Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2016) 40: 1374 – 1385
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doi:10.3906/mat-1505-55

Derivation-homomorphisms

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Received: 22.05.2015 • **Accepted/Published Online:** 24.02.2016 • **Final Version:** 02.12.2016

Abstract: In this paper, we introduce notions of (n,m)-derivation-homomorphisms and Boolean n-derivations. Using Boolean n-derivations and m-homomorphisms, we describe structures of (n,m)-derivation-homomorphisms.

Key words: Derivation-homomorphism, Boolean n-derivation, (n,m)-derivation-homomorphism

1. Introduction

In this paper, by a ring we shall always mean an associative ring with an identity.

Homomorphisms and derivations are important in the course of researching rings. Multiderivations (e.g., biderivation, 3-derivation, or n-derivation in general) have been explored in (semi-) rings. In 1989, Vukman [8] researched Posner's theorems [7] for the trace map of symmetric biderivations on (semi-) prime rings. Brešar [1, 2] characterized biderivations on prime and semiprime rings, respectively, explaining the reason why Vukman's results hold. In 2007, Jung and Park [3] investigated Posner's theorems for the trace of permuting 3-derivations on prime and semiprime rings. In cases of permuting 4-derivations and symmetric n-derivations, similar results were obtained in [5] and [6]. It was proved in [10] that a skew n-derivation $(n \geq 3)$ on a semiprime ring R must map into the center of R. Wang et al. [9] also investigated n-derivations $(n \geq 3)$ on triangular algebras. In a recent paper, Li and Xu [4] described multihomomorphisms.

In this paper, we consider a kind of multimapping that is either a derivation or a homomorphism for each component when the other components are fixed by any given elements. Such a multimapping is called an (n,m)-derivation-homomorphism and will be described in this paper.

Let $m \geq 0$, $n \geq 0$, and $m+n > 0$ in $\mathbb{Z}$. Let R_k be rings, where $k \in \{1, \dots, n+m\}$. Let S be a ring and a bimodule ${}_{R_k}S_{R_k}$ for $1 \leq k \leq m$ such that $r_k(st) = (r_k s)t$, $(st)r_k = s(tr_k)$, and $(sr_k)t = s(r_k t)$ for $r_k \in R_k$, $s,t \in S$. Then we call $f : R_1 \times \cdots \times R_{n+m} \to S$ an (n,m)-derivation-homomorphism from $R_1 \times \cdots \times R_{n+m}$ to S, if the following conditions hold:

(i) For $i \in \{1, \dots, n+m\}$

$$f(a_1, \dots, a_i + b, \dots, a_{n+m}) = f(a_1, \dots, a_i, \dots, a_{n+m}) + f(a_1, \dots, b, \dots, a_{n+m});$$

(ii) For $i \in \{1, \dots, n\}$

$$f(a_1, \dots, a_i b, \dots, a_{n+m}) = a_i f(a_1, \dots, b, \dots, a_{n+m}) + f(a_1, \dots, a_i, \dots, a_{n+m})b;$$

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2010 *AMS Mathematics Subject Classification:* 16W25.

(iii) For $i \in \{n+1, \ldots, n+m\}$

$$f(a_1, \ldots, a_i b, \ldots, a_{n+m}) = f(a_1, \ldots, a_i, \ldots, a_{n+m}) f(a_1, \ldots, b, \ldots, a_{n+m}).$$

It is easy to see that an $(m,0)$-derivation-homomorphism is an m-derivation, and a $(0,n)$-derivation-homomorphism is an n-homomorphism. In this paper, our concern will focus on the case $mn \neq 0$, i.e. the case that both m and n are positive.

An n-derivation $\phi : R_1 \times \cdots \times R_n \to S$ is said to be a Boolean n-derivation, if $\phi(x_1, \ldots, x_n) = \phi(x_1, \ldots, x_n)^2$ holds for all $(x_1, \ldots, x_n) \in R_1 \times \cdots \times R_n$. In particular, a Boolean 1-derivation is also called a Boolean derivation.

Let $\phi_i : R_i \to S$ be mappings, $i = 1, \ldots, n$. Then we define $\phi_1 * \cdots * \phi_n : R_1 \times \cdots \times R_n \to S$ as follows:

$$(\phi_1 * \cdots * \phi_n)(a_1, \ldots, a_n) = \phi_1(a_1) \cdots \phi_n(a_n),$$

where $(a_1, \ldots, a_n) \in R_1 \times \cdots \times R_n$.

We call $f : R_1 \times \cdots \times R_n \times R_{n+1} \times \cdots \times R_{n+m} \to S$ an (n,m)-derivation-homomorphism of S, if $R_i = S$ for all $i \in \{1, \ldots, n+m\}$.

2. Main result

Firstly, we consider the case of $(1,1)$-derivation-homomorphisms.

Lemma 2.1 *Let f be a $(1,1)$-derivation-homomorphism from $R_1 \times R_2$ to S. Then for $a, b, c \in R_1$ and $x, y \in R_2$,*

(I) $f(a,x) = -f(a,x)$;

(II) $f(a,x)f(b,y) = f(b,x)f(a,y)$;

(III) $af(b,x) = f(b,x)a$;

(IV) $[a,c]f(b,x) + [b,c]f(a,x) = 0$. In particular, $[a,b]f(b,x) = 0$.

