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On generalized Ostrowski-type inequalities for functions whose first derivatives absolute values are convex

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Abstract: In this paper, we establish some generalized Ostrowski-type inequalities for functions whose first derivatives absolute values are convex.

Key words: Ostrowski-type inequalities, Hölder's inequality, convex functions

1. Introduction

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [11]:

Theorem 1 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then we have the inequality*

$$\left| f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

This inequality is well known in the literature as the *Ostrowski inequality.*

Definition 1 *The function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is said to be convex if the following inequality holds:*

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Let $f:I\subseteq R\to\mathrm{R}$ be a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds [13]:

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}. \tag{1.2}$$

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In [8], Dragomir and Agarwal gave the following important inequality for convex functions:

Theorem 2 *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)du \right| \leq (b-a) \frac{|f'(a)| + |f'(b)|}{8}. \tag{1.3}$$

In [12], Ozdemir et al. gave the following Ostrowski-type inequalities for functions whose derivatives are convex:

Theorem 3 *Let $I \subset \mathbb{R}$ be an open interval and $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function where $a, b \in I$ with $a < b$. If $|f'|^q$ is a convex function for $\lambda \in [0,1]$, $x \in [a, b]$, and $q \in [1, \infty)$, then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(u)du \right. \qquad (1.4)\\
&\left. - \frac{(b-x)\left[(1-\lambda) f(x) + \lambda f(b)\right] + (x-a)\left[(1-\lambda) f(x) + \lambda f(a)\right]}{(b-a} \right| \\
\leq\ & (b-a) \left(\frac{2\lambda^2 - 2\lambda + 1}{2} \right)^{\frac{q-1}{q}} \left\{ \left[\left(\frac{2\lambda^3 - 3\lambda + 2}{6} \right) \left(\frac{b-x}{b-a} \right)^{2q+1} |f'(a)|^q \right. \right. \\
&+ \left. \left(\frac{\left[6\lambda^2 - 6\lambda + 3\right] - \left[2\lambda^3 - 3\lambda + 2\right] \left(\frac{b-x}{b-a} \right)}{6} \right) \left(\frac{b-x}{b-a} \right)^{2q} |f'b|^q \right]^{\frac{1}{q}} \\
&+ \left[\left(\frac{\left[6\lambda^2 - 6\lambda + 3\right] - \left[2\lambda^3 - 3\lambda + 2\right] \left(\frac{x-a}{b-a} \right)}{6} \right) \left(\frac{x-a}{b-a} \right)^{2q} |f'(a)|^q \right. \\
&+ \left. \left. \left(\frac{2\lambda^3 - 3\lambda + 2}{6} \right) \left(\frac{x-a}{b-a} \right)^{2q+1} |f'(b)|^q \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

For more information and recent advances on Ostrowski-type inequalities, please refer to [1-10, 12, 14-18].

The aim of this paper is to establish generalization of the inequality (1.4) and give some special results.

2. Main Results

First, we will give the following calculated integrals used as the main results:

$$\int_0^{\frac{b-x}{b-a}} \left| t - \lambda \frac{b-x}{b-a} \right| t dt = \left(\frac{2\lambda^3 - 3\lambda + 2}{6} \right) \left(\frac{b-x}{b-a} \right)^3, \tag{2.1}$$

$$\int_0^{\frac{b-x}{b-a}} \left| t-\lambda\frac{b-x}{b-a}\right| (1-t)\,dt \tag{2.2}$$
$$= \left(\frac{\left[6\lambda^2-6\lambda+3\right]-\left[2\lambda^3-3\lambda+2\right]\left(\frac{b-x}{b-a}\right)}{6}\right)\left(\frac{b-x}{b-a}\right)^2,$$

$$\int_{\frac{b-x}{b-a}}^{1} \left| t-1+\lambda\frac{x-a}{b-a}\right| tdt \tag{2.3}$$
$$= \left(\frac{\left[6\lambda^2-6\lambda+3\right]-\left[2\lambda^3-3\lambda+2\right]\left(\frac{x-a}{b-a}\right)}{6}\right)\left(\frac{x-a}{b-a}\right)^2,$$

$$\int_{\frac{b-x}{b-a}}^{1} \left| t-1+\lambda\frac{x-a}{b-a}\right| (1-t)\,dt = \left(\frac{2\lambda^3-3\lambda+2}{6}\right)\left(\frac{x-a}{b-a}\right)^3, \tag{2.4}$$

