varnothing \\ \varnothing & ,\ X = \varnothing \end{cases}. \tag{2.2}$$

The notation $\lambda_n \searrow 0$ shows that $\lambda_n > 0$ and $\lambda_n \to 0$ [9].

Proposition 2.1 ([9]) *A set $X \subset \mathbb{R}^m$ is bounded if and only if $A(X) = \varnothing$.*

3. Rough asymptotic cones

In this section, we introduce the concept of the rough asymptotic cone of a set X in $\mathbb{R}^m$. Then we investigate the properties of these cones. Finally, we explore the effect of the asymptotic cone of the limit set of a sequence that is rough Wijsman convergent.

Definition 3.1 *Let $r > 0$. The rough asymptotic cone of a set X is the set*

$$A^r(X) = \left\{ax : a \geq 0,\ \frac{x_n}{\|x_n\|} \xrightarrow{r} x,\ x_n \in X,\ \|x_n\| \to \infty\right\}. \tag{3.1}$$

It clear that $A(X) \subseteq A^r(X)$ for each $r \geq 0$.

An alternative definition of the rough asymptotic cone can be given via the following

Proposition 3.1 *The rough asymptotic cone of the set X is*

$$\widetilde{A}^r(X) = \begin{cases} \left\{x \in \mathbb{R}^m : \exists x_n \in X,\ \lambda_n \searrow 0,\ \lambda_n x_n \xrightarrow{r} x\right\} & ,\ X \neq \varnothing \\ \varnothing & ,\ X = \varnothing \end{cases}. \tag{3.2}$$

Proof First we show that $\widetilde{A}^r(X) \subseteq A^r(X)$. Let $x \in \widetilde{A}^r(X)$. Then there exists a sequence $\{x_n\}$ in X such that $\lambda_n \searrow 0$ and $\lambda_n x_n \xrightarrow{r} x$. That is, $\|\lambda_n x_n - x\| < r + \varepsilon$ for almost all n (i.e all but finitely many). Define $\lambda_n := \frac{1}{\|x_n\|}$. Since $\lambda_n > 0$ for all $n \in \mathbb{N}$ and $\|x_n\| \to \infty$ as $n \to \infty$, we have $\lambda_n \to 0$. We obtain $\|\lambda_n x_n - x\| = \left\| \frac{1}{\|x_n\|} x_n - x \right\| < r + \varepsilon$ for almost all n, because $\frac{1}{\|x_n\|} x_n \xrightarrow{r} x$. From (3.1), we have $ax \in A^r(X)$ for all $a \geq 0$. If we choose $a = 1$, then $x \in A^r(X)$. Hence we get $\widetilde{A}^r(X) \subseteq A^r(X)$.

Now let $ax \in A^r(X)$. From (3.1), there exists a sequence $\{x_n\}$ in X such that $\left\| \frac{x_n}{\|x_n\|} - x \right\| < r + \varepsilon$ for almost all n, $x_n \in X$ and $\|x_n\| \to \infty$. We take $\lambda_n := \frac{1}{\|x_n\|}$. Since $\lambda_n > 0$ for all $n \in \mathbb{N}$ and $\|x_n\| \to \infty$ as $n \to \infty$, we have $\lambda_n \to 0$. Since

$$\begin{aligned} \left\| \frac{x_n}{\|x_n\|} - x \right\| &= \left\| \frac{1}{\|x_n\|} x_n - x \right\| \\ &= \|\lambda_n x_n - x\| \\ &< r + \varepsilon, \end{aligned}$$

for almost all n, we obtain $x \in \widetilde{A}^r(X)$. We also have $ax \in \widetilde{A}^r(X)$ for all $a \geq 0$. Hence we get $A^r(X) \subseteq \widetilde{A}^r(X)$, which completes the proof. □

Definition 3.2 *The rough closure of a set X is*

$$\overline{X}^r := \left\{ x \in \mathbb{R}^m : \exists \{x_n\} \subset X \text{ such that } x_n \xrightarrow{r} x, \text{ as } n \to \infty \right\}.$$

It is clear that $\overline{X}^r = B(\overline{X}, r)$, where $B(\overline{X}, r) := \left\{ y \in \mathbb{R}^m : d(y, \overline{X}) \leq r \right\}$.

Proposition 3.2 *If a set $X \subset \mathbb{R}^m$ is convex, then its rough closure $\overline{X}^r$ is convex.*

Proof Assume that $x^0, x^1 \in \overline{X}^r$. Then there exist two sequences $\{x_n^0\}$ and $\{x_n^1\}$ in X such that $x_n^0 \xrightarrow{r} x^0$ and $x_n^1 \xrightarrow{r} x^1$, as $n \to \infty$. Since the set X is convex, we have

$$(1-t)x_n^0 + t x_n^1 \in X \text{ for all } n \text{ and all } t \in [0,1].$$

Since $x_n^0 \xrightarrow{r} x^0$ and $x_n^1 \xrightarrow{r} x^1$, we have $(1-t)x_n^0 + t x_n^1 \xrightarrow{r} (1-t)x^0 + t x^1 \in \overline{X}^r$. □

Theorem 3.1 *Let X be a nonempty subset of $\mathbb{R}^m$. Then $A^r(\overline{X}) = A^r(X)$.*

Proof Since $X \subseteq \overline{X}$, it is clear that $A^r(X) \subseteq A^r(\overline{X})$. Then we need to show that $A^r(\overline{X}) \subseteq A^r(X)$.

