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Invariant structures and gauge transformation of almost contact metric manifolds

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Abstract: In this paper, conditions for K-contact, Sasakian, and cosymplectic structures to be invariant under gauge transformation are found. Moreover, the same question is studied for 3-Sasakian, 3-almost contact, and 3-cosymplectic manifolds. Finally, it is shown that a slant submanifold of an almost contact metric manifold is invariant by gauge transformation.

Key words: Gauge transformation, cosymplectic manifold, almost contact 3-structure, 3-Sasakian manifold, 3-cosymplectic manifold, slant submanifold

1. Introduction

Gauge transformation of a contact metric manifold was introduced by Tanno [12]. He obtained a contact metric structure invariant under gauge transformation. Sakamoto and Takemora introduced the gauge transformation of an almost contact metric manifold in [9]. The study of gauge transformation has been considered by several authors (see for instance [7, 8, 11]).

In this paper, we find conditions in which k-contact structures, Sasakian structures, and cosymplectic structures are invariant under gauge transformation and we see that if the gauge transformation of a Sasakian manifold is k-contact then it is Sasakian. We study gauge transformation of 3-Sasakian structures, almost contact 3-structures, and 3-cosymplectic structures and take conditions in which these structures are invariant under gauge transformation. Finally, we study gauge transformation of slant submanifolds of almost contact structures. We see that these submanifolds are invariant under gauge transformation.

2. Preliminaries

An almost contact metric manifold is an odd-dimensional manifold M that carries a field φ of endomorphisms of the tangent spaces, a vector field ξ, called characteristic or Reeb vector field, a 1-form η satisfying $\varphi^2 = -I+\eta\otimes\xi$ and $\eta(\xi) = 1$, where $I : TM \longrightarrow TM$ is the identity map [3]. From the definition it also follows that $\varphi\xi = 0$, $\eta o \varphi = 0$ and the $(1, 1)$-tensor field φ has a constant rank $2n$. It is well known that any almost contact manifold (M, φ, ξ, η) admits a Riemannian metric g such that

$$g(\varphi E, \varphi F) = g(E, F) - \eta(E)\eta(F), \qquad (1)$$

holds for all $E, F \in \Gamma(TM)$. The metric g is called compatible and the manifold M together with the structure (φ, ξ, η, g) is called an almost contact metric manifold. As an immediate consequence of (1), one has $\eta = g(\cdot, \xi)$.

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The 2-form Φ on M defined by $\Phi(E,F) = g(E,\varphi F)$ is called the fundamental 2-form of the almost contact metric manifold. An almost contact metric manifold (M,φ,ξ,η,g) is said to be normal when the tensor field $N = [\varphi,\varphi] + 2d\eta\otimes\xi$ vanishes identically, where $[\varphi,\varphi]$ denotes the Nijenhuis tensor of φ. Almost contact metric manifolds such that $d\eta = 0$ and $d\Phi = 0$ are called almost cosymplectic manifolds. A normal almost cosymplectic manifold is called a cosymplectic manifold. Almost contact metric manifolds such that $d\eta = \Phi$ are called contact metric manifolds. A contact metric manifold is called K-contact if the tensor $h = \frac{1}{2}\pounds_\xi\varphi$ vanishes [1]. Finally, a normal contact metric manifold is said to be a Sasakian manifold [10].

Let $\tilde{M}$ be an immersed submanifold of M, $T\tilde{M}$ the Lie algebra of vector fields on $\tilde{M}$, and $T\tilde{M}^\perp$ the set of all vector fields normal to $\tilde{M}$. For any $X \in T\tilde{M}$, we write

$$\varphi X = TX + NX,$$

where TX is the tangential components of the tangent bundle and N is a normal-bundle valued 1-form on the tangent bundle.

The submanifold $\tilde{M}$ is said to be invariant if N is identically zero, that is, $\varphi X \in T\tilde{M}$ for any $X \in T\tilde{M}$. On the other hand, $\tilde{M}$ is said to be an anti-invariant submanifold if T is identically zero, that is, $\varphi X \in T^\perp\tilde{M}$, for any $X \in T\tilde{M}$. According Lotta's definition, $\tilde{M}$ is slant if $\theta(X) \in [0,\frac{\pi}{2}]$ and the angle between φX and $T_x\tilde{M}$ is a constant that is independent of the choice of $x \in \tilde{M}$ and $X \in T_x\tilde{M} - <\xi_x>$ [4].