$$\int_0^{\frac{b-x}{b-a}} \left| t-\lambda\frac{b-x}{b-a}\right| dt = \left(\frac{2\lambda^2-2\lambda+1}{2}\right)\left(\frac{b-x}{b-a}\right)^2, \tag{2.5}$$

$$\int_{\frac{b-x}{b-a}}^{1} \left| t-1+\lambda\frac{x-a}{b-a}\right| dt = \left(\frac{2\lambda^2-2\lambda+1}{2}\right)\left(\frac{x-a}{b-a}\right)^2, \tag{2.6}$$

$$\int_0^{\frac{b-x}{b-a}} \left| t-\lambda\frac{b-x}{b-a}\right|^p dt = \left(\frac{\lambda^{p+1}-(1-\lambda)^{p+1}}{p+1}\right)\left(\frac{b-x}{b-a}\right)^{p+1}, \tag{2.7}$$

and

$$\int_{\frac{b-x}{b-a}}^{1} \left| t-1+\lambda\frac{x-a}{b-a}\right|^p dt = \left(\frac{\lambda^{p+1}-(1-\lambda)^{p+1}}{p+1}\right)\left(\frac{x-a}{b-a}\right)^{p+1}. \tag{2.8}$$

We give a important integral identity for differentiable functions:

Lemma 1 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$ for $\lambda\in[0,1]$,*

then for all $x \in [a,b]$ *we have*

$$(b-a)\int_0^1 h(t,\lambda)f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]dt \tag{2.9}$$

$$= \frac{(1-\lambda)\, f\left(\mu x+(1-\mu)\left(a+b-x\right)\right)}{(1-2\mu)}$$

$$+\lambda\frac{(b-x)\, f\left(\mu b+(1-\mu)a\right)+(x-a)\, f\left(\mu a+(1-\mu)b\right)}{(b-a)\,(1-2\mu)}$$

$$-\frac{1}{(b-a)\,(1-2\mu)^2}\int_{\mu b+(1-\mu)a}^{\mu a+(1-\mu)b} f(u)du$$

for $\mu \in [0,1]\,/\,\{1/2\}$*, where*

$$h(t,\lambda)=\begin{cases} t-\lambda\dfrac{b-x}{b-a}, & t\in\left[0,\dfrac{b-x}{b-a}\right] \\[2ex] t-1+\lambda\dfrac{x-a}{b-a}, & t\in\left(\dfrac{b-x}{b-a},1\right]. \end{cases}$$

Proof Denote

$$\begin{aligned} I &= \int_0^1 h(t,\lambda)f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]dt \\ &= \int_0^{\frac{b-x}{b-a}}\left[t-\lambda\frac{b-x}{b-a}\right]f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]dt \\ &\quad+\int_{\frac{b-x}{b-a}}^1\left[t-1+\lambda\frac{x-a}{b-a}\right]f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]dt \\ &= I_1+I_2. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} I_1 &= \int_0^{\frac{b-x}{b-a}}\left[t-\lambda\frac{b-x}{b-a}\right]f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]dt \\ &= \frac{(1-\lambda)\,(b-x)\, f\left(\mu x+(1-\mu)\left(a+b-x\right)\right)}{(b-a)^2\,(1-2\mu)}+\lambda\frac{(b-x)\, f\left(\mu b+(1-\mu)a\right)}{(b-a)^2\,(1-2\mu)} \end{aligned}$$

$$-\frac{1}{(b-a)(1-2\mu)}\int_{0}^{\frac{b-x}{b-a}} f\left[t(\mu a+(1-\mu)b)+(1-t)(\mu b+(1-\mu)a)\right]dt$$

and

$$\begin{aligned}
I_2 &= \int_{\frac{b-x}{b-a}}^{1}\left[t-1+\lambda\frac{x-a}{b-a}\right]f'\left[t(\mu a+(1-\mu)b)+(1-t)(\mu b+(1-\mu)a)\right]dt \\
&= \frac{(1-\lambda)(x-a)f(\mu x+(1-\mu)(a+b-x))}{(b-a)^2(1-2\mu)}+\lambda\frac{(x-a)f(\mu a+(1-\mu)b)}{(b-a)^2(1-2\mu)} \\
&\quad -\frac{1}{(b-a)(1-2\mu)}\int_{\frac{b-x}{b-a}}^{1} f\left[t(\mu a+(1-\mu)b)+(1-t)(\mu b+(1-\mu)a)\right]dt.
\end{aligned}$$