Proof (I) Observing the different expansions of $f(a+b, xy)$, we get

$$\begin{cases} f(a+b,xy) = f(a,xy) + f(b,xy), \\ f(a+b,xy) = f(a+b,x)f(a+b,y) \\ \qquad\qquad\quad = (f(a,x) + f(b,x))(f(a,y) + f(b,y)) \\ \qquad\qquad\quad = f(a,xy) + f(a,x)f(b,y) + f(b,x)f(a,y) + f(b,xy). \end{cases}$$

Then

$$f(a,x)f(b,y) = -f(b,x)f(a,y). \tag{2.1}$$

Taking $y = 1$ and $b = a$ in (2.1), we have $f(a,x)f(a,1) = -f(a,x)f(a,1)$. Hence, $f(a,x) = -f(a,x)$.

(II) It is easy to see from (I) and (2.1).

(III) We write (2.1) as

$$f(a,x)f(b,y) + f(b,x)f(a,y) = 0. \tag{2.2}$$

Replacing a by ab in (2.2), we obtain

$$f(ab,x)f(b,y) + f(b,x)f(ab,y) = 0,$$

that is,

$$af(b,x)f(b,y)+f(a,x)bf(b,y)+f(b,x)af(b,y)+f(b,x)f(a,y)b=0. \tag{2.3}$$

Replacing b by b^2 in (2.2), we obtain

$$f(a,x)f(b^2,y)+f(b^2,x)f(a,y)=0,$$

that is,

$$f(a,x)bf(b,y)+f(a,x)f(b,y)b+bf(b,x)f(a,y)+f(b,x)bf(a,y)=0. \tag{2.4}$$

With (I) and (II), it follows from (2.3) and (2.4) that

$$af(b,x)f(b,y)+f(b,x)af(b,y)+bf(b,x)f(a,y)+f(b,x)bf(a,y)=0. \tag{2.5}$$

Replacing a by ba in (2.2), we get

$$f(ba,x)f(b,y)+f(b,x)f(ba,y)=0,$$

that is,

$$bf(a,x)f(b,y)+f(b,x)af(b,y)+f(b,x)bf(a,y)+f(b,x)f(b,y)a=0. \tag{2.6}$$

With (I) and (II), it follows from (2.5) and (2.6) that

$$af(b,x)f(b,y)+f(b,x)f(b,y)a=0. \tag{2.7}$$

Taking $y=1$, we get

$$af(b,x)+f(b,x)a=0.$$

Then by (I), $af(b,x)=f(b,x)a$.

(IV) Using different expansions of $f(abc,x)$ and (III), we have

$$\begin{cases} f(abc,x)=af(bc,x)+bcf(a,x)=abf(c,x)+acf(b,x)+bcf(a,x), \\ f(abc,x)=abf(c,x)+cf(ab,x)=abf(c,x)+caf(b,x)+cbf(a,x). \end{cases}$$

Therefore,

$$[a,c]f(b,x)+[b,c]f(a,x)=0.$$

Setting $c=b$, we obtain $[a,b]f(b,x)=0$. □

Theorem 2.2 *Let f be a $(1,1)$-derivation-homomorphism from $R_1\times R_2$ to S. Assume that there exists $a_0\in R_1$ such that $f(a_0,1)f(b,1)=f(b,1)f(a_0,1)=f(b,1)$ holds for each $b\in R_1$. Then there exist a Boolean derivation $\phi: R_1\to S$ and a homomorphism $\lambda: R_2\to S$ such that $f=\phi*\lambda$ and $a\lambda(x)-\lambda(x)a=[\phi(a),\lambda(x)]=0$ for $a\in R_1$ and $x\in R_2$. Furthermore, if the identity element of S has an inverse image, then f has a unique decomposition.*

Proof Let $\phi(a) = f(a,1)$ for $a \in R_1$ and $\lambda(x) = f(a_0, x)$ for $x \in R_2$. It is easy to see that ϕ is a Boolean derivation from R_1 to S. Obviously, λ is a homomorphism from R_2 to S. Then by (II) of Lemma 2.1 we have

$$\begin{aligned}(\phi * \lambda)(a,x) &= \phi(a)\lambda(x) = f(a,1)f(a_0,x) = f(a_0,1)f(a,x)\\ &= f(a_0,1)f(a,1)f(a,x) = f(a,1)f(a,x) = f(a,x).\end{aligned}$$

For $a \in R_1$, $x \in R_2$, $a\lambda(x) - \lambda(x)a = 0$ follows from (III) of Lemma 2.1. Then

$$\begin{aligned}\lambda(x)\phi(a) &= f(a_0,x)f(a,1) = f(a,x)f(a_0,1)\\ &= f(a,x)f(a,1)f(a_0,1) = f(a,x)f(a,1)\\ &= f(a,x) = \phi(a)\lambda(x).\end{aligned}$$

Thus the proof of the existence is finished.

