Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2016) 40: 1221 – 1230

doi:10.3906/mat-1507-79

On Hermite–Hadamard type inequalities via generalized fractional integrals

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Received: 21.07.2015 • **Accepted/Published Online:** 29.01.2016 • **Final Version:** 02.12.2016

Abstract: New Hermite–Hadamard type inequalities are obtained for convex functions via generalized fractional integrals. The results presented here are generalizations of those obtained in earlier works.

Key words: Hermite–Hadamard inequality, convex function, generalized fractional integral, Riemann–Liouville fractional integral, Hadamard fractional integral

1. Introduction

Let I be an interval of real numbers and $a, b \in I$ with $a < b$. If $f : I \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This result is known in the literature as the Hermite–Hadamard inequality [13]. Such an inequality has received some attention in recent years and many generalizations and extensions can be found in the literature; see [1–12, 14, 15, 19–24] and the references therein.

In [10], Dragomir and Agarwal established the following result connected with the right part of (1.1).

Theorem 1.1 *Let $f : \mathring{I} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$, $a, b \in \mathring{I}$ with $a < b$. If $f' \in L^1[a,b]$ and $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{8}\left(|f'(a)| + |f'(b)|\right). \tag{1.2}$$

Recently some authors obtained Hermite–Hadamard-type inequalities involving fractional integrals for various classes of functions; see [3, 5, 6, 14, 15, 22, 23] and the references therein. In [23], Sarikaya et al. established the following Hermite–Hadamard-type inequalities involving Riemann–Liouville fractional integrals.

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2010 *AMS Mathematics Subject Classification:* 26D15, 26D10, 26A33.

Theorem 1.2 *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L^1[a,b]$. If f is a convex function on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)] \le \frac{f(a)+f(b)}{2}, \tag{1.3}$$

where $\alpha > 0$, Γ is the Gamma function, and $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ are the left-sided and right-sided Riemann–Liouville fractional integrals of order $\alpha > 0$.

Note that for $\alpha = 1$, (1.3) reduces to the classical Hermite–Hadamard inequality (1.1).

Theorem 1.3 *Let $f : \mathring{I} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$, $a, b \in \mathring{I}$ with $a < b$. If $f' \in L^1[a,b]$ and $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)]\right| \le \frac{(b-a)}{2(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)(|f'(a)| + |f'(b)|), \tag{1.4}$$

with $\alpha > 0$.

Observe that for $\alpha = 1$, (1.4) reduces to (1.2).

In this paper, we obtain generalizations of Theorems 1.2 and 1.3 using the generalized fractional integrals introduced recently by Katugampola in [16].

First we recall some definitions and mathematical preliminaries that will be used in this paper.

Let $f : [a,b] \to \mathbb{R}$ be a given function, where $0 < a < b < \infty$.

Definition 1.4 (see [25]) *The left-sided Riemann–Liouville fractional integral J^α_{a+} of order $\alpha > 0$ of f is defined by*

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad x > a, \tag{1.5}$$

provided that the integral exists. The right-sided Riemann–Liouville fractional integral J^α_{b-} of order $\alpha > 0$ of f is defined by

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (\tau-x)^{\alpha-1} f(\tau)\, d\tau, \quad x < b, \tag{1.6}$$

provided that the integral exists.

Definition 1.5 *(see [18, 25]) The left-sided Hadamard fractional integral $\mathbf{J}^\alpha_{a+}$ of order $\alpha > 0$ of f is defined by*

$$\mathbf{J}^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{\tau}\right)^{\alpha-1} \frac{f(\tau)}{\tau}\, d\tau, \quad x > a, \tag{1.7}$$

provided that the integral exists. The right-sided Hadamard fractional integral $\mathbf{J}^\alpha_{b-}$ of order $\alpha > 0$ of f is defined by

$$\mathbf{J}^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{\tau}{x}\right)^{\alpha-1} \frac{f(\tau)}{\tau}\, d\tau, \quad x < b, \tag{1.8}$$

provided that the integral exists.

In [16], Katugampola introduced a new fractional integration that generalizes the Riemann–Liouville and Hadamard fractional integrals into a single form.