Adding I_1 and I_2, then we have

$$\begin{aligned}
I &= I_1+I_2 \\
&= \frac{(1-\lambda)f(\mu x+(1-\mu)(a+b-x))}{(b-a)(1-2\mu)} \\
&\quad +\lambda\frac{(b-x)f(\mu b+(1-\mu)a)+(x-a)f(\mu a+(1-\mu)b)}{(b-a)^2(1-2\mu)} \\
&\quad -\frac{1}{(b-a)(1-2\mu)}\int_{0}^{1} f\left[t(\mu a+(1-\mu)b)+(1-t)(\mu b+(1-\mu)a)\right]dt.
\end{aligned}$$

If we use the change in the variable $u=t(\mu a+(1-\mu)b)+(1-t)(\mu b+(1-\mu)a)$ with $du=b-a)(1-2\mu)\,dt$, then we have

$$\begin{aligned}
I &= \frac{(1-\lambda)f(\mu x+(1-\mu)(a+b-x))}{(b-a)(1-2\mu)} \\
&\quad +\lambda\frac{(b-x)f(\mu b+(1-\mu)a)+(x-a)f(\mu a+(1-\mu)b)}{(b-a)^2(1-2\mu)} \\
&\quad -\frac{1}{(b-a)^2(1-2\mu)^2}\int_{\mu b+(1-\mu)a}^{\mu a+(1-\mu)b} f(u)du
\end{aligned}$$

which completes the proof. □

Remark 1 *If we choose* $\mu = 1$ *in (2.9), then Lemma 1 reduces to the Lemma 1 in [12].*

Theorem 4 *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable mapping on* (a, b) *with* $a < b$. *If* $|f'|^q$, $q \geq 1$, *is convex on* $[a, b]$ *for* $\lambda \in [0, 1]$ *and* $x \in [a, b]$, *then we have the following inequality:*

$$\begin{aligned}
&|T(f, \lambda, \mu, x)| \qquad (2.10)\\
\leq\ & (b-a)\left(\frac{2\lambda^2 - 2\lambda + 1}{2}\right)^{1-\frac{1}{q}} \\
&\times \left\{ \left[\left(\frac{2\lambda^3 - 3\lambda + 2}{6}\right)\left(\frac{b-x}{b-a}\right)^{2q+1} |f'(\mu a + (1-\mu)b)|^q \right.\right. \\
&\left. + F(x, \lambda)\left(\frac{b-x}{b-a}\right)^{2q} |f'(\mu b + (1-\mu)a)|^q \right]^{\frac{1}{q}} \\
&+ \left[G(x, \lambda)\left(\frac{x-a}{b-a}\right)^{2q} |f'(\mu a + (1-\mu)b)|^q \right. \\
&\left.\left. + \left(\frac{2\lambda^3 - 3\lambda + 2}{6}\right)\left(\frac{x-a}{b-a}\right)^{2q+1} |f'(\mu b + (1-\mu)a)|^q \right]^{\frac{1}{q}} \right\}
\end{aligned}$$

where $\mu \in [0, 1] / \{1/2\}$. *Here*

$$F(x, \lambda) = \frac{\left[6\lambda^2 - 6\lambda + 3\right] - \left[2\lambda^3 - 3\lambda + 2\right]\left(\frac{b-x}{b-a}\right)}{6},$$

$$G(x, \lambda) = \frac{\left[6\lambda^2 - 6\lambda + 3\right] - \left[2\lambda^3 - 3\lambda + 2\right]\left(\frac{x-a}{b-a}\right)}{6}$$

and

$$\begin{aligned}
T(f, \lambda, \mu, x) &= \frac{(1-\lambda) f(\mu x + (1-\mu)(a+b-x))}{(1-2\mu)} \\
&+ \lambda \frac{(b-x) f(\mu b + (1-\mu)a) + (x-a) f(\mu a + (1-\mu)b)}{(b-a)(1-2\mu)} \\
&- \frac{1}{(b-a)(1-2\mu)^2} \int_{\mu b + (1-\mu)a}^{\mu a + (1-\mu)b} f(u)du
\end{aligned}$$

Proof Firstly, we suppose that $q=1$. Taking the modulus in (2.9), we have

$$\begin{aligned}
&|T(f,\lambda,\mu,x)| \\
&\leq (b-a)\left\{\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\frac{b-x}{b-a}\right|\left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right|dt\right. \\
&\left.+\int_{\frac{b-x}{b-a}}^{1}\left|t-1+\lambda\frac{x-a}{b-a}\right|\left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right|dt\right\} \\
&= (b-a)\,K_1.
\end{aligned}$$