Now we prove the uniqueness. Suppose that there exist a Boolean derivation $\phi' : R_1 \to S$ and a homomorphism $\lambda' : R_2 \to S$ such that $f = \phi * \lambda = \phi' * \lambda'$, $a\lambda'(x) - \lambda'(x)a = [\phi'(a), \lambda'(x)] = 0$ for $a \in R_1$, $x \in R_2$, and the identity element of S has an inverse image under f. Then there exists $(a_0, x_0) \in R_1 \times R_2$ such that $f(a_0, x_0) = 1$. Moreover, $1 = f(a_0, x_0) = f(a_0,1)f(a_0,x_0) = f(a_0,1)$. Hence

$$\begin{aligned}&f(a_0,1)(\phi'(a)\lambda'(1) - \phi'(a))\\ =&\phi'(a_0)\lambda'(1)(\phi'(a)\lambda'(1) - \phi'(a))\\ =&\phi'(a_0)\phi'(a)\lambda'(1) - \phi'(a_0)\phi'(a)\lambda'(1)\\ =&0,\end{aligned}$$

that is, $\phi'(a)\lambda'(1) = \phi'(a)$. Furthermore, we obtain

$$\phi(a) = f(a,1) = (\phi' * \lambda')(a,1) = \phi'(a)\lambda'(1) = \phi'(a).$$

Similarly, we get $f(a_0,1)(\phi'(a_0)\lambda'(x) - \lambda'(x)) = 0$, which implies $\phi'(a_0)\lambda'(x) = \lambda'(x)$. Then

$$\lambda(x) = f(a_0,x) = (\phi' * \lambda')(a_0,x) = \phi'(a_0)\lambda'(x) = \lambda'(x).$$

□

The following example shows that it is possible that f has two different decompositions without the assumption that the identity element of S has an inverse image.

Example 2.3 *Let $R = S = \mathbb{F}_2[a,b]/(a^2-1, b^2-b)$, where $\mathbb{F}_2[a,b]$ is the polynomial ring in variables a,b over the field $\mathbb{F}_2$ and $I = (a^2-1, b^2-b)$ is the ideal generated by a^2-1 and b^2-b. Let ϕ be a derivation of $\mathbb{F}_2[a,b]$ by $\phi(a) = b$ and $\phi(b) = 0$. It is easy to see that $\phi(I) \subseteq I$. Therefore, ϕ induces a derivation ϕ of R. It is obvious that $\phi(R) = \{0, \bar{b}\}$, and so ϕ is a Boolean derivation. For all $x \in R$, we define $\lambda : R \to S$ and $\lambda' : R \to S$ by*

$$\lambda(x) = x, \ \ \lambda'(x) = \bar{b}x.$$

It is easy to show that both λ and λ' are homomorphisms from R to S, and $\lambda \neq \lambda'$. Meanwhile, $\phi*\lambda = \phi*\lambda'$. Let $f = \phi*\lambda$. It is clear that f is a $(1,1)$-derivation-homomorphism from $R \times R$ to S, but f has no unique decomposition.

For the derivation-homomorphism of a semiprime ring, we get the following result.

Theorem 2.4 *Let R be a semiprime ring. Then any derivation-homomorphism of R must be zero.*

Proof Let f be a derivation-homomorphism of R, that is, a $(1,1)$-derivation-homomorphism from $R\times R$ to R. By the definition of $(1,1)$-derivation-homomorphism, for any $a,b,c\in R$, we have $f(ab,x)=f(ab,x)f(ab,1)$. It follows from Lemma 2.1 that

$$\begin{aligned}
& af(b,x)+f(a,x)b\\
=&(af(b,x)+f(a,x)b)(af(b,1)+f(a,1)b)\\
=&af(b,x)af(b,1)+af(b,x)f(a,1)b+f(a,x)baf(b,1)+f(a,x)bf(a,1)b\\
=&a^2f(b,x)+abf(a,1)f(b,x)+baf(a,1)f(b,x)+f(a,x)b^2\\
=&a^2f(b,x)+[a,b]f(a,1)f(b,x)+f(a,x)b^2\\
=&a^2f(b,x)+f(a,x)b^2.
\end{aligned}$$

Then

$$af(b,x)+f(a,x)b=a^2f(b,x)+f(a,x)b^2. \tag{2.8}$$

By (I) and (II) of Lemma 2.1, it is easy to show that $f(a^2,x)=0$. Taking $x=1$ and $b=a^2$ in (2.8), we get

$$a^2f(a,1)=a^4f(a,1). \tag{2.9}$$

For any $a,r\in R$, it can be checked from (III), (IV) of Lemma 2.1 and (2.9) that

$$\begin{aligned}
&(a^2-a)f(a,1)r(a^2-a)f(a,1)\\
=&(a^2f(a,1)-af(a,1))r(a^2f(a,1)-af(a,1))\\
=&a^2f(a,1)ra^2f(a,1)-a^2f(a,1)raf(a,1)-af(a,1)ra^2f(a,1)+af(a,1)raf(a,1)\\
=&a^2ra^2f(a,1)-a^2raf(a,1)-ara^2f(a,1)+araf(a,1)\\
=&-a(a^2ra)f(a,1)-a^2raf(a,1)+a(ara)f(a,1)-a(ar)f(a,1)\\
=&-a^3raf(a,1)-a^2rf(a,1)\\
=&a(a^3r)f(a,1)-a^2rf(a,1)\\
=&0.
\end{aligned}$$

Since R is a semiprime ring, we have $(a^2-a)f(a,1)=0$. Therefore

$$(af(a,1))^2=a^2f(a,1)=af(a,1),$$

that is, $af(a,1)$ is an idempotent element. By the definition of derivation-homomorphism, we get

$$\begin{aligned}
f(a,x)&=f(a,x)f(a,1)\\
&=f(af(a,1),x)-af(f(a,1),1)\\
&=f((af(a,1))^2,x)-af((f(a,1))^2,1)\\
&=0.
\end{aligned}$$

□

In order to describe (n,m)-derivation-homomorphisms of a given ring, we first give two lemmas.