A gauge transformation of a contact metric manifold $(M^{2n+1},\varphi,\xi,\eta,g)$ is a contact metric manifold $(M^{2n+1},\bar{\varphi},\bar{\xi},\bar{\eta},\bar{g})$ defined by

$$\bar{\eta} = f\eta, \quad \bar{\xi} = f^{-1}\xi + Z, \quad \bar{\varphi} = \varphi - \bar{\eta}\otimes\varphi Z,$$

$$\bar{g} = fg + f(f + f^2|Z|^2 - 1)\eta\otimes\eta - f^2\eta\otimes g(Z,.) - f^2 g(Z,.)\otimes\eta,$$

with $Z = \frac{1}{2}f^{-2}\varphi(\nabla f)$, for $f \in C^\infty(M)$, $f > 0$.

A gauge transformation of an almost contact metric manifold (M,φ,ξ,η,g) is an almost contact metric manifold $(M^{2n+1},\bar{\varphi},\bar{\xi},\bar{\eta},\bar{g})$ defined by

$$\bar{\eta} = e^\sigma\eta, \quad d\bar{\eta} = e^\sigma(d\eta + d\eta\wedge\eta), \quad \bar{\xi} = e^{-\sigma}(\xi + \varphi A), \quad \bar{\varphi} = \varphi + \eta\otimes A,$$

$$\bar{g} = e^{2\sigma}(g - \eta\otimes g(\varphi A,.) - g(\varphi A,.)\otimes\eta + g(A,A)\eta\otimes\eta),$$

where $\sigma \in C^\infty(M)$ and A is a vector field in which $d\eta(\varphi A, X) = d\sigma(hX)$, where hX is the projection of X in $\ker\eta$.

An almost 3-contact manifold is a (4n + 3)-dimensional smooth manifold M endowed with three almost contact structures $(\varphi_1,\xi_1,\eta_1),(\varphi_2,\xi_2,\eta_2),(\varphi_3,\xi_3,\eta_3)$ satisfying the following relations, for every $\alpha,\beta \in \{1,2,3\}$,

$$\varphi_\alpha\varphi_\beta - \eta_\beta\otimes\xi_\alpha = \sum_{\gamma=1}^{3}\epsilon_{\alpha\beta\gamma}\varphi_\gamma - \delta_{\alpha\beta}I, \quad \varphi_\alpha\xi_\beta = \sum_{\gamma=1}^{3}\epsilon_{\alpha\beta\gamma}\xi_\gamma, \quad \eta_\alpha o\varphi_\beta = \sum_{\gamma=1}^{3}\epsilon_{\alpha\beta\gamma}\eta_\gamma, \quad \eta_\alpha(\xi_\beta) = 0, \tag{2}$$

where $\epsilon_{\alpha\beta\gamma}$ is the totally antisymmetric symbol. This notion was introduced by Kuo [6] and, independently, by Udriste [13]. In [6] Kuo proved that given an almost contact 3-structure $(\varphi_\alpha,\xi_\alpha,\eta_\alpha)$, there exists a

Riemannian metric g compatible with $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$ and hence we can consider almost contact metric 3-structures. It is well known that in any almost 3-contact metric manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal with respect to the compatible metric g. Moreover, by setting $\mathcal{H} = \bigcap_{\alpha=1}^{3} ker\eta_\alpha$ one obtains a 4n-dimensional distribution on M and the tangent bundle splits to the orthogonal sum $TM = \mathcal{H} \oplus < \xi_1, \xi_2, \xi_3 >$.

When the three structures $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ are contact metric structures, we say that M is a 3-contact metric manifold and when they are Sasakian, that is when each structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is also normal, we call M a 3-Sasakian manifold. Indeed as it has been proved in 2001 by Kashiwada [5], every contact metric 3-structure is 3-Sasakian. A manifold with an almost contact 3-structure (φ, ξ, η, g) is called an almost 3-cosymplectic manifold if each almost contact structure is an almost cosymplectic structure. If each almost contact structure is a cosymplectic structure the manifold is a 3-cosymplectic manifold.

Theorem 2.1 *[2] Any almost 3-cosymplectic manifold is 3-cosymplectic.*

3. Gauge transformation of almost contact metric manifolds

In this section we investigate some conditions under which K-contact manifolds, Sasakian manifolds, and cosymplectic manifolds are invariant under gauge transformation.

