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Perturbational self-similar solutions for the 2-component Degasperis–Procesi system via a characteristic method

Hongli AN[1,*], **Ka-Luen CHEUNG**[2], **Manwai YUEN**[2]
[1]College of Sciences, Nanjing Agricultural University, Nanjing, P.R. China
[2]Department of Mathematics and Information Technology, The Hong Kong Institute of Education, Tai Po, New Territories, Hong Kong

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Abstract: In this paper, the two-component Degasperis–Procesi system arising in shallow water theory is investigated. By using a special transformation and the characteristic method, a class of perturbational self-similar solutions is constructed. Such solutions are not only more general than those obtained by Yuen in 2011, but also they may have potential applications in the modeling of tsunamis. In addition, the method proposed can be extended to other mathematical physics models like the two-component Camassa–Holm equations.

Key words: 2-Component Degasperis–Procesi system, special transformation, characteristic method, perturbational solution

1. Introduction

In the present work, we consider the following 2-component Degasperis–Procesi shallow water system:

$$\begin{cases} \rho_t + k_2 u\rho_x + (k_1+k_2)\rho u_x = 0, \\ u_t - u_{xxt} + 4uu_x - 3uu_{xx} - uu_{xxx} + k_3\rho\rho_x = 0, \end{cases} \tag{1}$$

where k_1, k_2, and k_3 are constants. It is noted that when $\rho = 0$ system (1) reduces to the classical Degasperis–Procesi (DP) equation, which has been extensively studied by many authors (see references [6-8, 10, 13, 19, 20, 22, 31, 41]). For example, Degasperis et al. proved that the DP model was integrable [7]. Lundmark et al. showed that such a system allowed multipeakon solutions [22]. Later, Lin and Liu proved that such peakon solutions were stable under small perturbations [19]. Zhou and Liu et al. demonstrated that the DP equation possessed blow-up phenomena [10, 20, 41]. Constantin and Lannes found that there was some hydrodynamical relevance between the Camassa–Holm (CH) and DP equations [6]. Important contributions to the study of the DP equation were also made by Matsuno and Henry et al. (see e.g. [13, 31]).

However, our main concern here is with the interesting case when $\rho \neq 0$, namely the 2-component DP system (1). To the best of our knowledge, except for the work mentioned in [15, 33, 39], little related work has been carried out. In particular, no work has been done on the constructions of analytical solutions to the 2-component DP system. In fact, it is of significance to investigate the 2-component DP equation because such

*Correspondence: kaixinguoan@163.com.cn

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a model representing breaking waves and peaked traveling waves is of great importance in hydrodynamics [36], and the traveling wave solutions of large amplitude to the governing equations for water waves are peaked waves [4]. Furthermore, such a model is relevant to the modeling of tsunamis (see references [5, 18]).

There exist many powerful methods that have been used to construct analytical solutions of nonlinear partial differential equations, which contain inverse scattering method, Darboux transformation, bilinear technique, variable separation approach, Wronskian and Casoratian techniques, various *tanh* methods, transformed rational function method, and multiple exp-function method [1,11,14,23-30,37]. Among these methods, the transformed rational method is the most general solution algorithm to construct traveling wave solutions by using a rational function transformation [28]. The multiple exp-function method and the Casoratian techniques are the most effective approaches to produce multiple wave solutions and mixed rational-solition solutions, respectively [29, 30].

Here, we would like to use the characteristic method [21] to seek solutions of self-similar type for the 2-component DP system. Investigation on such type solutions is a current hot topic in various models and much work has been done. For instance, Barna [3] presented the self-similar solutions for the 3-dimensional Navier–Stokes (NS) equations via a group theoretical method. Yuen and An [2, 40] derived some self-similar solutions with elliptic symmetry for the N-dimensional NS equations via the separation method. Since we notice that the system (1) shares some similarities with the NS equations in form, it is natural to inquire whether one can construct self-similar solutions of perturbational type for (1). In this paper, by introducing a special transformation, we successfully obtain perturbational self-similar solutions for the 2-component DP equation via the characteristic method. The main result is described in the following theorem:

Theorem *For the 2-component DP system, there exists a family of perturbational self-similar solutions:*

$$\begin{cases} \rho(x,t) = \max\left\{ \dfrac{f(\eta)}{a(\bar{t})^{\frac{k_1+k_2}{k_2}}} \ , \ 0 \right\} \\ u(x,t) = \dfrac{a'(\bar{t})}{a(\bar{t})}\left(\bar{x} - d(\bar{t})\right) + d'(\bar{t}) \end{cases} \tag{2}$$

where

$$f(\eta) = \pm\sqrt{-\frac{k_2\xi}{k_3}\,\eta + \eta_0} \qquad \text{with} \qquad \eta = \left(\frac{\bar{x} - d(\bar{t})}{a(\bar{t})}\right)^2$$

with

$$\bar{x} = \frac{4}{k_2}x\ , \quad \bar{t} = 4t\ .$$

In the above, prime denotes $\dfrac{d}{d\bar{t}}$ *and* $\eta_0, \xi \neq 0$ *are constants. The auxiliary function* $a(\bar{t})$ *is governed by the Emden dynamical system [16, 17]:*

$$\left(a(\bar{t})^{\frac{4}{k_2}}\right)'' = \frac{4\,\xi}{k_2 a(\bar{t})^{\frac{4}{k_2}(\frac{k_1}{2}+k_2-1)}}\ , \quad a(0) = a_0 \neq 0, \quad a'(0) = a_1 \tag{3}$$

and $d(\bar{t})$ *is governed by*

$$d(\bar{t}) = \int^{\bar{t}} \frac{c}{a(\mathrm{t})^{\frac{4}{k_2}-1}}\mathrm{dt} \tag{4}$$

with constants a_0, a_1 *, and* c*.*

Remark 1 It is observed that when the time-dependent function $d(\bar{t})$ degenerates to zero, the solution described by (2) coincides with those obtained by Yuen in [39]. Thus, in this sense, we can conclude that the solution obtained here is more general than Yuen's. What is more important is that such a perturbational solution may have potential applications in the modeling of tsunamis as suggested in [5, 18].

Remark 2 What we want to emphasize here is that there exist two special cases wherein exact solutions of the 2-component DP system can be constructed explicitly. Details will be given in the following section.

Remark 3 The method proposed can also be extended to construct perturbational solutions of the 2-component CH equations [38] as well as other models whose forms are analogous to the 2-component DP system.

2. Perturbational solutions of the 2-component DP system

In this section, it is shown that the perturbational solutions of the 2-component DP system can be constructed via the characteristic method. However, this progress is achieved mainly due to the novel important lemma together with a special transformation introduced:

Lemma 1 *For the* $1+1$*-dimensional continuity equation of (1):*

$$\rho_t + k_2 u\rho_x + (k_1+k_2)\rho u_x = 0\ , \tag{5}$$

there exist solutions

$$\rho(t,x) = \frac{f(\eta)}{a(\bar{t})^{\frac{k_1+k_2}{k_2}}},\ u(t,x) = \frac{a'(\bar{t})}{a(\bar{t})}\left(\bar{x} - d(\bar{t})\right) + d'(\bar{t}) \tag{6}$$

with $\bar{t} = 4t\,,\ \bar{x} = \frac{4}{k_2}x\,,\ \eta = \frac{\bar{x}-d(\bar{t})}{a(\bar{t})}\,,\ f(\eta) \in C^1$ *and* $a(\bar{t}) \in C^1$.