Using the convexity of $|f'|$, we get

$$\begin{aligned}
&K_1 \qquad (2.11)\\
&\leq \int_0^{\frac{b-x}{b-a}}\left|t-\lambda\frac{b-x}{b-a}\right|\left[t\left|f'\left(\mu a+(1-\mu)b\right)\right|+(1-t)\left|f'\left(\mu b+(1-\mu)a\right)\right|\right]dt \\
&+\int_{\frac{b-x}{b-a}}^{1}\left|t-1+\lambda\frac{x-a}{b-a}\right|\left[t\left|f'\left(\mu a+(1-\mu)b\right)\right|+(1-t)\left|f'\left(\mu b+(1-\mu)a\right)\right|\right]dt \\
&= \left|f'\left(\mu a+(1-\mu)b\right)\right|\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\frac{b-x}{b-a}\right|tdt \\
&+\left|f'\left(\mu b+(1-\mu)a\right)\right|\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\frac{b-x}{b-a}\right|(1-t)\,dt \\
&+\left|f'\left(\mu a+(1-\mu)b\right)\right|\int_{\frac{b-x}{b-a}}^{1}\left|t-1+\lambda\frac{x-a}{b-a}\right|tdt \\
&+\left|f'\left(\mu b+(1-\mu)a\right)\right|\int_{\frac{b-x}{b-a}}^{1}\left|t-1+\lambda\frac{x-a}{b-a}\right|(1-t)\,dt.
\end{aligned}$$

If we use the equalities (2.1)–(2.4) in (2.11), then we complete the proof for the case $q = 1$.

Secondly, we suppose that $q > 1$. Using Lemma 1 and power mean inequality, we obtain

$$
\begin{aligned}
&|T(f,\lambda,\mu,x)| \\
\leq\ & (b-a)\left\{ \int_0^{\frac{b-x}{b-a}} \left|t-\lambda\frac{b-x}{b-a}\right| \left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right| dt \right. \\
& \left. + \int_{\frac{b-x}{b-a}}^{1} \left|t-1+\lambda\frac{x-a}{b-a}\right| \left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right| dt \right\} \\
=\ & (b-a)\left\{ \left(\int_0^{\frac{b-x}{b-a}} \left|t-\lambda\frac{b-x}{b-a}\right| dt \right)^{1-\frac{1}{q}} \right. \\
& \times \left(\int_0^{\frac{b-x}{b-a}} \left|t-\lambda\frac{b-x}{b-a}\right| \left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right|^q dt \right)^{\frac{1}{q}} \\
& + \left(\int_{\frac{b-x}{b-a}}^{1} \left|t-1+\lambda\frac{x-a}{b-a}\right| dt \right)^{1-\frac{1}{q}} \\
& \left. \left(\int_{\frac{b-x}{b-a}}^{1} \left|t-1+\lambda\frac{x-a}{b-a}\right| \left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right|^q dt \right)^{\frac{1}{q}} \right\} \\
=\ & (b-a)K_2.
\end{aligned}
$$

Using the convexity of $|f'|^q$, we obtain

$$
K_2 \leq \left(\int_0^{\frac{b-x}{b-a}} \left|t-\lambda\frac{b-x}{b-a}\right| dt \right)^{1-\frac{1}{q}} \tag{2.12}
$$

$$\times\left(\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\frac{b-x}{b-a}\right|\left[t\left|f'\left(\mu a+(1-\mu)b\right)\right|^q+(1-t)\left|f'\left(\mu b+(1-\mu)a\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{b-x}{b-a}}^1\left|t-1+\lambda\frac{x-a}{b-a}\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{b-x}{b-a}}^1\left|t-1+\lambda\frac{x-a}{b-a}\right|\left[t\left|f'\left(\mu a+(1-\mu)b\right)\right|^q+(1-t)\left|f'\left(\mu b+(1-\mu)a\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$=\left(\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\frac{b-x}{b-a}\right|dt\right)^{1-\frac{1}{q}}\left(\left|f'\left(\mu a+(1-\mu)b\right)\right|^q\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\frac{b-x}{b-a}\right|tdt\right.$$

$$\left.+\left|f'\left(\mu b+(1-\mu)a\right)\right|^q\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\frac{b-x}{b-a}\right|(1-t)\,dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{b-x}{b-a}}^1\left|t-1+\lambda\frac{x-a}{b-a}\right|dt\right)^{1-\frac{1}{q}}\left(\left|f'\left(\mu a+(1-\mu)b\right)\right|^q\int_{\frac{b-x}{b-a}}^1\left|t-1+\lambda\frac{x-a}{b-a}\right|tdt\right.$$

$$\left.+\left|f'\left(\mu b+(1-\mu)a\right)\right|^q\int_{\frac{b-x}{b-a}}^1\left|t-1+\lambda\frac{x-a}{b-a}\right|(1-t)\,dt\right)^{\frac{1}{q}}.$$