Let $y = ax \in A^r(\overline{X})$. If $x_n \in \overline{X}$, then for every $\varepsilon > 0$ and for all $n \in \mathbb{N}$ there exists an $\widetilde{x}_n \in X$ such that $\|x_n - \widetilde{x}_n\| < \frac{\varepsilon}{2}$. Since $\|x_n\| \to \infty$, we have $\|\widetilde{x}_n\| \to \infty$. By definition of $A^r(\overline{X})$, for every $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that $\left\| \frac{x_n}{\|x_n\|} - x \right\| < r + \frac{\varepsilon}{2}$ for all $n \geq N(\varepsilon)$. It is clear that if $\|\widetilde{x}_n - x_n\| < \frac{\varepsilon}{2}$, we have

$\left\| \frac{\widetilde{x}_n}{\|\widetilde{x}_n\|} - \frac{x_n}{\|x_n\|} \right\| < \frac{\varepsilon}{2}$. Hence we obtain

$$\begin{aligned} \left\| \frac{\widetilde{x}_n}{\|\widetilde{x}_n\|} - x \right\| &= \left\| \frac{\widetilde{x}_n}{\|\widetilde{x}_n\|} - \frac{x_n}{\|x_n\|} + \frac{x_n}{\|x_n\|} - x \right\| \\ &\leq \left\| \frac{\widetilde{x}_n}{\|\widetilde{x}_n\|} - \frac{x_n}{\|x_n\|} \right\| + \left\| \frac{x_n}{\|x_n\|} - x \right\| \\ &< \frac{\varepsilon}{2} + r + \frac{\varepsilon}{2} \\ &= r + \varepsilon. \end{aligned}$$

Therefore, we get $y = ax \in A^r(X)$ for all $a \geq 0$. This implies that $A^r(\overline{X}) \subseteq A^r(X)$. □

On the other hand, we know that if the set X is a cone, then $A(X) = \overline{X}$ (see page 26 in [1]). As can be seen in this example, this fact is not true for rough asymptotic cones. Indeed, $A^r(X) \neq \overline{X}^r$.

Example 3.1 *Let $X = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0\}$. Then the $1-$rough asymptotic cone of the set X is*

$$A^1(X) = \mathbb{R}^2 - \left\{(x, y) \in \mathbb{R}^2 : x \leq 0,\ y \leq 0\right\}.$$

The $1-$rough closure of the set X is

$$\overline{X}^1 = \left\{(x, y) \in \mathbb{R}^2 : x \geq -1,\ y \geq -1\right\}.$$

Hence, we have $A^1(X) \neq \overline{X}^1$.

Proposition 3.3 *A set $X \subset \mathbb{R}^m$ is bounded if and only if $A^r(X) = \varnothing$.*

Proof (Necessity) Let X be a bounded set. Then, from Proposition 2.1, we have $A(X) = \varnothing$. By definition of $A(X)$, there does not exist any sequence $\{x_n\}$ in X such that $\|x_n\| \to \infty$ as $n \to \infty$. That is why we have $A^r(X) = \varnothing$ for all r.

(Sufficiency) Assume that $A^r(X) = \varnothing$. Since $A(X) \subseteq A^r(X) = \varnothing$, we have $A(X) = \varnothing$. From Proposition 2.1, the set X is bounded. □

We know by Theorem 3.6 in [9] that if the set X is convex then the set $A(X)$ is also convex. However, the set $A^r(X)$ may not be convex as can be seen in the following example.

Example 3.2 *Let $X = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R},\ y \geq 0\}$. Then the $\frac{1}{2}-$rough asymptotic cone of the set X is*

$$A^{\frac{1}{2}}(X) = \mathbb{R}^2 - \left\{ \left\{ (x, y) \in \mathbb{R}^2 : y < \frac{\sqrt{3}}{3} x \right\} \cap \left\{ (x, y) \in \mathbb{R}^2 : y < -\frac{\sqrt{3}}{3} x \right\} \right\}.$$

This set is not convex.

Now we will give an example that will help to explain the concept of rough asymptotic cones.