Proof For later use, we introduce a special transformation first

$$\rho(x,t) = \rho(\bar{x},\bar{t}\,)\ , \quad u(x,t) = u(\bar{x},\bar{t}\,), \quad \bar{t} = 4t, \quad \bar{x} = \frac{4}{k_2}x, \tag{7}$$

so that under transformation (7), equation (5) becomes

$$\rho_{\bar{t}} + u\rho_{\bar{x}} + \frac{k_1+k_2}{k_2}\rho u_{\bar{x}} = 0. \tag{8}$$

Inspired by the work of [2, 35], we perturb the velocity as this form

$$\rho = \rho(\bar{x},\bar{t}\,)\ , \quad u = \frac{a'(\bar{t})}{a(\bar{t})}\left(\bar{x} - d(\bar{t})\right) + d'(\bar{t})\ . \tag{9}$$

Substituting the above ansatz into (8) yields

$$\rho_{\bar{t}} + u\rho_{\bar{x}} + \frac{k_1+k_2}{k_2}\rho u_{\bar{x}} = \rho_{\bar{t}} + \left[\frac{a'(\bar{t})}{a(\bar{t})}\left(\bar{x} - d(\bar{t})\right) + d'(\bar{t})\right]\rho_{\bar{x}} + \frac{(k_1+k_2)a'(\bar{t})}{k_2 a(\bar{t})}\rho = 0\ .$$

According to the characteristic method [21], we have

$$\frac{d\bar{t}}{1}=\frac{d\bar{x}}{\dfrac{a'(\bar{t})}{a(\bar{t})}\left(\bar{x}-d(\bar{t})\right)+d'(\bar{t})}=\frac{d\rho}{-\dfrac{(k_1+k_2)a'(\bar{t})}{k_2a(\bar{t})}\rho}\ ,$$

where the solution is

$$\Phi\left(\frac{\bar{x}-d(\bar{t})}{a(\bar{t})}\ ,\ \ \rho\, a(\bar{t})^{\frac{k_1+k_2}{k_2}}\right)=0 \tag{10}$$

with an arbitrary function $\Phi\in C^1$. It is convenient to write (10) in an explicit form

$$\rho(x,t)=\frac{f\left(\frac{\bar{x}-d(\bar{t})}{a(\bar{t})}\right)}{a(\bar{t})^{\frac{k_1+k_2}{k_2}}}. \tag{11}$$

Therefore, the proof is completed. □

Remark 4 It is necessary to point out that the variable transformation made by (7) as well as the negative symbol in the perturbational function $d(\bar{t})$ is crucial to guarantee the use of the characteristic method.

On applications of the above lemma, we obtain the generalized perturbational solutions for the 2-component DP system, namely

Theorem *For the 2-component DP system (1), there exists a family of perturbational solutions:*

$$\begin{cases} \rho(x,t)=\max\left\{\dfrac{f(\eta)}{a(\bar{t})^{\frac{k_1+k_2}{k_2}}}\ ,\ \ 0\right\} \\ u(x,t)=\dfrac{a'(\bar{t})}{a(\bar{t})}\left(\bar{x}-d(\bar{t})\right)+d'(\bar{t}) \end{cases} \tag{12}$$

where

$$f(\eta)=\pm\sqrt{-\frac{k_2\xi}{k_3}\,\eta+\eta_0}\qquad\text{with}\qquad \eta=\left(\frac{\bar{x}-d(\bar{t})}{a(\bar{t})}\right)^2. \tag{13}$$

In the above, $'=\dfrac{d}{d\bar{t}}$, $\bar{x}=\dfrac{4}{k_2}x$, $\bar{t}=4t$ *and* η_0, $\xi\neq 0$ *are constants. The auxiliary function* $a(\bar{t})$ *is governed by the Emden dynamical system:*

$$\left(a(\bar{t})^{\frac{4}{k_2}}\right)''=\frac{4\xi}{k_2a(\bar{t})^{\frac{4}{k_2}(\frac{k_1}{2}+k_2-1)}}\ ,\quad a(0)=a_0\neq 0,\quad a'(0)=a_1 \tag{14}$$

and $d(\bar{t})$ *is determined by*

$$d(\bar{t})=\int^{\bar{t}}\frac{c}{a(\mathrm{t})^{\frac{4}{k_2}-1}}\mathrm{dt} \tag{15}$$

with arbitrary constants a_0, a_1, *and* c.

Proof of Theorem It is clear, from the above Lemma, that the function (12) indeed satisfies the continuity equation of (1). In the sequel, we only need to validate the function (12) also holds for the second equation.