Definition 1.6 (see [16]) *Let* $\rho > 0$. *The generalized left-sided fractional integral* ${}^{\rho}I^{\alpha}_{a^+}f$ *of order* $\alpha > 0$ *of* f *is defined by*

$$ {}^{\rho}I^{\alpha}_{a^+}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}} f(\tau)\, d\tau, \quad x > a, $$

provided that the integral exists. The generalized right-sided fractional integral ${}^{\rho}I^{\alpha}_{b^-}f$ *of order* $\alpha > 0$ *of* f *is defined by*

$$ {}^{\rho}I^{\alpha}_{b^-}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{\tau^{\rho-1}}{(\tau^{\rho}-x^{\rho})^{1-\alpha}} f(\tau)\, d\tau, \quad x < b, $$

provided that the integral exists.

Lemma 1.7 (see [17]) *Let* $\alpha > 0$. *Then, for* $x > a$,

(i) $\lim_{\rho\to 1} {}^{\rho}I^{\alpha}_{a^+}f(x) = J^{\alpha}_{a^+}f(x)$,

(ii) $\lim_{\rho\to 0^+} {}^{\rho}I^{\alpha}_{a^+}f(x) = \mathbf{J}^{\alpha}_{a^+}f(x)$.

Similar results hold for right-sided operators as well.

By using an adequate change of variables, we obtain the following identities.

Lemma 1.8 *Let* $\alpha > 0$ *and* $\rho > 0$.

(i) For $x > a$, *we have*

$$ {}^{\rho}I^{\alpha}_{a^+}f(x) = (x-a)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \frac{(sx+(1-s)a)^{\rho-1}}{[x^{\rho}-(sx+(1-s)a)^{\rho}]^{1-\alpha}} f(sx+(1-s)a)\, ds. $$

(ii) For $x < b$, *we have*

$$ {}^{\rho}I^{\alpha}_{b^-}f(x) = (b-x)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \frac{(sb+(1-s)x)^{\rho-1}}{[(sb+(1-s)x)^{\rho}-x^{\rho}]^{1-\alpha}} f(sb+(1-s)x)\, ds. $$

2. Main results

Let $f : \mathring{I} \to \mathbb{R}$ be a given function, where $a, b \in \mathring{I}$ and $0 < a < b < \infty$. We suppose that $f \in L^{\infty}(a,b)$ in such a way that ${}^{\rho}I^{\alpha}_{a^+}f(x)$ and ${}^{\rho}I^{\alpha}_{b^-}f(x)$ are well defined. We define the functions

$$ \tilde{f}(x) = f(a+b-x), \quad x \in [a,b] $$

and

$$ F(x) = f(x) + \tilde{f}(x), \quad x \in [a,b]. $$

We have the following result.

Theorem 2.1 *Let* $\alpha > 0$ *and* $\rho > 0$. *If* f *is a convex function on* $[a, b]$, *then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\rho^{\alpha}\Gamma(\alpha+1)}{4(b^{\rho}-a^{\rho})^{\alpha}}[{}^{\rho}I^{\alpha}_{a^+}F(b) + {}^{\rho}I^{\alpha}_{b^-}F(a)] \leq \frac{f(a)+f(b)}{2}. \tag{2.1}$$

Proof For $s \in [0, 1]$, let $u = as + (1-s)b$ and $v = (1-s)a + bs$. The convexity of f yields

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{u+v}{2}\right) \leq \frac{1}{2}f(u) + \frac{1}{2}f(v),$$

that is,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}f(as+(1-s)b) + \frac{1}{2}f((1-s)a+bs). \tag{2.2}$$

Multiplying both sides of (2.2) by

$$(b-a)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{(sb+(1-s)a)^{\rho-1}}{[b^{\rho}-(sb+(1-s)a)^{\rho}]^{1-\alpha}}$$

and integrating over $(0, 1)$ with respect to s, we get

$$\begin{aligned}
&(b-a)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)\int_0^1 \frac{(sb+(1-s)a)^{\rho-1}}{[b^{\rho}-(sb+(1-s)a)^{\rho}]^{1-\alpha}}\,ds \\
&\leq \frac{1}{2}(b-a)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^1 \frac{(sb+(1-s)a)^{\rho-1}}{[b^{\rho}-(sb+(1-s)a)^{\rho}]^{1-\alpha}}f(as+(1-s)b)\,ds \\
&+ \frac{1}{2}(b-a)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^1 \frac{(sb+(1-s)a)^{\rho-1}}{[b^{\rho}-(sb+(1-s)a)^{\rho}]^{1-\alpha}}f((1-s)a+bs)\,ds.
\end{aligned}$$