Lemma 3.1 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a contact metric manifold and $0 < f \in C^\infty(M)$ and $\bar{\xi} = f^{-1}\xi + Z$ be the Reeb vector field of $\bar{\eta} = f\eta$; then*

$$X(f) = -2f^2 g(X, \varphi Z) + \xi(f)\eta(X), \quad Z(f) = 0. \tag{3}$$

Proof By using (1) we can write

$$\begin{aligned} X(f) &= g(\nabla f, X) \\ &= g(\varphi(\nabla f), \varphi(X)) + \eta(X)\eta(\nabla f) \\ &= 2f^2 g(Z, \varphi X) + \eta(X) g(\nabla f, \xi) \\ &= -2f^2 g(X, \varphi Z) + \xi(f)\eta(X), \end{aligned}$$

setting $X = Z$ we have $Z(f) = 0$. □

In the following theorems we find some conditions for which the gauge transformation of a K-contact manifold (Sasakian resp) is K-contact (Sasakian resp) as well.

Theorem 3.2 *The gauge transformation of a K-contact manifold $(M, \varphi, \xi, \eta, g)$ is also K-contact manifold iff $(\pounds_Z \varphi)X = g(X, Z)\xi$ for all $X \in \chi(M)$.*

Proof Let $(M, \varphi, \xi, \eta, g)$ be a K-contact manifold and $(M, \bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ the gauge transformation by f. Since $(\pounds_{\bar{\xi}} \bar{\varphi})\bar{\xi} = 0$, it is enough to check $\bar{h}X = 0$ for any $X \in \chi(M)$. We see

$$\begin{aligned} (\pounds_{\bar{\xi}} \bar{\varphi})X &= \varphi[X, \bar{\xi}] - [\varphi X, \bar{\xi}] \\ &= f^{-1}\varphi[X, \xi] + \varphi[X, Z] - f^{-1}[\varphi X, \xi] - [\varphi X, Z] - \varphi X(f^{-1})\xi \\ &= f^{-1}(\pounds_\xi \varphi)X + (\pounds_Z \varphi)X - \varphi X(f^{-1})\xi, \end{aligned}$$

now using (3) we have

$$\varphi X(f^{-1}) = -f^{-2}\varphi X(f) = g(X,Z).$$

Then $(M,\bar{\varphi},\bar{\xi},\bar{\eta},\bar{g})$ is also K-contact iff $(\pounds_Z\varphi)X - g(X,Z)\xi = 0$. □

Theorem 3.3 *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a Sasakian manifold. The gauge transformation defined by f is also Sasakian iff $\varphi(\pounds_Z\varphi)X = 0$ for all $X \in \chi(M)$.*

Proof Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a Sasakian manifold. Since we can write $T_pM = \mathcal{D}\oplus < \bar{\xi} >$, it is sufficient to prove for the following cases:

For all X,Y belonging to $\mathcal{D}$ we have

$$[\bar{\varphi},\bar{\varphi}](X,Y) = \varphi^2[X,Y] + [\bar{\varphi}X,\bar{\varphi}Y] - \bar{\varphi}[\bar{\varphi}X,Y] - \bar{\varphi}[X,\bar{\varphi}Y],$$

but we can see

$$\bar{\varphi}^2[X,Y] = \varphi^2[X,Y] + \bar{\eta}[X,Y]Z, \quad [\bar{\varphi}X,\bar{\varphi}Y] = [\varphi X,\varphi Y],$$

$$\bar{\varphi}[\bar{\varphi}X,Y] = \varphi[\varphi X,Y] - \bar{\eta}[\varphi X,Y]\varphi Z, \quad \bar{\varphi}[X,\bar{\varphi}Y] = \varphi[X,\varphi Y] - \bar{\eta}[X,\varphi Y]\varphi Z;$$

therefore

$$[\bar{\varphi},\bar{\varphi}](X,Y) = -2d\bar{\eta}(X,Y)\bar{\xi}.$$

For all X belonging to $\mathcal{D}$ we have

$$[\bar{\varphi},\bar{\varphi}](\bar{\xi},Y) = \varphi^2[\bar{\xi},Y] + [\bar{\varphi}\bar{\xi},\bar{\varphi}Y] - \bar{\varphi}[\bar{\varphi}\bar{\xi},Y] - \bar{\varphi}[\bar{\xi},\bar{\varphi}Y],$$

if we set $\bar{\xi} = f^{-1}\xi + Z$ and $\bar{\varphi}X = \varphi X - \bar{\eta}X\varphi Z$ in the above equation, we see

$$[\bar{\varphi},\bar{\varphi}](\bar{\xi},X) = -2d\bar{\eta}(\bar{\xi},X)\bar{\xi} + \varphi(L_Z\varphi)X.$$