Insertion of (12) into the second equation yields

$$
\begin{aligned}
&u_t - u_{xxt} + 4uu_x - 3uu_{xx} - uu_{xxx} + k_3\rho\rho_x \\
=&4(u_{\bar{t}} - \frac{16}{k_2^2}u_{\bar{x}\bar{x}\bar{t}} + \frac{4}{k_2}uu_{\bar{x}} - \frac{12}{k_2^2}uu_{\bar{x}\bar{x}} - \frac{16}{k_2^3}uu_{\bar{x}\bar{x}\bar{x}} + \frac{k_3}{k_2}\rho\rho_{\bar{x}}) \\
=&\frac{4(\bar{x}-d(\bar{t}))}{a(\bar{t})}\left[a''(\bar{t}) - (1-\frac{4}{k_2})\frac{a'^2(\bar{t})}{a(\bar{t})} + \frac{2k_3}{k_2}\frac{f(\eta)\dot{f}(\eta)}{a(\bar{t})^{3+\frac{2k_1}{k_2}}}\right] + 4\left[d''(\bar{t}) - (1-\frac{4}{k_2})\frac{a'(\bar{t})}{a(\bar{t})}d'(\bar{t})\right] \\
=&\frac{4}{a(\bar{t})^{4+\frac{2k_1}{k_2}}}\left(\xi + \frac{2k_3}{k_2}f(\eta)\dot{f}(\eta)\right)(\bar{x}-d(\bar{t}))
\end{aligned}
\tag{16}
$$

if we require that the auxiliary functions $d(\bar{t}),\ a(\bar{t})$ are governed by the following equations

$$
d(\bar{t}) = \int^{\bar{t}} \frac{c}{a(\mathrm{t})^{\frac{4}{k_2}-1}}\mathrm{dt}\ , \qquad c = \mathrm{const} \tag{17}
$$

and

$$
a''(\bar{t}) - (1-\frac{4}{k_2})\frac{a'^2(\bar{t})}{a(\bar{t})} = \frac{\xi}{a(\bar{t})^{3+\frac{2k_1}{k_2}}}\ , \quad a(0)=a_0\neq 0, \quad a'(0)=a_1 \tag{18}
$$

which can be reducible to the Emden equation

$$
A''(\bar{t}) = \frac{4\xi}{k_2 A(\bar{t})^{\frac{k_1}{2}+k_2-1}}\ , \qquad\qquad A(\bar{t}) = a(\bar{t})^{\frac{4}{k_2}}\ . \tag{19}
$$

So that the function $f(\eta)$ is determined by

$$
\xi + \frac{2k_3}{k_2}f(\eta)\dot{f}(\eta) = 0
$$

where

$$
f(\eta) = \pm\sqrt{-\frac{k_2\xi}{k_3}\,\eta + \eta_0}, \qquad \text{with} \qquad \eta = \left(\frac{\bar{x}-d(\bar{t})}{a(\bar{t})}\right)^2. \tag{20}
$$

The proof is completed. □

Here two special cases are discussed, wherein some exact perturbational solutions are constructed explicitly:

Case 1 When $\frac{k_1}{2}+k_2-1=0$, namely $k_1+2k_2=2$, the Emden equation (14) reduces to the following second-order ordinary differential equation:

$$
\left(a(\bar{t})^{\frac{4}{k_2}}\right)'' = \frac{4\xi}{k_2}\ , \qquad a(0)=a_0\neq 0\ , \qquad a'(0)=a_1
$$

whose solution takes the form of

$$
a(\bar{t}) = \left(\frac{2\xi}{k_2}\,\bar{t}^2 + \frac{4}{k_2}a_1 a_0^{\frac{4}{k_2}-1}\,\bar{t} + a_0^{\frac{4}{k_2}}\right)^{\frac{k_2}{4}}\ . \tag{21}
$$