Lemma 2.5 *Let $f : R_1 \times \cdots \times R_{n+1} \to S$ be an $(n,1)$-derivation-homomorphism. Then for any $(a_1, x_1, \ldots, a_n, x_n, b, c) \in R_1^2 \times \cdots \times R_n^2 \times R_{n+1}^2$,*

$$\sum_{u_1,\ldots,u_n} f(u_1,\ldots,u_n,b)f(v_1,\ldots,v_n,c) = 0,$$

where u_i is one component of (a_i, x_i) and v_i is the other component, and so the left-hand side of the above equation is the sum of 2^n terms.

Proof We prove this by induction on n. For $n = 1$, we have obtained the conclusion from (2.2).

Assume the lemma holds for $1, \ldots, n-1$, that is to say, for all $k \le n-1$, any $(k,1)$-derivation-homomorphism $g : R_1 \times \cdots \times R_k \times R_{k+1} \to S$ and any $(a_1, x_1, \ldots, a_k, x_k, b, c) \in R_1^2 \times \cdots \times R_k^2 \times R_{k+1}^2$, we have

$$\sum_{u_1,\ldots,u_k} g(u_1,\ldots,u_k,b)f(v_1,\ldots,v_k,c) = 0, \tag{2.10}$$

where u_i is one component of (a_i, x_i) and v_i is the other component, and so the left-hand side of the above equation is the sum of 2^k terms.

Let f be an $(n,1)$-derivation-homomorphism. For any

$$(a_1, x_1, \ldots, a_n, x_n, b, c) \in R_1^2 \times \cdots \times R_n^2 \times R_{n+1}^2,$$

expanding the first n variables of $f(a_1 + x_1, \ldots, a_n + x_n, bc)$ by addition, and then expanding the $(n+1)$-th variable by multiplication, we have

$$\begin{aligned} &f(a_1 + x_1, \ldots, a_n + x_n, bc) \\ =&\sum_{u_1,\ldots,u_n} f(u_1,\ldots,u_n,bc) \\ =&\sum_{u_1,\ldots,u_n} f(u_1,\ldots,u_n,b)f(u_1,\ldots,u_n,c), \end{aligned} \tag{2.11}$$

where u_i is one component of (a_i, x_i), and so the right-hand side of (2.11) is the sum of 2^n terms. On the other hand, expanding the $(n+1)$-th variable of $f(a_1 + x_1, \ldots, a_n + x_n, bc)$ by multiplication, and then expanding the first n variables by addition, we obtain

$$\begin{aligned} &f(a_1 + x_1, \ldots, a_n + x_n, bc) \\ =&f(a_1 + x_1, \ldots, a_n + x_n, b)f(a_1 + x_1, \ldots, a_n + x_n, c) \\ =&\sum_{y_1,\ldots,y_n} \sum_{z_1,\ldots,z_n} f(y_1,\ldots,y_n,b)f(z_1,\ldots,z_n,c), \end{aligned} \tag{2.12}$$

where y_i is one component of (a_i, x_i) and z_i is one component of (a_i, x_i), and so the right-hand side of (2.12) is the sum of 2^{2n} terms.

We shall now classify items on the right-hand side of (2.12). For any $s \in \{0, \ldots, n\}$, denote by A_s the sum of the item on the right-hand side of (2.12) that satisfies the following condition:

There exist $1 \leq j_1 < j_2 < \cdots < j_s \leq n$ such that y_{j_t} is one component of (a_k, x_k), and z_{j_t} is the other component for $t = 1, \ldots, s$; however, y_k and z_k are the same component of (a_k, x_k) for $k \in \{1, \ldots, n\} \backslash \{j_1, \ldots, j_s\}$.

Then by (2.12) we get

$$f(a_1 + x_1, \ldots, a_n + x_n, bc) = A_0 + \cdots + A_n. \tag{2.13}$$

If $s \in \{1, \ldots, n-1\}$, let $i_1, \ldots, i_{n-s} \in \{1, \ldots, n\}$ with $i_1 < \cdots < i_{n-s}$. Denote by $\{j_1, \ldots, j_s\}$ the complementary set of $\{i_1, \ldots, i_{n-s}\}$ in $\{1, \ldots, n\}$. Fixed positions $i_1, \ldots, i_{n-s}$ in f by $u_{i_1}, \ldots, u_{i_{n-s}}$, we obtain an $(s,1)$-derivation-homomorphism

$$g_{u_{i_1}, \ldots, u_{i_{n-s}}}(y_{j_1}, \ldots, y_{j_s}, b) = f(y_1, \ldots, y_n, b), \tag{2.14}$$

where $(y_{i_1}, \ldots, y_{i_{n-s}}) = (u_{i_1}, \ldots, u_{i_{n-s}})$. It follows from (2.10), (2.13), and (2.14) that

$$A_s = \sum_{i_1 < \cdots < i_{n-s}} \sum_{u_{i_1}, \ldots, u_{i_{n-s}}} \sum_{y_{j_1}, \ldots, y_{j_s}} g_{u_{i_1}, \ldots, u_{i_{n-s}}}(y_{j_1}, \ldots, y_{j_s}, b) \cdot g_{u_{i_1}, \ldots, u_{i_{n-s}}}(z_{j_1}, \ldots, z_{j_s}, c).$$