If we use the equalities (2.1)–(2.6) in (2.12), then we complete the proof completely. □

Remark 2 *If we choose* $\mu=1$ *in Theorem 4, then the inequality (2.10) reduces to the inequality (1.4).*

Corollary 1 *Under assumptions of Theorem 4, if we choose* $\mu=0$ *in (2.10), then we have the inequality*

$$\left|(1-\lambda)\,f\left(a+b-x\right)\right. \tag{2.13}$$

$$\left.+\lambda\frac{(b-x)\,f\left(a\right)+(x-a)\,f\left(b\right)}{(b-a)}-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leq (b-a)\left(\frac{2\lambda^2-2\lambda+1}{2}\right)^{1-\frac{1}{q}}\left\{\left[\left(\frac{2\lambda^3-3\lambda+2}{6}\right)\left(\frac{b-x}{b-a}\right)^{2q+1}\left|f'(b)\right|^q\right.\right.$$

$$\left.+\left(\frac{\left[6\lambda^2-6\lambda+3\right]-\left[2\lambda^3-3\lambda+2\right]\left(\frac{b-x}{b-a}\right)}{6}\right)\left(\frac{b-x}{b-a}\right)^{2q}\left|f'(a)\right|^q\right]^{\frac{1}{q}}$$

$$+\left[\left(\frac{\left[6\lambda^2-6\lambda+3\right]-\left[2\lambda^3-3\lambda+2\right]\left(\frac{x-a}{b-a}\right)}{6}\right)\left(\frac{x-a}{b-a}\right)^{2q}\left|f'(b)\right|^q\right.$$

$$\left.\left.+\left(\frac{2\lambda^3-3\lambda+2}{6}\right)\left(\frac{x-a}{b-a}\right)^{2q+1}\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Corollary 2 *If choose $\lambda=0$ in Corollary 1, then we have the inequality*

$$\left|f(a+b-x)-\frac{1}{b-a}\int_a^b f(u)du\right| \tag{2.14}$$

$$\leq \frac{b-a}{2^{1-\frac{1}{q}}}\left\{\left[\frac{1}{3}\left(\frac{b-x}{b-a}\right)^{2q+1}\left|f'(b)\right|^q\right.\right.$$

$$\left.+\left[\frac{1}{2}-\frac{1}{3}\left(\frac{b-x}{b-a}\right)\right]\left(\frac{b-x}{b-a}\right)^{2q}\left|f'(a)\right|^q\right]^{\frac{1}{q}}$$

$$+\left[\left[\frac{1}{2}-\frac{1}{3}\left(\frac{x-a}{b-a}\right)\right]\left(\frac{x-a}{b-a}\right)^{2q}\left|f'(b)\right|^q\right.$$

$$\left.\left.+\frac{1}{3}\left(\frac{x-a}{b-a}\right)^{2q+1}\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Remark 3 *If we choose $x=\frac{a+b}{2}$ in Corollary 2, then Corollary 2 reduces to the Theorem 2.1 in [9].*

Corollary 3 *If we take $\lambda=1$ in Corollary 1, then we have the following inequality:*

$$\left|\frac{(b-x)f(a)+(x-a)f(b)}{(b-a)}-\frac{1}{b-a}\int_a^b f(u)du\right| \tag{2.15}$$

(2.16)

$$\leq \frac{b-a}{2^{1-\frac{1}{q}}}\left\{\left[\frac{1}{6}\left(\frac{b-x}{b-a}\right)^{2q+1}\left|f'(b)\right|^q+\left[\frac{1}{2}-\frac{1}{6}\left(\frac{b-x}{b-a}\right)\right]\left(\frac{b-x}{b-a}\right)^{2q}\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\left[\frac{1}{2}-\frac{1}{6}\left(\frac{x-a}{b-a}\right)\right]\left(\frac{x-a}{b-a}\right)^{2q}\left|f'(b)\right|^q+\frac{1}{6}\left(\frac{x-a}{b-a}\right)^{2q+1}\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Corollary 4 *If we take $x=\frac{a+b}{2}$ in Corollary 3, the we have the following trapezoid inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \tag{2.17}$$

$$\leq \frac{b-a}{8}\left\{\left[\frac{5\left|f'(a)\right|^q+\left|f'(b)\right|^q}{6}\right]^{\frac{1}{q}}+\left[\frac{\left|f'(a)\right|^q+5\left|f'(b)\right|^q}{6}\right]^{\frac{1}{q}}\right\}$$