Finally $[\bar{\varphi},\bar{\varphi}](X,Y) = -2d\bar{\eta}(X,Y)\bar{\xi}$ for any $X,Y \in \chi(M)$ iff $\varphi(\pounds_Z\varphi)X = 0$. □

Corollary 3.4 *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a Sasakian manifold and f be a function on M such that $0 < f$; then the gauge transformation of this structure by f is Sasakian if the structure is K-contact.*

Now we find a condition for a cosymplectic structure to be invariant under gauge transformation.

Theorem 3.5 *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a cosymplectic manifold. The gauge transformation defined by e^{σ} is also cosymplectic iff*

$$d\sigma \wedge \Phi = 0, \quad d\sigma \wedge \eta = 0 \quad , \quad \varphi(\pounds_{\varphi A}\varphi) = 0.$$

Proof Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a cosymplectic manifold. We have

$$d\bar{\eta} = e^{2\sigma}(d\eta + d\sigma \wedge \eta),$$

Since $d\eta = 0$, $d\bar{\eta} = 0$ iff $d\sigma \wedge \eta = 0$. Now since $d\eta = 0$ and $d\Phi = 0$ we have

$$\begin{aligned} d\bar{\Phi} &= e^{2\sigma}[d\sigma \wedge \Phi - d\sigma \wedge (\eta \otimes g(A,.)) + d\sigma \wedge (g(A,.) \otimes \eta)] \\ &= e^{2\sigma}[-d\sigma \wedge (\eta \wedge g(A,.)) + d\sigma \wedge \Phi] \\ &= e^{2\sigma}[-(d\sigma \wedge \eta) \wedge g(A,.) + d\sigma \wedge \Phi], \end{aligned}$$

and so $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ has an almost cosymplectic structure iff $d\sigma \wedge \Phi = 0$ and $d\sigma \wedge \eta = 0$. If this new almost cosymplectic structure is normal it will be a cosymplectic structure. Since we can write $T_pM = \mathcal{D} \oplus < \bar{\xi} >$, it is sufficient to prove for the following cases:

For all X, Y belonging to $\mathcal{D}$ we have

$$[\bar{\varphi}, \bar{\varphi}](X, Y) = \varphi^2[X, Y] + [\bar{\varphi}X, \bar{\varphi}Y] - \bar{\varphi}[\bar{\varphi}X, Y] - \bar{\varphi}[X, \bar{\varphi}Y],$$

but we can see

$$\bar{\varphi}^2[X, Y] = \varphi^2[X, Y] + \bar{\eta}[X, Y]\varphi A, \quad [\bar{\varphi}X, \bar{\varphi}Y] = [\varphi X, \varphi Y],$$

$$\bar{\varphi}[\bar{\varphi}X, Y] = \varphi[\varphi X, Y] - \bar{\eta}[\varphi X, Y]A, \quad \bar{\varphi}[X, \bar{\varphi}Y] = \varphi[X, \varphi Y] - \bar{\eta}[X, \varphi Y]A;$$

therefore since $d\bar{\eta} = 0$ we have

$$[\bar{\varphi}, \bar{\varphi}](X, Y) = -2d\bar{\eta}(X, Y)\bar{\xi} = 0.$$

For all X belonging to $\mathcal{D}$ we have

$$[\bar{\varphi}, \bar{\varphi}](\bar{\xi}, Y) = \varphi^2[\bar{\xi}, Y] + [\bar{\varphi}\bar{\xi}, \bar{\varphi}Y] - \bar{\varphi}[\bar{\varphi}\bar{\xi}, Y] - \bar{\varphi}[\bar{\xi}, \bar{\varphi}Y],$$

if we set $\bar{\xi} = e^{-\sigma}(\xi + \varphi A)$ and $\bar{\varphi}X = \varphi X + \bar{\eta}(X)A$ in the above equation, we see

$$[\bar{\varphi}, \bar{\varphi}](\bar{\xi}, X) = \varphi(\pounds_{\varphi A}\varphi)X.$$