By the inductive assumption, we have

$$\sum_{y_{j_1}, \ldots, y_{j_s}} g_{u_{i_1}, \ldots, u_{i_{n-s}}}(y_{j_1}, \ldots, y_{j_s}, b) \cdot g_{u_{i_1}, \ldots, u_{i_{n-s}}}(z_{j_1}, \ldots, z_{j_s}, c) = 0.$$

Moreover, $A_s = 0$ for all $1 \leq s \leq n-1$. Looking back at (2.11) and (2.13), and noting that the right-hand side of (2.11) is A_0, we get $A_0 = A_0 + A_n$. Thus the proof is completed. □

Lemma 2.6 *Let f be an $(n,1)$-derivation-homomorphism of a ring S, that is, an $(n,1)$-derivation-homomorphism from $R_1 \times \cdots \times R_{n+1}$ to S, where $R_i = S$ for all $i \in \{1, \ldots, n+1\}$. Assume that the identity element of S has an inverse image, that is, there exists $(x_1, \cdots, x_n, x_{n+1}) \in R_1 \times \cdots \times R_{n+1}$ such that $f(x_1, \cdots, x_n, x_{n+1}) = 1$. Then there exist a unique Boolean n-derivation $\phi : R_1 \times \cdots \times R_n \to S$ and a unique homomorphism $\lambda : R_{n+1} \to Z(S)$ such that $f = \phi * \lambda$, where $\phi(a_1, \ldots, a_n) = f(a_1, \ldots, a_n, 1)$ and $\lambda(b) = f(x_1, \ldots, x_n, b)$.*

Proof Firstly we prove the existence. From now on, in the course of proof of this Lemma, we will always assume that $R_1 = \cdots = R_{n+1} = S$. In order to make the implication of the symbols clear, we go on to use all the symbols $R_1, \ldots, R_{n+1}$ except the symbol S. We shall prove that any $(n,1)$-derivation-homomorphism $f : R_1 \times \cdots \times R_n \times R_{n+1} \to S$ satisfies

$$f(a_1, \ldots, a_n, b) = f(a_1, \ldots, a_n, 1) f(x_1, \ldots, x_n, b),$$

for a given $(x_1, \ldots, x_n) \in R_1 \times \cdots \times R_n$ and any $(a_1, \ldots, a_n, b) \in R_1 \times \cdots \times R_n \times R_{n+1}$.

If $n = 1$, it is a part of conclusions in Theorem 2.2. We now proceed by induction on n.

Assume the lemma holds for $1, \ldots, n-1$, that is, for all $1 \leq k \leq n-1$ and any $(k,1)$-derivation-homomorphism $g : R_1 \times \cdots \times R_k \times R_{k+1} \to S$, we have

$$g(a_1, \ldots, a_k, b) = g(a_1, \ldots, a_k, 1) g(x_1, \ldots, x_k, b), \tag{2.15}$$

for a given $(x_1, \dots, x_k) \in R_1 \times \cdots \times R_k$ and any $(a_1, \dots, a_k, b) \in R_1 \times \cdots \times R_k \times R_{k+1}$.

Let f be an $(n, 1)$-derivation-homomorphism. Since the identity element of S has an inverse image, there exists $(x_1, \dots, x_n, 1) \in R_1 \times \cdots \times R_n \times R_{n+1}$ such that $f(x_1, \dots, x_n, 1) = 1$, since

$$1 = f(x_1, \dots, x_n, x_{n+1}) = f(x_1, \dots, x_n, 1)f(x_1, \dots, x_n, x_{n+1}) = f(x_1, \dots, x_n, 1).$$

Fixing the first $n-1$ variables in $f(a_1, \dots, a_n, b)$, then $f(a_1, \dots, a_n, b)$ can be viewed as a $(1,1)$-derivation-homomorphism from $R_n \times R_{n+1}$ to S. By Lemma 2.1, for any $(a_1, \dots, a_n, b) \in R_1 \times \cdots \times R_n \times R_{n+1}$ and $r \in S$, we get

$$f(a_1, \dots, a_n, b) = -f(a_1, \dots, a_n, b), \tag{2.16}$$

and

$$f(a_1, \dots, a_n, b)r = rf(a_1, \dots, a_n, b). \tag{2.17}$$

For any $(a_1, x_1, \dots, a_n, x_n, b, c) \in R_1^2 \times \cdots \times R_n^2 \times R_{n+1}^2$, by Lemma 2.5, we obtain

$$\sum_{u_1, \dots, u_n} f(u_1, \dots, u_n, b)f(v_1, \dots, v_n, 1) = 0, \tag{2.18}$$