$$\leq \left(\frac{6^{1-\frac{1}{q}}}{8}\right)(b-a)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Proof The proof of the first inequality is obvious. For the second inequality, let $a_1=5\left|f'(a)\right|^q$, $a_2=\left|f'(a)\right|^q$, $b_1=\left|f'(b)\right|^q$, $b_2=5\left|f'(b)\right|^q$. Here $0<\frac{1}{q}<1$, for $q>1$. Using the fact that

$$\sum_{k=1}^{n}(a_k+b_k)^s\leq\sum_{k=1}^{n}a_k^s+\sum_{k=1}^{n}b_k^s$$

for $(0<s<1)$ $a_1,a_2,...,a_n\geq 0$, $b_1,b_2,...,b_n\geq 0$, we have

$$\left(\frac{5\left|f'(a)\right|^q+\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+5\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}$$

$$=\frac{1}{6^{\frac{1}{q}}}\left[\left(5\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'(a)\right|^q+5\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\left(1+5^{\frac{1}{q}}\right)}{5^{\frac{1}{q}}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

$$\leq 6^{1-\frac{1}{q}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

which completes the proof. □

Theorem 5 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$, $q>1$ is convex on $[a,b]$ for $\lambda\in[0,1]$ and $x\in[a,b]$, then we have the following inequality:*

$$\begin{aligned}
&|T(f,\lambda,\mu,x)| \qquad (2.18)\\
\leq\ & (b-a)\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}\right)^{\frac{1}{p}}\\
&\times\left[\left(\frac{x-a}{b-a}\right)^{\frac{p+1}{p}}\left(\frac{1}{2}\left(\frac{b-x}{b-a}\right)^{2}\left|f'\left(\mu a+(1-\mu)b\right)\right|^{q}\right.\right.\\
&\left.+\left[\frac{1}{2}-\frac{1}{2}\left(\frac{x-a}{b-a}\right)^{2}\right]\left|f'\left(\mu b+(1-\mu)a\right)\right|^{q}\right)^{\frac{1}{q}}\\
&+\left(\frac{b-x}{b-a}\right)^{\frac{p+1}{p}}\left(\left[\frac{1}{2}-\frac{1}{2}\left(\frac{b-x}{b-a}\right)^{2}\right]\left|f'\left(\mu a+(1-\mu)b\right)\right|^{q}\right.\\
&\left.\left.+\frac{1}{2}\left(\frac{x-a}{b-a}\right)\left|f'\left(\mu b+(1-\mu)a\right)\right|^{q}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\mu\in[0,1]/\{1/2\}$.

Proof Taking the modulus in Lemma 1 and using Hölder's inequality, we have

$$\begin{aligned}
|T(f,\lambda,\mu,x)| \leq\ & (b-a)\left\{\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\frac{b-x}{b-a}\right|\left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right|dt\right.\\
&\left.+\int_{\frac{b-x}{b-a}}^{1}\left|t-1+\lambda\frac{x-a}{b-a}\right|\left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right|dt\right\}\\
\leq\ & (b-a)\left\{\left(\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\frac{b-x}{b-a}\right|^{p}dt\right)^{\frac{1}{p}}\right.
\end{aligned}$$

$$\times\left(\int_{0}^{\frac{b-x}{b-a}}\left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right|^{q}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{b-x}{b-a}}^{1}\left|t-1+\lambda\frac{x-a}{b-a}\right|^{p}dt\right)^{\frac{1}{p}}$$

$$\left.\times\left(\int_{\frac{b-x}{b-a}}^{1}\left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right|dt\right)^{\frac{1}{q}}\right\}$$

$$=\quad (b-a)K_3.$$

Using the convexity of $|f'|^q$, we obtain

$$K_3 \leq \left\{\left(\int_{0}^{\frac{b-x}{b-a}}\left|t-\lambda\frac{b-x}{b-a}\right|^{p}dt\right)^{\frac{1}{p}}\right.$$

$$\times\left(\left|f'\left(\mu a+(1-\mu)b\right)\right|^{q}\int_{0}^{\frac{b-x}{b-a}}tdt+\left|f'\left(\mu b+(1-\mu)a\right)\right|^{q}\int_{0}^{\frac{b-x}{b-a}}(1-t)\,dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{b-x}{b-a}}^{1}\left|t-1+\lambda\frac{x-a}{b-a}\right|^{p}dt\right)^{\frac{1}{p}}$$