Finally $[\bar{\varphi}, \bar{\varphi}](X, Y) = -2d\bar{\eta}(X, Y)\bar{\xi} = 0$ for any $X, Y \in \chi(M)$ iff $\varphi(\pounds_{\varphi A}\varphi)X = 0$. □

Corollary 3.6 *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost cosymplectic manifold. The gauge transformation defined by e^σ is also almost cosymplectic iff*

$$d\sigma \wedge \Phi = 0, \quad d\sigma \wedge \eta = 0.$$

4. Gauge transformation of almost contact 3-structures

Now we are in a position to investigate the gauge transformation of almost contact 3-structures. First, let $(M^{4n+3}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-Sasakian manifold. If we set $\bar{\eta}_\alpha = f\eta_\alpha$ for $\alpha \in \{1, 2, 3\}$ where f is a positive function on M, we have three new contact structures:

$$\bar{\eta_\alpha} = f\eta_\alpha, \bar{\xi_\alpha} = f^{-1}\xi_\alpha + Z_\alpha, \bar{\varphi_\alpha} = \varphi_\alpha - \bar{\eta_\alpha} \otimes \varphi_\alpha Z_\alpha,$$

$$\bar{g_\alpha} = fg + f(f + f^2|Z_\alpha|^2 - 1)\eta_\alpha \otimes \eta_\alpha - f^2\eta_\alpha \otimes g(Z_\alpha, .) - f^2 g(Z_\alpha, .) \otimes \eta_\alpha,$$

where $Z_\alpha = \frac{1}{2} f^{-2}\varphi_\alpha(\nabla f)$. We say that this transformation is a gauge transformation of a 3-Sasakian manifold.

These new contact metric structures have a 3-Sasakian structure if $\nabla f \in \bigcap ker\eta_\alpha$. Since $\bar{\eta_\alpha}(\bar{\xi_\beta}) = f^{-1}\eta_\gamma(\nabla f) = 0$ iff $\nabla f \in \bigcap ker\eta_\alpha$. Thus we have the following theorem:

Theorem 4.1 *The gauge transformation of a 3-Sasakian manifold has a 3-Sasakian structure iff* $\nabla f \in \bigcap ker\eta_\alpha$.

Remark 4.2 *If we set* $\bar{\eta_\alpha} = f_\alpha\eta_\alpha$, *we will see from (2) that this transformation of a 3-Sasakian structure cannot carry a 3-Sasakian structure unless* $f_1 = f_2 = f_3$. *Thus we can use theorem 4.1.*

Lemma 4.3 *If we set* $\bar{\eta} = f\eta$ *for* $\alpha \in \{1,2,3\}$ *where* f *is a positive function on* M *and* $\nabla f \in \bigcap ker\eta_\alpha$ *then we have* $g(Z_\alpha, Z_\beta) = 0$ *and* $|Z_\alpha| = |Z_\beta|\ \forall\ \alpha,\beta \in \{1,2,3\}$.

Proof We have

$$g(Z_\alpha, Z_\beta) = f^{-4} g(\varphi_\alpha \nabla f, \varphi_\beta \nabla f) = f^{-4} g(\varphi_\gamma \nabla f, \nabla f) = 0,$$

and

$$g(Z_\alpha, Z_\alpha) = f^{-4} g(\nabla f, \nabla f);$$

since $\nabla f \in \mathcal{H}$ we have $|Z_\alpha| = |Z_\beta|$ for all $\alpha,\beta \in \{1,2,3\}$. □

Theorem 4.4 *The compatible metric with gauge transformation of a 3-Sasakian manifold is*

$$\bar{g}(X,Y) = fg + \Sigma_{\alpha=1}^{3}[f(f + f^2|Z_\alpha|^2 - 1)\eta_\alpha \otimes \eta_\alpha - f^2\eta_\alpha \otimes g(Z_\alpha, .) - f^2 g(Z_\alpha, .) \otimes \eta_\alpha]. \tag{4}$$