Note we have

$$\int_0^1 \frac{(sb+(1-s)a)^{\rho-1}}{[b^{\rho}-(sb+(1-s)a)^{\rho}]^{1-\alpha}}\,ds = \frac{1}{\rho(b-a)\alpha}(b^{\rho}-a^{\rho})^{\alpha}.$$

Using the identity

$$\tilde{f}((1-s)a+bs) = f(as+(1-s)b)$$

and property (i) in Lemma 1.8, we obtain

$$(b-a)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^1 \frac{(sb+(1-s)a)^{\rho-1}}{[b^{\rho}-(sb+(1-s)a)^{\rho}]^{1-\alpha}}f(as+(1-s)b)\,ds = {}^{\rho}I^{\alpha}_{a^+}\tilde{f}(b)$$

and

$$(b-a)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^1 \frac{(sb+(1-s)a)^{\rho-1}}{[b^{\rho}-(sb+(1-s)a)^{\rho}]^{1-\alpha}}f((1-s)a+bs)\,ds = {}^{\rho}I^{\alpha}_{a^+}f(b).$$

As a consequence, we have

$$\frac{(b^{\rho}-a^{\rho})^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right) \leq \frac{{}^{\rho}I^{\alpha}_{a^+}F(b)}{2}. \tag{2.3}$$

Similarly, multiplying both sides of (2.2) by

$$(b-a)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{(sb+(1-s)a)^{\rho-1}}{[(bs+(1-s)a)^{\rho}-a^{\rho}]^{1-\alpha}},$$

integrating over $(0,1)$ with respect to s and using property (ii) in Lemma 1.8, we obtain

$$\frac{(b^{\rho}-a^{\rho})^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)\leq\frac{{}^{\rho}I_{b-}^{\alpha}F(a)}{2}. \tag{2.4}$$

By adding the above inequalities (2.3) and (2.4), we get

$$f\left(\frac{a+b}{2}\right)\leq\frac{\rho^{\alpha}\Gamma(\alpha+1)}{4(b^{\rho}-a^{\rho})^{\alpha}}[{}^{\rho}I_{a+}^{\alpha}F(b)+{}^{\rho}I_{b-}^{\alpha}F(a)]$$

and the left-hand side of the inequality (2.1) is established.

Since f is convex, for every $s\in[0,1]$, we have

$$f(as+(1-s)b)+f((1-s)a+bs)\leq f(a)+f(b). \tag{2.5}$$

Multiplying both sides of (2.5) by

$$(b-a)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{(sb+(1-s)a)^{\rho-1}}{[b^{\rho}-(sb+(1-s)a)^{\rho}]^{1-\alpha}}$$

and integrating over $(0,1)$ with respect to s, we get

$$\begin{aligned}
&(b-a)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^1\frac{(sb+(1-s)a)^{\rho-1}}{[b^{\rho}-(sb+(1-s)a)^{\rho}]^{1-\alpha}}f(as+(1-s)b)\,ds\\
&+(b-a)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^1\frac{(sb+(1-s)a)^{\rho-1}}{[b^{\rho}-(sb+(1-s)a)^{\rho}]^{1-\alpha}}f((1-s)a+bs)\,ds\\
&\leq(b-a)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}[f(a)+f(b)]\int_0^1\frac{(sb+(1-s)a)^{\rho-1}}{[b^{\rho}-(sb+(1-s)a)^{\rho}]^{1-\alpha}}\,ds,
\end{aligned}$$

i.e.

$${}^{\rho}I_{a+}^{\alpha}F(b)\leq\frac{(b^{\rho}-a^{\rho})^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}[f(a)+f(b)]. \tag{2.6}$$