$$\left.\times\left(\left|f'\left(\mu a+(1-\mu)b\right)\right|^{q}\int_{\frac{b-x}{b-a}}^{1}tdt+\left|f'\left(\mu b+(1-\mu)a\right)\right|^{q}\int_{\frac{b-x}{b-a}}^{1}(1-t)\,dt\right)^{\frac{1}{q}}\right\}$$

$$= \left\{ \left(\int_0^{\frac{b-x}{b-a}} \left| t - \lambda \frac{b-x}{b-a} \right|^p dt \right)^{\frac{1}{p}} \left(\frac{1}{2} \left(\frac{b-x}{b-a} \right)^2 |f'(\mu a + (1-\mu)b)|^q \right. \right.$$

$$\left. + \left[\frac{1}{2} - \frac{1}{2} \left(\frac{x-a}{b-a} \right)^2 \right] |f'(\mu b + (1-\mu)a)|^q \right)^{\frac{1}{q}}$$

$$+ \left(\int_{\frac{b-x}{b-a}}^1 \left| t - 1 + \lambda \frac{x-a}{b-a} \right|^p dt \right)^{\frac{1}{p}} \left(\left[\frac{1}{2} - \frac{1}{2} \left(\frac{b-x}{b-a} \right)^2 \right] |f'(\mu a + (1-\mu)b)|^q \right.$$

$$\left. \left. + \frac{1}{2} \left(\frac{x-a}{b-a} \right)^2 |f'(\mu b + (1-\mu)a)|^q \right)^{\frac{1}{q}} \right\}.$$

If we use equalities (2.7) and (2.8), then we obtain the required result. □

Remark 4 *If we choose $\mu = 1$ in (2.18), then the inequality Theorem 5 reduces to the Theorem 2 in [12].*

Corollary 5 *Under the assumptions of Theorem 5, choosing $\mu = 0$, we get the inequality*

$$\left| (1-\lambda) f(a+b-x) + \lambda \frac{(b-x) f(a) + (x-a) f(b)}{(b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \tag{2.19}$$

$$\leq (b-a) \left(\frac{\lambda^{p+1} + (1-\lambda)^{p+1}}{p+1} \right)^{\frac{1}{p}}$$

$$\times \left[\left(\frac{b-x}{b-a} \right)^{\frac{p+1}{p}} \left(\frac{1}{2} \left(\frac{b-x}{b-a} \right)^2 |f'(b)|^q + \left[\frac{1}{2} - \frac{1}{2} \left(\frac{x-a}{b-a} \right)^2 \right] |f'(a)|^q \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\frac{b-x}{b-a} \right)^{\frac{p+1}{p}} \left(\left[\frac{1}{2} - \frac{1}{2} \left(\frac{b-x}{b-a} \right)^2 \right] |f'(b)|^q + \frac{1}{2} \left(\frac{x-a}{b-a} \right) |f'(a)|^q \right)^{\frac{1}{q}} \right].$$

Corollary 6 *If we take $\lambda = 1$ and $x = \frac{a+b}{2}$ in Corollary 5, then we have the following trapezoid inequality:*

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du \right| \\
&\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left[\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} \right] \\
&\leq \left(\frac{b-a}{4} \right) \left(\frac{4}{p+1} \right)^{\frac{1}{p}} \left[|f'(a)| + |f'(b)| \right].
\end{aligned}$$

Proof The proof of the first inequality is obvious. For the second inequality, let $a_1 = 3|f'(a)|^q$, $a_2 = |f'(a)|^q$, $b_1 = |f'(b)|^q$, $b_2 = 3|f'(b)|^q$. Here $0 < \frac{1}{q} < 1$, for $q > 1$. Using the fact that

$$\sum_{k=1}^{n} (a_k + b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s$$

for $(0 < s < 1)$ $a_1, a_2, ..., a_n \geq 0$, $b_1, b_2, ..., b_n \geq 0$, we have

$$\begin{aligned}
\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} &= \frac{1}{4^{\frac{1}{q}}} \left[\left(3|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}} + \left(|f'(a)|^q + 3|f'(b)|^q\right)^{\frac{1}{q}} \right] \\
&\leq \frac{\left(1 + 3^{\frac{1}{q}}\right)}{4^{\frac{1}{q}}} \left[|f'(a)| + |f'(b)| \right] \\
&\leq 4^{1-\frac{1}{q}} \left[|f'(a)| + |f'(b)| \right].
\end{aligned}$$

This completes the proof. □

Remark 5 *If we take $\lambda = 0$ and $x = \frac{a+b}{2}$ in Corollary 5, then Corollary 5 reduces to the Theorem 2.4 in [10].*