Proof It is sufficient to prove $\bar{g}(\bar{\varphi}_\alpha X, \bar{\varphi}_\alpha Y) = \bar{g}(X,Y) - \bar{\eta}_\alpha(X)\bar{\eta}_\alpha(Y)$. From (4) we have

$$\bar{g}(X,Y) = \bar{g}_1 + \Sigma_{\alpha=2}^{3}[f(f + f^2|Z_\alpha|^2 - 1)\eta_\alpha \otimes \eta_\alpha - f^2\eta_\alpha \otimes g(Z_\alpha, .) - f^2 g(Z_\alpha, .) \otimes \eta_\alpha],$$

Therefore we have

$$\begin{aligned}\bar{g}(\bar{\varphi}_1 X, \bar{\varphi}_1 Y) = \bar{g}_1(X,Y) - \bar{\eta}_1(X)\bar{\eta}_1(Y) \\ + f(f + f^2|Z_2|^2 - 1)\eta_3(X)\eta_3(Y) - f^2\eta_3(X)g(Z_3,Y) - f^2 g(Z_3,X)\eta_3(Y) \\ + f(f + f^2|Z_3|^2 - 1)\eta_2(X)\eta_2(Y) - f^2\eta_2(X)g(Z_2,Y) - f^2 g(Z_2,X)\eta_2(Y).\end{aligned}$$

From lemma 4.3 and (2) we can see that this metric is compatible with a new 3-Sasakian structure. □

Now let $(M, \xi_\alpha, \varphi_\alpha, \eta_\alpha, g)$ with $\alpha \in \{1,2,3\}$ be a manifold with an almost contact 3-structure. If we set $\bar{\eta}_\alpha = e^\sigma\eta_\alpha$ then we have three almost contact structures:

$$\bar{\eta}_\alpha = e^\sigma\eta_\alpha, \quad d\bar{\eta}_\alpha = e^\sigma(d\eta_\alpha + d\sigma \wedge \eta_\alpha), \quad \bar{\xi}_\alpha = e^{-\sigma}(\xi_\alpha + \varphi A_\alpha), \quad \bar{\varphi}_\alpha = \varphi_\alpha + \eta \otimes A_\alpha,$$

$$\bar{g}_\alpha = e^{2\sigma}(g - \eta_\alpha \otimes g(\varphi A_\alpha, .) - g(\varphi A_\alpha, .) \otimes \eta_\alpha + g(A,A)\eta_\alpha \otimes \eta_\alpha).$$

We say that this transformation is a gauge transformation of an almost contact 3-structure. In the following theorem we find a condition in which gauge transformation of an almost contact structure has an almost contact 3-structure.

Theorem 4.5 *A gauge transformation of an almost contact 3-structure has an almost contact 3-structure iff* $A_1 = A_2 = A_3 \in \mathcal{H}$.

Proof We have

$$\begin{aligned}\bar{\eta}_\alpha(\bar{\xi}_\beta) &= \eta_\alpha(\varphi_\beta A_\beta) \\ &= \eta_\gamma(A_\beta),\end{aligned}$$

Therefore $\bar{\eta}_\alpha(\bar{\xi}_\beta) = 0$ iff $A_\alpha \in \mathcal{H}, \quad \forall \alpha \in \{1, 2, 3\}$.

$$\bar{\varphi}_\alpha \bar{\xi}_\beta = e^{-\sigma}(\xi_\gamma + \varphi_\gamma A_\beta),$$

and so $\bar{\varphi}_\alpha \bar{\xi}_\beta = \bar{\xi}_\gamma$ iff $A_1 = A_2 = A_3$. With direct computation we can see that

$$\bar{\varphi}_\alpha \bar{\varphi}_\beta - \bar{\eta}_\beta \otimes \bar{\xi}_\alpha = \varphi_\gamma + \eta_\gamma \otimes A_\alpha + \eta_\beta \otimes \varphi_\alpha A_\beta - \eta_\beta \otimes \varphi_\alpha A_\alpha.$$

By setting $X \in ker\eta_\gamma$ and $X = \xi_\gamma$ we see that $\bar{\varphi}_\alpha \bar{\varphi}_\beta - \bar{\eta}_\beta \otimes \bar{\xi}_\alpha = \bar{\varphi}_\gamma$ iff $A_1 = A_2 = A_3$. Therefore, the almost contact 3-structure is invariant under the gauge transformation iff $A_1 = A_2 = A_3 \in \mathcal{H}$. □