Similarly, multiplying both sides of (2.5) by

$$(b-a)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{(sb+(1-s)a)^{\rho-1}}{[(bs+(1-s)a)^{\rho}-a^{\rho}]^{1-\alpha}}$$

and integrating over $(0,1)$ with respect to s, we get

$${}^{\rho}I_{b-}^{\alpha}F(a)\leq\frac{(b^{\rho}-a^{\rho})^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}[f(a)+f(b)]. \tag{2.7}$$

Adding the inequalities (2.6) and (2.7), we obtain

$$\frac{\rho^\alpha\Gamma(\alpha+1)}{4(b^\rho-a^\rho)^\alpha}[{}^\rho I^\alpha_{a+}F(b)+{}^\rho I^\alpha_{b-}F(a)]\le\frac{f(a)+f(b)}{2}.$$

The proof is complete. □

Remark 2.2 *Theorem 2.1 is a generalization of Theorem 1.2. Indeed, letting* $\rho\to 1$ *in* (2.1) *and using property (i) in Lemma 1.7, we obtain immediately inequality* (1.3)*. Moreover, in Theorem 2.1 we do not suppose that* f *is a positive function, which is required in Theorem 1.2.*

Letting $\rho\to 0^+$ in (2.1) and using property (ii) in Lemma 1.7, we obtain the following Hermite–Hadamard inequality for Hadamard fractional integrals.

Corollary 2.3 *Let* $\alpha>0$*. If* f *is a convex function, then*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{4\left(\ln\frac{b}{a}\right)^\alpha}[\mathbf{J}^\alpha_{a+}F(b)+\mathbf{J}^\alpha_{b-}F(a)]\le\frac{f(a)+f(b)}{2}.$$

For $\alpha>0$ and $\rho>0$, let $\Xi_{\alpha,\rho}:[0,1]\to\mathbb{R}$ be the function defined by

$$\begin{aligned}\Xi_{\alpha,\rho}(t) &= [(ta+(1-t)b)^\rho-a^\rho]^\alpha-[(bt+(1-t)a)^\rho-a^\rho]^\alpha\\ &\quad [b^\rho-(bt+(1-t)a)^\rho]^\alpha-[b^\rho-(ta+(1-t)b)^\rho]^\alpha.\end{aligned}$$

Before stating and proving our next result, we need the following lemma.

Lemma 2.4 *Let* $\alpha>0$ *and* $\rho>0$*. If* $f\in C^1(\mathring{I})$*, then*

$$\frac{f(a)+f(b)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{4(b^\rho-a^\rho)^\alpha}[{}^\rho I^\alpha_{a+}F(b)+{}^\rho I^\alpha_{b-}F(a)]=\frac{(b-a)}{4(b^\rho-a^\rho)^\alpha}\int_0^1\Xi_{\alpha,\rho}(t)f'(ta+(1-t)b)\,dt. \tag{2.8}$$

Proof Using integration by parts, we obtain

$${}^\rho I^\alpha_{a+}F(b)=\frac{(b^\rho-a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}F(a)+\frac{(b-a)}{\rho^\alpha\Gamma(\alpha+1)}\int_0^1[b^\rho-(bs+(1-s)a)^\rho]^\alpha F'(bs+(1-s)a)\,ds. \tag{2.9}$$

Similarly, we have

$${}^\rho I^\alpha_{b-}F(a)=\frac{(b^\rho-a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}F(b)-\frac{(b-a)}{\rho^\alpha\Gamma(\alpha+1)}\int_0^1[(bs+(1-s)a)^\rho-a^\rho]^\alpha F'(bs+(1-s)a)\,ds. \tag{2.10}$$

Using (2.9) and (2.10), we obtain

$$\begin{aligned}&\frac{4(b^\rho-a^\rho)^\alpha}{(b-a)}\left(\frac{f(a)+f(b)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{4(b^\rho-a^\rho)^\alpha}[{}^\rho I^\alpha_{a+}F(b)+{}^\rho I^\alpha_{b-}F(a)]\right) \qquad (2.11)\\ &=\int_0^1[((bs+(1-s)a)^\rho-a^\rho)^\alpha-(b^\rho-(bs+(1-s)a)^\rho)^\alpha]F'(bs+(1-s)a)\,ds.\end{aligned}$$