Theorem 6 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|^q$, $q > 1$ is convex on $[a,b]$ for $\lambda \in [0,1]$ and $x \in [a,b]$, then we have the following inequality:*

$$\begin{aligned}
|T(f,\lambda,\mu,x)| \leq{}& \frac{b-a}{2^{\frac{1}{q}}} \left(\frac{\lambda^{p+1} + (1-\lambda)^{p+1}}{p+1} \right)^{\frac{1}{p}} \\
&\times \left[\left(\frac{b-x}{b-a} \right)^{\frac{p+1}{p}} \left(|f'(\mu x + (1-\mu)(a+b-x))|^q + |f'(\mu b + (1-\mu)a)|^q \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{x-a}{b-a} \right)^{\frac{p+1}{p}} \left(|f'(\mu a + (1-\mu)b)|^q + |f'(\mu x + (1-\mu)(a+b-x))|^q \right)^{\frac{1}{q}} \right]
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\mu \in [0,1] / \{1/2\}$.

Proof Taking the modulus in Lemma 1 and using Hölder's inequality, we have

$$\begin{aligned}
|T(f,\lambda,\mu,x)| &\leq (b-a)\left\{ \int_0^{\frac{b-x}{b-a}} \left| t-\lambda\frac{b-x}{b-a}\right| \left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right| dt \right. \\
&\left. + \int_{\frac{b-x}{b-a}}^1 \left| t-1+\lambda\frac{x-a}{b-a}\right| \left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right| dt \right\} \\
&\leq (b-a)\left\{ \left(\int_0^{\frac{b-x}{b-a}} \left| t-\lambda\frac{b-x}{b-a}\right|^p dt \right)^{\frac{1}{p}} \right. \\
&\times \left(\int_0^{\frac{b-x}{b-a}} \left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right|^q dt \right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{b-x}{b-a}}^1 \left| t-1+\lambda\frac{x-a}{b-a}\right|^p dt \right)^{\frac{1}{p}} \\
&\left. \times \left(\int_{\frac{b-x}{b-a}}^1 \left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right| dt \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

With convexity of $|f'|^q$, using the Hermite–Hadamard inequality we have

$$\begin{aligned}
&\int_0^{\frac{b-x}{b-a}} \left|f'\left[t\left(\mu a+(1-\mu)b\right)+(1-t)\left(\mu b+(1-\mu)a\right)\right]\right|^q dt \qquad (2.20) \\
&= \frac{1}{(b-a)(1-2\mu)} \int_{\mu b+(1-\mu)a}^{\mu a+(1-\mu)(a+b-x)} f(u)du \\
&\leq \frac{\left|f'\left(\mu x+(1-\mu)\left(a+b-x\right)\right)\right|^q+\left|f'\left(\mu b+(1-\mu)a\right)\right|^q}{2}
\end{aligned}$$

and

$$\int_{\frac{b-x}{b-a}}^{1} |f'[t(\mu a+(1-\mu)b)+(1-t)(\mu b+(1-\mu)a)]|\,dt \tag{2.21}$$

$$= \frac{1}{(b-a)(1-2\mu)} \int_{\mu a+(1-\mu)(a+b-x)}^{\mu a+(1-\mu)b} f(u)du$$

$$+\frac{|f'(\mu a+(1-\mu)b)|^q+|f'(\mu x+(1-\mu)(a+b-x))|^q}{2}$$

If we put (2.7)–(2.8) and (2.20)–(2.21) in (2.20), then we complete the proof. □

Corollary 7 *Under the assumption of Theorem 6, if we choose $\mu=1$, then we have the inequality*

$$\left|(1-\lambda)f(x)+\lambda\frac{(b-x)f(b)+(x-a)f(a)}{(b-a)}-\frac{1}{(b-a)}\int_a^b f(u)du\right| \tag{2.22}$$

$$\leq \frac{b-a}{2^{\frac{1}{q}}}\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{b-x}{b-a}\right)^{\frac{p+1}{p}}\left(|f'(x)|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{x-a}{b-a}\right)^{\frac{p+1}{p}}\left(|f'(a)|^q+|f'(x)|^q\right)^{\frac{1}{q}}\right].$$

Remark 6 *If we choose $\lambda=0$ in Corollary 7, then Corollary 7 reduces to the Theorem 2 in [3].*

Corollary 8 *If we choose $\lambda=1$ and $x=\frac{a+b}{2}$ in Corollary 7, then we have the following trapezoid inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(u)du\right| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

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