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Optimality conditions via weak subdifferentials in reflexive Banach spaces

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Abstract: In this paper the relation between the weak subdifferentials and the directional derivatives, as well as optimality conditions for nonconvex optimization problems in reflexive Banach spaces, are investigated. It partly generalizes several related results obtained for finite dimensional spaces.

Key words: Supporting cone, weak subdifferential and nonconvex optimization

1. Introduction

The generalization of concepts of ordinary derivatives and normal cones plays an important role in the study of necessary and sufficient conditions of optimality for nonsmooth and nonconvex optimization problems. The notion of subdifferentials was introduced by Rockafellar [17] to deal with optimization problems involving convex and nonsmooth functions. Since then, different notions of subdifferentials and normal cones have been introduced, which are applicable for different classes of optimization problems. We mention here the concepts of the Fréchet subdifferential [3, 15], Clarke's subdifferential [4], and limiting Fréchet subdifferentials [15, 16].

In [1, 2], the notion of a supporting cone was introduced and led to so-called weak subdifferentials. To eliminate the duality gap in nonconvex programming, an augmented Lagrangian is used that is constructed by supporting cones [2, 5, 6]. Later in [12], the concept of an augmented dual cone was introduced in Banach spaces and a special class of sublinear functions was defined by using the elements of the augmented dual cone; it was shown that two closed cones possessing a separation property can be separated by using a zero sublevel set of some function from this class. Recently, these concepts were used in [13, 14] to obtain necessary and sufficient conditions of optimality for a wide range of nonconvex and nonsmooth problems in Euclidean space.

In this paper, we study optimality conditions for nonconvex nonsmooth problems in reflexive Banach spaces by applying augmented normal cones and weak subdifferentials. The main purpose is to establish the analogies of the main results obtained in [14] for infinite dimensional normed spaces by using the notion of the supporting cone introduced in [8].

The paper is organized as follows. The main notations, definitions, and preliminaries are presented in the next section. In Section 3, we establish the relation of weak subdifferentials with the directional derivatives in reflexive Banach spaces. Optimality conditions in infinite dimensional normed spaces by applying weak subdifferentials are presented in section 4.

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2. Notations

Throughout the paper we assume that $\mathbf{X}$ is a reflexive Banach space with norm $\|\cdot\|$ unless otherwise stated. Let $\Omega \subset \mathbf{X}$ and $\bar{x} \in \Omega$. We will use the notation $\mathbf{K} = \text{cl}\,(\text{cone}(\Omega - \bar{x}))$ where "cl" stands for the closure of a set, and "cone(A)" for a given set $A \subset \mathbf{X}$ stands for

$$\text{cone}(A) = \{\lambda x : \ \lambda \geq 0, \ x \in A\}.$$

The unit sphere and the unit ball of $\mathbf{X}$ are denoted by $\mathbf{U}$ and $\mathbf{B}$, respectively:

$$\mathbf{U} := \{x \in \mathbf{X} : \ \|x\| = 1\}, \ \ \mathbf{B} := \{x \in \mathbf{X} : \ \|x\| \leq 1\}.$$

The dual norm of $\mathbf{X}$ is denoted by $\|\cdot\|_*$, where $\|\cdot\|_* := \max\{\langle \cdot, x\rangle : \ x \in \mathbf{U}\}$ where $\langle \cdot, \cdot \rangle$ is the scalar product (note that any continuous linear function attains its supremum on a unit ball of reflexive Banach space [10]). The unit sphere and unit ball of dual space of $\mathbf{X}$ are denoted by $\mathbf{U}^*$ and $\mathbf{B}^*$, respectively.