Note we have

$$F'(bs+(1-s)a)=f'(bs+(1-s)a)-f'(as+(1-s)b),\quad s\in[0,1].$$

Then we obtain

$$\begin{aligned}&\int_0^1((bs+(1-s)a)^\rho-a^\rho)^\alpha F'(bs+(1-s)a)\,ds\\&=\int_0^1((ta+(1-t)b)^\rho-a^\rho)^\alpha f'(ta+(1-t)b)\,dt\\&-\int_0^1((bt+(1-t)a)^\rho-a^\rho)^\alpha f'(ta+(1-t)b)\,dt\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}&\int_0^1(b^\rho-(bs+(1-s)a)^\rho)^\alpha F'(bs+(1-s)a)\,ds\\&=\int_0^1(b^\rho-(at+(1-t)b)^\rho)^\alpha f'(ta+(1-t)b)\,dt\\&-\int_0^1(b^\rho-(bt+(1-t)a)^\rho)^\alpha f'(ta+(1-t)b)\,dt.\end{aligned}\tag{2.13}$$

Finally, (2.8) follows from (2.11), (2.12), and (2.13). □

Remark 2.5 *Letting $\rho\to1$ in (2.8), we obtain Lemma 2 in [23].*

For $\alpha>0$, we introduce the following operator:

$$\mathcal{L}^\alpha(\rho,x,y)=\int_a^{\frac{a+b}{2}}|x-u||y^\rho-u^\rho|^\alpha\,du-\int_{\frac{a+b}{2}}^b|x-u||y^\rho-u^\rho|^\alpha\,du,\quad \rho>0,\ x,y\in[a,b].$$

We have the following result.

Theorem 2.6 *Let $\alpha>0$ and $\rho>0$. If $f\in C^1(\mathring{I})$ and $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{4(b^\rho-a^\rho)^\alpha}[{}^\rho I_{a^+}^\alpha F(b)+{}^\rho I_{b^-}^\alpha F(a)]\right|\le\frac{I_\rho^\alpha(a,b)}{4(b^\rho-a^\rho)^\alpha(b-a)}\left(|f'(a)|+|f'(b)|\right),\tag{2.14}$$

where

$$I_\rho^\alpha(a,b)=\mathcal{L}^\alpha(\rho,b,b)+\mathcal{L}^\alpha(\rho,a,b)-\mathcal{L}^\alpha(\rho,b,a)-\mathcal{L}^\alpha(\rho,a,a).$$

Proof Using Lemma 2.4 and the convexity of $|f'|$, we obtain

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{4(b^\rho-a^\rho)^\alpha}[{}^\rho I_{a^+}^\alpha F(b)+{}^\rho I_{b^-}^\alpha F(a)]\right|\\&\le\frac{(b-a)}{4(b^\rho-a^\rho)^\alpha}\int_0^1|\Xi_{\alpha,\rho}(t)||f'(ta+(1-t)b)|\,dt\\&\le\frac{(b-a)}{4(b^\rho-a^\rho)^\alpha}\left(|f'(a)|\int_0^1t|\Xi_{\alpha,\rho}(t)|\,dt+|f'(b)|\int_0^1(1-t)|\Xi_{\alpha,\rho}(t)|\,dt\right).\end{aligned}\tag{2.15}$$

Note

$$\int_0^1 t|\Xi_{\alpha,\rho}(t)|\,dt = \frac{1}{(b-a)^2}\int_a^b |\varphi(u)|(b-u)\,du,$$

where

$$\varphi(u) = (u^\rho - a^\rho)^\alpha - ((b+a-u)^\rho - a^\rho)^\alpha + (b^\rho - (a+b-u)^\rho)^\alpha - (b^\rho - u^\rho)^\alpha, \quad u \in [a,b].$$

Observe that φ is a nondecreasing function on $[a,b]$. Moreover, we have

$$\varphi(a) = -2(b^\rho - a^\rho)^\alpha < 0$$

and

$$\varphi\left(\frac{a+b}{2}\right) = 0.$$

As a consequence, we have

$$\begin{cases} \varphi(u) \le 0 & \text{if} \quad a \le u \le \frac{a+b}{2}, \\ \varphi(u) > 0 & \text{if} \quad \frac{a+b}{2} < u \le b. \end{cases}$$