where u_i is one component of (a_i, x_i), v_i is the other component, and so the left-hand side of (2.18) is the sum of 2^n terms. If $k \in \{1, \dots, n-1\}$, let $i_1, \dots, i_k \in \{1, \dots, n\}$ with $i_1 < \cdots < i_k$. We denote by $\{j_1, \dots, j_s\}$ the complementary set of $\{i_1, \dots, i_k\}$ in $\{1, \dots, n\}$. Fixing variables $i_1, \dots, i_k$ in f through $x_{i_1}, \dots, x_{i_k}$, we obtain an $(n-k, 1)$-derivation-homomorphism

$$h_{x_{i_1}, \dots, x_{i_k}}(a_{j_1}, \dots, a_{j_{n-k}}, b) = f(u_1, \dots, u_n, b), \tag{2.19}$$

where $(u_{i_1}, \dots, u_{i_k}) = (x_{i_1}, \dots, x_{i_k})$ and $(u_{j_1}, \dots, u_{j_{n-k}}) = (a_{j_1}, \dots, a_{j_{n-k}})$. By (2.19), we write (2.18) as

$$f(a_1, \dots, a_n, b)f(x_1, \dots, x_n, 1) + f(x_1, \dots, x_n, b)f(a_1, \dots, a_n, 1) + \sum_{k=1}^{n-1} B_k = 0, \tag{2.20}$$

where $B_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} h_{x_{i_1}, \dots, x_{i_k}}(a_{j_1}, \dots, a_{j_{n-k}}, b)h_{x_{j_1}, \dots, x_{j_{n-k}}}(a_{i_1}, \dots, a_{i_k}, 1)$. It follows from (2.15), (2.17), and (2.19) that

$$\begin{aligned} B_k &= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} h_{x_{i_1}, \dots, x_{i_k}}(a_{j_1}, \dots, a_{j_{n-k}}, b)h_{x_{j_1}, \dots, x_{j_{n-k}}}(a_{i_1}, \dots, a_{i_k}, 1) \\ &= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} h_{x_{i_1}, \dots, x_{i_k}}(a_{j_1}, \dots, a_{j_{n-k}}, 1)h_{x_{i_1}, \dots, x_{i_k}}(x_{j_1}, \dots, x_{j_{n-k}}, b) \\ &\qquad \cdot h_{x_{j_1}, \dots, x_{j_{n-k}}}(a_{i_1}, \dots, a_{i_k}, 1) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\le i_1<i_2<\cdots<i_k\le n} h_{x_{i_1},\ldots,x_{i_k}}(a_{j_1},\ldots,a_{j_{n-k}},1)h_{x_{j_1},\ldots,x_{j_{n-k}}}(a_{i_1},\ldots,a_{i_k},1)\\
&\qquad\qquad\qquad\qquad\cdot h_{x_{i_1},\ldots,x_{i_k}}(x_{j_1},\ldots,x_{j_{n-k}},b)\\
&= \sum_{1\le i_1<i_2<\cdots<i_k\le n} h_{x_{i_1},\ldots,x_{i_k}}(a_{j_1},\ldots,a_{j_{n-k}},1)h_{x_{j_1},\ldots,x_{j_{n-k}}}(a_{i_1},\ldots,a_{i_k},1)\\
&\qquad\qquad\qquad\qquad\cdot h_{x_{j_1},\ldots,x_{j_{n-k}}}(x_{i_1},\ldots,x_{i_k},b) \qquad (2.21)\\
&= \sum_{1\le i_1<i_2<\cdots<i_k\le n} h_{x_{i_1},\ldots,x_{i_k}}(a_{j_1},\ldots,a_{j_{n-k}},1)h_{x_{j_1},\ldots,x_{j_{n-k}}}(a_{i_1},\ldots,a_{i_k},b)\\
&= \sum_{j_1,\ldots,j_{n-k}} h_{x_{j_1},\ldots,x_{j_{n-k}}}(a_{i_1},\ldots,a_{i_k},b)h_{x_{i_1},\ldots,x_{i_k}}(a_{j_1},\ldots,a_{j_{n-k}},1)\\
&= B_{n-k}.
\end{aligned}
$$

If n is odd, k and $n-k$ are one-to-one and then we have $\sum_{k=1}^{n-1} B_k = 0$. If n is even, then $\sum_{k=1}^{n-1} B_k = B_m$, where $m = n/2$. From (2.15), (2.16), (2.19), and (2.21), the items in B_m satisfy

$$
\begin{aligned}
&h_{x_{i_1},\ldots,x_{i_m}}(a_{j_1},\ldots,a_{j_m},b)h_{x_{j_1},\ldots,x_{j_m}}(a_{i_1},\ldots,a_{i_m},1)\\
&+h_{x_{j_1},\ldots,x_{j_m}}(a_{i_1},\ldots,a_{i_m},b)h_{x_{i_1},\ldots,x_{i_m}}(a_{j_1},\ldots,a_{j_m},1) = 0.
\end{aligned} \tag{2.22}
$$

Thus $B_m = 0$. Hence, $\sum_{k=1}^{n-1} B_k = 0$. Then by (2.20) we obtain

$$
\begin{aligned}
f(a_1,\ldots,a_n,b) =& f(a_1,\ldots,a_n,b)f(x_1,\ldots,x_n,1)\\
=& f(a_1,\ldots,a_n,1)f(x_1,\ldots,x_n,b).
\end{aligned} \tag{2.23}
$$