We say that Ω has a conic gap at $\bar{x}$ if $\mathbf{K} \neq \mathbf{X}$ (this property for set Ω at $\bar{x}$ was called "cone-shaped" in [14]). In [7, 8] a new supporting function was introduced to characterize the class of nonconvex sets having conic gaps. Given $x^* \in \mathbf{U}^*$, this supporting function $\sigma_\Omega(x^*; \bar{x})$ for the set Ω at $\bar{x}$ is defined as:

$$\sigma_\Omega(x^*; \bar{x}) := \sup_{y \in \mathbf{K} \cap \mathbf{U}} \langle x^*, y \rangle. \tag{2.1}$$

We present the definition of strictly convex spaces and three propositions used in the remainder of this paper.

Definition 2.1 *(page 112, [18]) Normed space $\mathbf{X}$ is called strictly convex if its unit ball is a strictly convex set, i.e. if $x \neq y$, $x, y \in \mathbf{U}$, and $h = \frac{1}{2}(x + y)$ then $\|h\| < 1$.*

The following proposition is an extension of the Hahn–Banach theorem [18, Theorem 5.20]:

Proposition 2.2 *Let $x^{'} \in \mathbf{U}$. There exists $x^* \in \mathbf{U}^*$ such that*

$$\langle x^*, x^{'} \rangle = \max_{x \in \mathbf{U}} \langle x^*, x \rangle = 1.$$

Proposition 2.3 *[9] Let $\mathbf{X}$ be reflexive strictly convex space and $x^* \in \mathbf{X}^*$. Then the maximum of x^* on unit sphere $\mathbf{U}$ is unique.*

Proposition 2.4 *[11, Theorem 7] Unit ball $\mathbf{U}$ of reflexive space is weakly sequentially compact.*

A new supporting cone, a "*$\sigma-$supporting cone*", constructed by using function $\sigma_\Omega(x^*; \bar{x})$ is introduced in the next definition.

Definition 2.5 *A σ-supporting cone for the set Ω at $\bar{x}$ is defined as follows:*

$$N^\sigma(\bar{x}; \Omega) = cone\{x^* \in \mathbf{U}^* : \ \sigma_\Omega(x^*; \bar{x}) = \sup_{y \in \mathbf{K} \cap \mathbf{U}} \langle x^*, y \rangle < 1\}. \tag{2.2}$$

We show that the following representation is true for a σ-supporting cone:

$$N^{\sigma}(\bar{x};\Omega) = \{x^* \in \mathbf{X}^* : \ \sigma_\Omega(x^*;\bar{x}) = \sup_{y\in \mathbf{K}\cap\mathbf{U}} \langle x^*, y\rangle < \|x^*\|_*\}. \tag{2.3}$$

Denote

$$\mathbf{C} = \text{cone}\{x^* \in \mathbf{U}^* : \ \sigma_\Omega(x^*;\bar{x}) = \sup_{y\in \mathbf{K}\cap\mathbf{U}} \langle x^*, y\rangle < 1\};$$

$$\mathbf{D} = \{x^* \in \mathbf{X}^* : \ \sigma_\Omega(x^*;\bar{x}) = \sup_{y\in \mathbf{K}\cap\mathbf{U}} \langle x^*, y\rangle < \|x^*\|_*\},$$

and let $x^* \in \mathbf{C}$. Clearly, we have $\frac{x^*}{\|x^*\|_*} \in \mathbf{U}^*$ and therefore

$$\sigma_\Omega(\frac{x^*}{\|x^*\|_*};\bar{x}) = \sup_{y\in \mathbf{K}\cap\mathbf{U}} \langle \frac{x^*}{\|x^*\|_*}, y\rangle = \frac{1}{\|x^*\|_*} \sup_{y\in \mathbf{K}\cap\mathbf{U}} \langle x^*, y\rangle < 1. \tag{2.4}$$

From (2.4), we obtain $\sup_{y\in \mathbf{K}\cap\mathbf{U}} \langle x^*, y\rangle < \|x^*\|_*$ and that means $x^* \in \mathbf{D}$.