Hence, we obtain

$$(b-a)^2\int_0^1 t|\Xi_{\alpha,\rho}(t)|\,dt = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= \int_a^{\frac{a+b}{2}} (b-u)(b^\rho - u^\rho)^\alpha\,du - \int_{\frac{a+b}{2}}^b (b-u)(b^\rho - u^\rho)^\alpha\,du, \\ I_2 &= -\int_a^{\frac{a+b}{2}} (b-u)(u^\rho - a^\rho)^\alpha\,du + \int_{\frac{a+b}{2}}^b (u^\rho - a^\rho)^\alpha(b-u), \\ I_3 &= \int_a^{\frac{a+b}{2}} ((b+a-u)^\rho - a^\rho)^\alpha(b-u)\,du - \int_{\frac{a+b}{2}}^b ((b+a-u)^\rho - a^\rho)^\alpha(b-u)\,du, \\ I_4 &= -\int_a^{\frac{a+b}{2}} (b^\rho - (a+b-u)^\rho)^\alpha(b-u)\,du + \int_{\frac{a+b}{2}}^b (b^\rho - (a+b-u)^\rho)^\alpha(b-u)\,du. \end{aligned}$$

Observe that

$$I_1 = \mathcal{L}^\alpha(\rho,b,b) \quad \text{and} \quad I_2 = -\mathcal{L}^\alpha(\rho,b,a).$$

Using the change of variable $v = a+b-u$, we get

$$I_3 = -\mathcal{L}^\alpha(\rho,a,a) \quad \text{and} \quad I_4 = \mathcal{L}^\alpha(\rho,a,b).$$

Thus, we obtain

$$\int_0^1 t|\Xi_{\alpha,\rho}(t)|\,dt = \frac{\mathcal{L}^\alpha(\rho,b,b) + \mathcal{L}^\alpha(\rho,a,b) - \mathcal{L}^\alpha(\rho,b,a) - \mathcal{L}^\alpha(\rho,a,a)}{(b-a)^2}. \tag{2.16}$$

Similarly, we obtain

$$\int_0^1 (1-t)|\Xi_{\alpha,\rho}(t)|\,dt = \frac{\mathcal{L}^\alpha(\rho,b,b)+\mathcal{L}^\alpha(\rho,a,b)-\mathcal{L}^\alpha(\rho,b,a)-\mathcal{L}^\alpha(\rho,a,a)}{(b-a)^2}. \tag{2.17}$$

Finally, the desired result follows from (2.15), (2.16), and (2.17). □

Remark 2.7 *Letting $\rho \to 1$ in (2.14), we obtain Theorem 1.3.*

For $\alpha > 0$, let us introduce the operator

$$\mathcal{W}^\alpha(x,y) = \int_a^{\frac{a+b}{2}} |x-u| \left|\ln \frac{u}{y}\right|^\alpha du - \int_{\frac{a+b}{2}}^b |x-u| \left|\ln \frac{u}{y}\right|^\alpha du, \quad x,y \in [a,b].$$

Letting $\rho \to 0$ in (2.14), we obtain the following result.

Corollary 2.8 *Let $\alpha > 0$. If $f \in C^1(\mathring{I})$ and $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{4\left(\ln\frac{b}{a}\right)^\alpha}[\mathbf{J}_{a+}^\alpha F(b) + \mathbf{J}_{b-}^\alpha F(a)]\right| \le \frac{K^\alpha(a,b)}{4(b-a)\left(\ln\frac{b}{a}\right)^\alpha}\left(|f'(a)|+|f'(b)|\right),$$

where

$$K^\alpha(a,b) = \mathcal{W}^\alpha(b,b) + \mathcal{W}^\alpha(a,b) - \mathcal{W}^\alpha(b,a) - \mathcal{W}^\alpha(a,a).$$

Acknowledgements

We thank the reviewers for carefully reading our manuscript. The first and third authors are supported by Distinguished Scientist Fellowship Program (DSFP), King Saud University, Saudi Arabia.

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