Then the value of $d(\bar{t})$ in (15) may be readily obtained by classical methods as described in Ref. [12]. Thus, the solution of the 2-component DP equation (1) is now written as

$$\begin{cases} \rho(x,t) = \max\left\{ \pm \dfrac{\sqrt{\dfrac{-\xi}{k_2k_3}\left(\dfrac{4x - k_2 d(\bar{t})}{a(\bar{t})}\right)^2 + \eta_0}}{a(\bar{t})^{\frac{k_1+k_2}{k_2}}}\ ,\ 0 \right\} \\ u(x,t) = \dfrac{a'(\bar{t})}{a(\bar{t})}\left(\dfrac{4}{k_2}x - d(\bar{t})\right) + \dfrac{c}{a(\bar{t})^{\frac{4}{k_2}-1}} \end{cases} \tag{22}$$

with $d(\bar{t})$ and $a(\bar{t})$ given by (15) and (21), respectively.

Case 2 When $\frac{k_1}{2} + k_2 - 1 = 3$, namely $k_1 + 2k_2 = 8$, the Emden equation (14) is reducible to a particular system of Ermakov–Pinney type [9, 32]

$$\ddot{\Omega} + w^2(t)\Omega = \frac{\mathbb{K}}{\Omega^3} \tag{23}$$

whose solution admits the nonlinear superposition principle [34, 35]

$$\Omega = \sqrt{\lambda\Omega_1^2 + 2\mu\Omega_1\Omega_2 + \nu\Omega_2^2} \tag{24}$$

with Ω_1 and Ω_2 being linearly independent solutions of

$$\ddot{\Omega} + 4w^2(t)\Omega = 0 \tag{25}$$

with unit Wronskian and $\lambda\nu - \mu^2 = \mathbb{K}$. Therefore, the exact solution of $a(\bar{t})$ is

$$a(\bar{t}) = \left(\frac{4\xi + k_2c_1^2(\bar{t}^2 + 2\bar{t}c_2 + c_2^2)}{k_2c_1}\right)^{\dfrac{k_2}{8}} \tag{26}$$

with $c_1 = \dfrac{\xi}{a_0^2} + a_1^2$ and $c_2 = \dfrac{a_0^3a_1}{\xi + a_0^2a_1^2}$.

Thus, in the case of $k_1 + 2k_2 = 8$, the above analysis leads to the solution of the 2-component DP equation:

$$\begin{cases} \rho(x,t) = \max\left\{ \pm \dfrac{\sqrt{\dfrac{-\xi}{k_2k_3}\left(\dfrac{4x - k_2 d(\bar{t})}{a(\bar{t})}\right)^2 + \eta_0}}{a(\bar{t})^{\frac{k_1+k_2}{k_2}}}\ ,\ 0 \right\} \\ u(x,t) = \dfrac{a'(\bar{t})}{a(\bar{t})}\left(\dfrac{4}{k_2}x - d(\bar{t})\right) + \dfrac{c}{a(\bar{t})^{\frac{4}{k_2}-1}} \end{cases} \tag{27}$$

with $d(\bar{t})$ and $a(\bar{t})$ governed by (15) and (26), respectively.

Remark 5 The perturbational solutions obtained in this paper belong to the self-similar type and share the same properties (like blowup, global existence) as that given by Yuen. Interested readers may refer to [39].

3. Conclusions

It is known that the two-component DP system is an important model that has been widely used in fluids, hydrodynamics, and modeling tsunamis (see [5, 18, 36] and references therein). In [33], Popowicz constructed the Hamiltonian structures for certain parameters. In [15], Lin and Guo analyzed some aspects of blowup mechanism, traveling wave solutions, and the persistence properties of the system. In this paper, we construct some perturbational self-similar solutions to this model by using the special transformation and characteristic method. These solutions constitute a generalization of what has been obtained by Yuen in [39]. What is more important is that the method proposed can be extended to construct perturbational solutions of other mathematical physics models like 2-component CH equations. In addition, it is hoped that the perturbational self-similar solutions derived can be used to predict or model tsunamis in oceans. However, there are still some problems left, for example, whether the 2-component DP system has Lax pairs and bilinear forms as well as multi-soliton solutions? What are the properties of the solutions? All these interesting questions are worthy of deep investigations in the future.

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