Remark 4.6 *If we set* $\bar{\eta_\alpha} = e^{\sigma_\alpha}\eta_\alpha$, *we will see from (2) that this transformation of an almost contact 3-structure cannot result in an almost contact 3-structure unless* $\sigma_1 = \sigma_2 = \sigma_3$. *Then we can use theorem 4.5*

Theorem 4.7 *A compatible metric with this new almost contact 3-structure is*

$$\bar{g} = e^{2\sigma}(g + \Sigma_{\alpha=1}^3[-\eta_\alpha \otimes g(\varphi A_\alpha, .) - g(\varphi A_\alpha, .) \otimes \eta_\alpha + g(A, A)\eta_\alpha \otimes \eta_\alpha]). \quad (5)$$

Proof It is sufficient to prove $\bar{g}(\bar{\varphi}_\alpha X, \bar{\varphi}_\alpha Y) = \bar{g}(X, Y) - \bar{\eta}_\alpha(X)\bar{\eta}_\alpha(Y)$. From (5) we have

$$\bar{g}(X, Y) = \bar{g}_1 + e^{2\sigma}(\Sigma_{\alpha=2}^3[-\eta_\alpha \otimes g(\varphi A_\alpha, .) - g(\varphi A_\alpha, .) \otimes \eta_\alpha + g(A, A)\eta_\alpha \otimes \eta_\alpha)]),$$

and from (2) and theorem 4.5 we can see that this metric is compatible with the gauge transformation of an almost contact 3-structure. □

Now with direct computation we have the following theorem

Theorem 4.8 *A gauge transformation of a 3-cosymplectic results in a 3-cosymplectic structure iff*

$$\forall \alpha \in \{1, 2, 3\}, \quad d\sigma \wedge \eta_\alpha = 0, \quad d\sigma \wedge \Phi_\alpha = 0, \quad A_1 = A_2 = A_3 \in \mathcal{H}.$$

Proof From theorem 2.1 we can see that gauge transformation of a 3-cosymplectic manifold is 3-cosymplectic iff each structure is almost cosymplectic. Therefore a gauge transformation of a 3-cosymplectic manifold is 3-cosymplectic iff

$$\forall \alpha \in \{1, 2, 3\}, \quad d\sigma \wedge \eta_\alpha = 0, \quad d\sigma \wedge \Phi_\alpha = 0.$$

□

5. Gauge transformation and slant submanifolds of an almost contact structure

Let $\tilde{M}$ be a slant submanifold of the almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ and $\theta(X)$ be the angle between φX and $T_x\tilde{M}$. It can easily be seen that the distribution $\mathcal{D}$ is invariant under the gauge transformation. Thus we can prove the following theorem:

Theorem 5.1 *Slant submanifolds of an almost contact metric manifold are invariant under gauge transformations.*

Proof We know that $X \in T_x\tilde{M} - < \xi_x >$ are vector fields in $T_x\tilde{M} \cap \mathcal{D}$. Since the distribution $\mathcal{D}$ is invariant under gauge transformation we have $\bar{\varphi}X = \varphi X$ and therefore $\bar{T}X = TX$ and $\bar{N}X = NX$.

We compute $\bar{\theta}(X)$

$$\cos\bar{\theta} = \frac{\bar{g}(\varphi X, TX)}{|\varphi X|_{\bar{g}}|TX|_{\bar{g}}},$$

Since $\bar{g} = e^{2\sigma}(g - \eta \otimes g(\varphi A, .) - g(\varphi A, .) \otimes \eta + g(A, A)\eta \otimes \eta)$, $\varphi X \in \mathcal{D}$ and $TX \in \mathcal{D}$ we have

$$\bar{g}(\varphi X, TX) = g(\varphi X, TX), \quad |TX|_{\bar{g}} = |TX|_g, \quad |\varphi X|_{\bar{g}} = |\varphi X|_g,$$

and so

$$cos\bar{\theta}(X) = cos\theta(X).$$

Finally since $\bar{\theta} \in [0, \frac{\pi}{2}]$, for any $x \in \tilde{M}$ and $X \in T_x\tilde{M} - < \bar{\xi} >$ we have

$$\bar{\theta}(X) = \theta(X).$$

□

Remark 5.2 *Invariant and anti-invariant submanifolds are invariant under the gauge transformation of an almost contact structure since for any $X \in T\tilde{M}$ we have $\bar{\varphi}X = \varphi X$.*

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