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Every norm is a restriction of an order-unit norm

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Abstract: We point out the equivalence of the fact that every norm on a vector space is a restriction of an order-unit norm to that of Paulsen's construction concerning generalization of operator systems.

Key words: Norm, ordered vector space, order-unit

1. Introduction

The purpose of this very short expository note is to bring a widely unnoticed fact concerning normed spaces to the readers' attention by pointing out that it is equivalent to Paulsen's construction in quantum analysis given in [4]. We refer to [1] for the general theory of ordered vector spaces.

A subset K of a vector space E is called a ***cone*** if

$$K + K \subseteq K,\ \mathbb{R}^+ K \subseteq K,\ \text{and } K \cap (-K) = \{0\},$$

in which case the pair (E, K) is called an ***ordered vector space***. We write $x \leqslant y$, or $y \geqslant x$ in E, whenever $y - x \in K$. An element $e \in K \setminus \{0\}$ is called an ***order-unit*** if for each $x \in E$ there exists a $\lambda > 0$ such that $x \leqslant \lambda e$. The notion of order-unit is due to Kadison [3]. An ordered vector space E is called ***almost Archimedean*** if $-\varepsilon x \leqslant y \leqslant \varepsilon x$ for all $\varepsilon > 0$; then $y = 0$. Similarly, E is called ***Archimedean*** if $\mathbb{N}x \leqslant y$ implies $x \leqslant 0$. It is obvious that Archimedeanness implies almost Archimedeanness, but not vice versa. If (E, K) is an almost Archimedean vector space with an order-unit $e > 0$, then

$$\|x\|_e = \inf\{\varepsilon > 0 : -\varepsilon e \leqslant x \leqslant \varepsilon e\}$$

defines a norm on the ordered vector space (E, K). Let us call this norm as the ***norm generated by the order unit*** e.

Theorem 1 *Let $(E, \|\cdot\|)$ be a normed space. Then there exists an Archimedean ordered vector space F with an order-unit $e > 0$ such that E is isomorphic to a vector subspace of F, and in this case the equality*

$$\|x\| = \|x\|_e$$

holds for all $x \in E$.

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Proof Let $F := E \times \mathbb{R}$, and consider F as a vector space under the pointwise algebraic operations. One can easily show that F is an Archimedean ordered vector space with respect to the cone

$$K = \{(x, r) : x \in B(0, r)\},$$

where $B(0, r)$ is the closed ball with center zero and radius r in the normed space $(E, \|\cdot\|_H)$. Moreover, the element $e = (0, 1)$ is an order unit of F. It is also obvious that F can be embedded into F as a vector subspace via the map $x \mapsto (x, 0)$. Therefore, we may write x instead of $(x, 0)$. From the following equality

$$-\|x\|\, e \leqslant (x, 0) \leqslant \|x\|\, e,$$

it follows that

$$\|x\|_e = \|(x, 0)\|_e \leqslant \|x\|.$$

Now suppose that $\|(x, 0)\|_e < \|x\|_H$. Choose $\alpha > 0$ so that

$$\|(x, 0)\|_e < \alpha < \|x\|, \text{ and } (x, 0) \leqslant \alpha e.$$

Then $0 \leqslant (-x, \alpha)$ in F, whence

$$\|x\| = \|-x\| \leqslant \alpha < \|x\|,$$

a contradiction. This completes the proof. □

Let us point out that Theorem 1 follows directly from the following construction of Paulsen [4, pp. 178, 182]: Let V be a (real or complex) vector space equipped with a norm $\|\cdot\|$. Define $\mathcal{S} = \left\{ \begin{bmatrix} \lambda & v \\ w^* & \mu \end{bmatrix} : \lambda, \mu \in \mathbb{C},\ v, w \in V \right\} \subseteq \mathbb{C} \oplus \mathbb{C} \oplus V \oplus \overline{V}$, where $\overline{V}$ is the conjugate space to V. Note that $\mathcal{S}$ is a $*$-vector space with the involution $\begin{bmatrix} \lambda & v \\ w^* & \mu \end{bmatrix}^* = \begin{bmatrix} \overline{\lambda} & w \\ v^* & \overline{\mu} \end{bmatrix}$, and $\mathcal{S}_h = \{x \in \mathcal{S} : x^* = x\}$ is a real vector subspace in $\mathcal{S}$. Define

$$\mathcal{C} := \left\{ \begin{bmatrix} \lambda & v \\ v^* & \mu \end{bmatrix} : \lambda, \mu \geqslant 0,\ \|v\|^2 \leqslant \lambda\mu \right\} \subseteq \mathcal{S} \text{ and } \varepsilon := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and note that $\mathcal{C}$ is a cone in $\mathcal{S}_h$, that e is an order-unit, and that $\mathcal{S}_h$ is an ordered vector space that is almost Archimedean. In particular, $\|x\|_e = \inf\{\varepsilon > 0 : -\varepsilon e \leqslant x \leqslant \varepsilon e\}$ is the norm on $\mathcal{S}_h$ generated by the order-unit e. Moreover, the (real part) vector space V can be embedded into $\mathcal{S}_h$ via the real linear mapping $v \mapsto \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}$. Lastly, the equality

$$\|v\| = \inf \left\{ \varepsilon > 0 : \begin{bmatrix} \varepsilon & v \\ v^* & \varepsilon \end{bmatrix} \in \mathcal{C} \right\} = \left\| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right\|_e$$

holds, whence the proof of Theorem is complete 1.

We also note, for the sake of completeness, that similar arguments underlying the proof of Theorem 1 in the setting of normed spaces are also used in [2].

References

[1] Aliprantis CD, Tourky R. Cones and Duality. Providence, RI: American Mathematical Society Graduate Studies in Mathematics, Vol. 84, 2007.

[2] Fakhar M, Koushesh MR, Raoofi M. Embedding normed linear spaces into $C(X)$. arXiv:1502.06008.

[3] Kadison RV. A representation theory for commutative topological algebra. Mem Amer Math Soc 1971; 21: 39 pp.

[4] Paulsen V. Completely Bounded Maps and Operator Algebras. Cambridge, UK: Cambridge University Press, 2003.