Example 3.3 *Let* $X = \{(x, y) : x \leq 0,\ y \leq 0\} \subset \mathbb{R}^2$. *Thus we have* $A(X) = X$. *The rough asymptotic cone of the set* X *is*

$$A^r(X) = \begin{cases} \left\{(x, y) : y \leq \frac{r\sqrt{1-r^2}}{r^2-1}x\right\} \cap \left\{(x, y) : y \leq \frac{r^2-1}{r\sqrt{1-r^2}}x\right\} & , \text{ if } 0 < r < 1 \\ \mathbb{R}^2 - \{(x, y) : x \geq 0, y \geq 0\} & , \text{ if } r = 1 \\ \mathbb{R}^2 & , \text{ if } r > 1 \end{cases}.$$

On the other hand, we know that the set $A(X)$ *is a closed cone (see page 26 in [1]). As can be seen in this example, this fact is not true for rough asymptotic cones. Indeed, the cone* $A^1(X)$ *is not closed.*

Now we will give a lemma that will be used in the proof of Theorem 3.2. The lemma and its proof are similar to those in [13, Lemma 3.1].

Lemma 3.1 *Let* X *be a set in* $\mathbb{R}^m$. *If the cone* C *is open such that* $C \supset A^r(X)$, *then there exists an* R *such that* $X \subset C \cup B(0, R)$, *where* $0 < R < \infty$.

Proof On the contrary, assume that there exists a sequence $\{x_n\}$ in X such that $x_n \notin C$ and $\|x_n\| \to \infty$. Since C is an open cone, C^c is a closed cone, where C^c is the complement of C. Since the sequence $\{(x_n)\}$ of rays lies in the compact set C^c, there exists a subsequence $\{(x_{n_k})\}$ of $\{(x_n)\}$ such that $\{(x_{n_k})\}$ is convergent to a ray $(x) \in C^c$. By definition of $A^r(X)$, we have $(x) \in A^r(X)$. This contradicts the fact that $C \supset A^r(X)$; thus the proof is complete. □

Lemma 3.2 ([13]) *Let* C_1 *and* C_2 *be open cones such that* $\overline{C_1} \subset C_2$. *Then for all* $a \in \mathbb{R}^m$ *there exists a positive real number* R *such that* $C_1 + a \subset C_2 \cup B(0, R)$.

The next theorem shows that if a sequence $\{X_n\}$ of convex sets is rough convergent to a set X, then all of them are eventually contained in $A^r(X)$ except for a bounded region. Its proof is similar to the proof of [13, Theorem 3.2].

Theorem 3.2 *Let* X_n *and* X *be nonempty convex subsets of* $\mathbb{R}^m$ *for each* n. *If* $X_n \xrightarrow{r} X$ *then for every open cone* C *such that* $C \supset A^r(X)$ *there exist* R *and* N *such that* $X_n \subset C \cup B(0, R)$ *for all* $n > N$.

Proof Without loss of generality, we can select the origin 0. Assume that the theorem is false for 0 but true for $0' = 0 + a$. Then there exists an open cone C_2 such that $C_2 \supset A^r(X)$. In addition, there is a sequence $\{x_n\}$ such that $x_n \in X_n$, $x_n \notin C_2$ and $\|x_n\| \to \infty$ as $n \to \infty$. We can select an open cone C_1 such that $A^r(X) \subset C_1 \subset \overline{C_1} \subset C_2$. The rough asymptotic cone of the set X for the origin $0'$ is $A^r(X) + a$. Furthermore, $A^r(X) + a \subset C_1 + a$. If we apply the theorem for the origin $0'$, we obtain $x_n \in C_1 + a$ for all $n > N$. From Lemma 3.2, we have $x_n \in C_2$ for all $n > N$. This case contradicts the fact that $x_n \notin C_2$; hence the proof is complete.

Let the origin 0 be an arbitrary point of X. On the contrary, assume that the theorem does not hold. Then there is a sequence $\{x_n\}$ in X_n such that $x_n \in X_n$, $\|x_n\| \to \infty$ and $x_n \notin C$. If we take $y_n := \frac{x_n}{\|x_n\|}$, we have $\|y_n\| = 1$. Then we can assume that there exists an $x \neq 0$ such that $\|y_n - x\| < r + \varepsilon$. Since $X_n \xrightarrow{r} X$

and $0 \in X$, there is a sequence $\{s_n\}$ such that $s_n \in X_n$ and $s_n \xrightarrow{r} 0$. Let $a > 0$ be arbitrary. Since X_n is convex, we have

$$t_n = \left(1 - \frac{a}{\|x_n\|}\right) s_n + \left(\frac{a}{\|x_n\|}\right) x_n.$$

Thus $t_n \in X_n$ for all $n > N$. Since $X_n \xrightarrow{r} X$, we obtain $t_n \xrightarrow{r} ax$ and $ax \in \overline{X}$. The case holds for all $a > 0$. Hence $x \in A^r(X)$. On the other hand, since $x_n \notin C$ and $y_n \notin C$, we have $y_n \xrightarrow{r} x \in \overline{C^c} = C^c \cup \{0\}$. Since $x \neq 0$ we get $x \in C^c$, but $x \in A^r(X)$. This contradiction completes the proof. □

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