Let $\phi(a_1,\ldots,a_n) = f(a_1,\ldots,a_n,1)$ and $\lambda(b) = f(x_1,\ldots,x_n,b)$. It is obvious that ϕ is a Boolean n-derivation from $R_1\times\cdots\times R_n$ to S and λ is a homomorphism from R_{n+1} to $Z(S)$. By (2.23), we get

$$
f(a_1,\ldots,a_n,b) = \phi(a_1,\ldots,a_n)\lambda(b) = (\phi*\lambda)(a_1,\ldots,a_n,b).
$$

Now we prove the uniqueness. Suppose that there exist a Boolean n-derivation $\phi' : R_1\times\cdots\times R_n \to S$ and a homomorphism $\lambda' : R_{n+1} \to Z(S)$ such that $f = \phi*\lambda = \phi'*\lambda'$. Assume the identity element of S has an inverse image under f. Then there exists $(x_1,\ldots,x_n,1) \in R_1\times\cdots\times R_{n+1}$ such that $f(x_1,\ldots,x_n,1) = 1$. From the definition of ϕ' and λ', it is easy to see that

$$
\begin{aligned}
&f(x_1,\ldots,x_n,1)(\phi'(a_1,\ldots,a_n)\lambda'(1) - \phi'(a_1,\ldots,a_n))\\
=&\phi'(x_1,\ldots,x_n)\lambda'(1)(\phi'(a_1,\ldots,a_n)\lambda'(1) - \phi'(a_1,\ldots,a_n))\\
=&\phi'(x_1,\ldots,x_n)\lambda'(1)\lambda'(1)\phi'(a_1,\ldots,a_n) - \phi'(x_1,\ldots,x_n)\lambda'(1)\phi'(a_1,\ldots,a_n)\\
=&0,
\end{aligned}
$$

that is, $\phi'(a_1,\dots,a_n)\lambda'(1)=\phi'(a_1,\dots,a_n)$. Furthermore, we have

$$\begin{aligned}\phi(a_1,\dots,a_n)&=f(a_1,\dots,a_n,1)\\&=(\phi'*\lambda')(a_1,\dots,a_n,1)\\&=\phi'(a_1,\dots,a_n)\lambda'(1)\\&=\phi'(a_1,\dots,a_n).\end{aligned}$$

In a similar way, we can prove that

$$f(x_1,\dots,x_n,1)(\phi'(x_1,\dots,x_n)\lambda'(b)-\lambda'(b))=0,$$

which implies $\phi'(x_1,\dots,x_n)\lambda'(b)=\lambda'(b)$. Then

$$\begin{aligned}\lambda(b)&=f(x_1,\dots,x_n,b)\\&=(\phi'*\lambda')(x_1,\dots,x_n,b)\\&=\phi'(x_1,\dots,x_n)\lambda'(b)\\&=\lambda'(b).\end{aligned}$$

□

In order to prove Theorem 2.8, we also need the following lemma, which can be obtained from the proof of Corollary 2 in [4].

Lemma 2.7 *Let f be a mapping from $R_1\times\cdots\times R_n$ to S, where $R_1=\cdots=R_n=S$ is a ring. Then f is an n-homomorphism if and only if there exist pairwise commutative Boolean homomorphisms $\phi_i:R_i\to S$ for $i\in\{1,\dots,n\}$ such that $f=\phi_1*\cdots*\phi_n$, where $\phi_i(a_i)=f(1,\dots,1,a_i,1,\dots,1)$, $i=1,\dots,n$.*

Theorem 2.8 *Let f be an (n,m)-derivation-homomorphism of a ring S, that is, an (n,m)-derivation-homomorphism from $R_1\times\cdots\times R_{n+m}$ to S, where $R_i=S$ for all $i\in\{1,\dots,n+m\}$. Assume that the identity element of S has an inverse image. Then there exist a unique Boolean n-derivation $\phi:R_1\times\cdots\times R_n\to S$ and a unique m-homomorphism $\lambda:R_{n+1}\times\cdots\times R_{n+m}\to Z(S)$ such that $f=\phi*\lambda$.*

Proof Firstly we prove the existence. Fixing the first n variables in an (n,m)-derivation-homomorphism $f:R_1\times\cdots\times R_{n+m}\to S$, we can view f as an m-homomorphism from $R_{n+1}\times\cdots\times R_{n+m}$ to S.

As the identity element of S has an inverse image, by Lemma 2.7, there exists $(x_1,\dots,x_n,1,\dots,1)\in R_1\times\cdots\times R_{n+m}$ such that

$$f(x_1,\dots,x_n,1,\dots,1)=1,$$

since

$$\begin{aligned}1=&f(x_1,\dots,x_n,x_{n+1},\dots,x_{n+m})\\=&f(x_1,\dots,x_n,x_{n+1},1,\dots,1)\cdots f(x_1,\dots,x_n,1,\dots,1,x_{n+m})\\=&f(x_1,\dots,x_n,1,\dots,1)f(x_1,\dots,x_n,x_{n+1},1,\dots,1)\\&\qquad\cdots f(x_1,\dots,x_n,1,\dots,1,x_{n+m})\\=&f(x_1,\dots,x_n,1,\dots,1)f(x_1,\dots,x_n,x_{n+1},\dots,x_{n+m})\\=&f(x_1,\dots,x_n,1,\dots,1).\end{aligned}$$

Fixing $m-1$ variables among the last m variables in $f(a_1,\ldots,a_n,b_1,\ldots,b_m)$, we can view f as an $(n,1)$-derivation-homomorphism. Then Lemma 2.6 implies that $f(x_1,\ldots,x_n,1,\ldots,1,b_i,1\ldots,1)\in Z(S)$. Hence

$$\begin{aligned}
&f(a_1,\ldots,a_n,b_1,\ldots,b_m)\\
=&f(a_1,\ldots,a_n,b_1,1,\ldots,1)\cdots f(a_1,\ldots,a_n,1,\ldots,1,b_m)\\
=&f(a_1,\ldots,a_n,1,\ldots,1)f(x_1,\ldots,x_n,b_1,1,\ldots,1)\\
&\qquad\cdots f(a_1,\ldots,a_n,1,\ldots,1)f(x_1,\ldots,x_n,1,\ldots,1,b_m)\\
=&(f(a_1,\ldots,a_n,1,\ldots,1))^m f(x_1,\ldots,x_n,b_1,1,\ldots,1)\cdots f(x_1,\ldots,x_n,1,\ldots,1,b_m)\\
=&f(a_1,\ldots,a_n,1,\ldots,1)f(x_1,\ldots,x_n,b_1,\ldots,b_m).
\end{aligned}\tag{2.24}$$

Let

$$\phi(a_1,\ldots,a_n)=f(a_1,\ldots,a_n,1,\ldots,1),$$

and

$$\lambda(b_1,\ldots,b_m)=f(x_1,\ldots,x_n,b_1,\ldots,b_m).$$

It is easy to show that ϕ is a Boolean n-derivation from $R_1\times\cdots\times R_n$ to S and λ is an m-homomorphism from $R_{n+1}\times\cdots\times R_{n+m}$ to $Z(S)$. Then by (2.24) we obtain

$$\begin{aligned}
f(a_1,\ldots,a_n,b_1,\ldots,b_m)&=\phi(a_1,\ldots,a_n)\lambda(b_1,\ldots,b_m)\\
&=(\phi*\lambda)(a_1,\ldots,a_n,b_1,\ldots,b_m).
\end{aligned}$$

Now we prove the uniqueness. Suppose that there exist a Boolean n-derivation $\phi':R_1\times\cdots\times R_n\to S$ and an m-homomorphism $\lambda':R_{n+1}\times\cdots\times R_{n+m}\to Z(S)$ such that $f=\phi*\lambda=\phi'*\lambda'$. Assume the identity element of S has an inverse image under f. Thus, there exists $(x_1,\ldots,x_n,1,\ldots,1)\in R_1\times\cdots\times R_{n+m}$ such that

$$f(x_1,\ldots,x_n,1,\ldots,1)=1.$$

From the definition of ϕ' and λ', it is easy to see that

$$\begin{aligned}
&f(x_1,\ldots,x_n,1,\ldots,1)(\phi'(a_1,\ldots,a_n)\lambda'(1,\ldots,1)-\phi'(a_1,\ldots,a_n))\\
=&\phi'(x_1,\ldots,x_n)\lambda'(1,\ldots,1)(\phi'(a_1,\ldots,a_n)\lambda'(1,\ldots,1)-\phi'(a_1,\ldots,a_n))\\
=&\phi'(x_1,\ldots,x_n)\lambda'(1,\ldots,1)\lambda'(1,\ldots,1)\phi'(a_1,\ldots,a_n)\\
&\qquad-\phi'(x_1,\ldots,x_n)\lambda'(1,\ldots,1)\phi'(a_1,\ldots,a_n)\\
=&0.
\end{aligned}$$

Therefore, $\phi'(a_1,\ldots,a_n)\lambda'(1,\ldots,1)=\phi'(a_1,\ldots,a_n)$. Hence, we have

$$\begin{aligned}
\phi(a_1,\ldots,a_n)&=f(a_1,\ldots,a_n,1,\ldots,1)\\
&=(\phi'*\lambda')(a_1,\ldots,a_n,1,\ldots,1)\\
&=\phi'(a_1,\ldots,a_n)\lambda'(1,\ldots,1)\\
&=\phi'(a_1,\ldots,a_n).
\end{aligned}$$

Similarly, it can be checked that

$$f(x_1, \ldots, x_n, 1, \ldots, 1)(\phi'(x_1, \ldots, x_n)\lambda'(b_1, \ldots, b_m) - \lambda'(b_1, \ldots, b_m)) = 0,$$

that is $\phi'(x_1, \ldots, x_n)\lambda'(b_1, \ldots, b_m) = \lambda'(b_1, \ldots, b_m)$. Then we get

$$\begin{aligned}\lambda(b_1, \ldots, b_m) &= f(x_1, \ldots, x_n, b_1, \ldots, b_m) \\ &= (\phi' * \lambda')(x_1, \ldots, x_n, b_1, \ldots, b_m) \\ &= \phi'(x_1, \ldots, x_n)\lambda'(b_1, \ldots, b_m) \\ &= \lambda'(b_1, \ldots, b_m).\end{aligned}$$

□

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