Thus, $\mathbf{C} \subset \mathbf{D}$. To show the inverse inclusion, take any $x^* \in \mathbf{D}$; that is,

$$\sup_{y\in \mathbf{K}\cap\mathbf{U}} \langle x^*, y\rangle < \|x^*\|_*. \tag{2.5}$$

Dividing both sides of equation (2.5) by $\|x^*\|_*$, we obtain

$$\sup_{y\in \mathbf{K}\cap\mathbf{U}} \langle \frac{x^*}{\|x^*\|_*}, y\rangle < 1,$$

and that means $\frac{x^*}{\|x^*\|_*} \in \{x^* \in \mathbf{U}^* : \ \sigma_\Omega(x^*;\bar{x}) = \sup_{y\in \mathbf{K}\cap\mathbf{U}} \langle x^*, y\rangle < 1\}$ and consequently $x^* \in \mathbf{C}$ and $\mathbf{D} \subset \mathbf{C}$.

3. Directional derivatives and weak subdifferentials

The notion of a weak subdifferential, introduced in [1] for any normed spaces, will be used to establish optimality conditions in the next section. One of the important properties of this notion is its relation with the directional derivatives. This property was established in [14] for the Euclidean norm. In this section we prove this property for any reflexive Banach space that is strictly convex.

Let $f : \Omega \to R$ be a single-valued function. We start with the definition of weak subdifferential.

Definition 3.1 *A pair of $(x^*, \alpha) \in \mathbf{X}^* \times R$ is called a weak subgradient of f at $\bar{x}$ on Ω if*

$$f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle + \alpha\|x - \bar{x}\|, \ \ \forall x \in \Omega. \tag{3.1}$$

The set

$$\partial^w_\Omega f(\bar{x}) = \{(x^*, \alpha) \in \mathbf{X}^* \times R : (3.1) \text{ is satisfied}\} \tag{3.2}$$

of all subgradients is called the weak subdifferential of f at $\bar{x}$ on Ω.

The directional derivative of function f at $\bar{x}$ on direction $x - \bar{x}$ is defined as follows:

$$f^{'}(\bar{x}; x - \bar{x}) := \lim_{t\downarrow 0} \frac{f(\bar{x} + t(x - \bar{x})) - f(\bar{x})}{t}.$$

We will use the following assumption in order to establish some properties of the weak subdifferential and to derive optimality condition.

Assumption 1. *Suppose that* $\mathbf{K} = cone(\Omega - \bar{x})$ *is a closed set and* f *has a directional derivative* $f^{'}(\bar{x}; h)$ *at* $\bar{x} \in \Omega$ *for all* $h \in \mathbf{K}$. *Moreover,* $f^{'}(\bar{x}; \cdot)$ *is lower semicontinuous on* $\mathbf{K}$ *and there exists* $\delta > 0$ *such that*

$$f(x) - f(\bar{x}) \geq \delta f^{'}(\bar{x}; x - \bar{x}), \; \forall x \in \Omega. \tag{3.3}$$

The next theorem is about the relation between weak subdifferentials and directionally differentiable functions in reflexive Banach spaces that are strictly convex.

Theorem 3.2 *Let* $\mathbf{X}$ *be a reflexive strictly convex space,* $\Omega \subseteq \mathbf{X}$ *and* $\bar{x} \in \Omega$. *Assume that Assumption 1 holds and*

$$inf\{f^{'}(\bar{x}; h) : \; h \in \mathbf{K} \cap \mathbf{U}\} > -\infty. \tag{3.4}$$

Then f *is weakly subdifferentiable at* $\bar{x}$*; that is,* $\partial^w_\Omega f(\bar{x}) \neq \emptyset$. *Moreover, if* $\delta = 1$ *in Assumption 1, then*

$$\sup\{\langle x^*, h\rangle + \alpha\|h\| : \; (x^*, \alpha) \in \partial^w_\Omega f(\bar{x})\} = f^{'}(\bar{x}; h), \;\; \forall \; h \in \mathbf{K}. \tag{3.5}$$

Proof: Let $h \in \mathbf{K} \cap \mathbf{U}$. By Proposition 2.2, there exists $x^* \in \mathbf{U}^*$ such that

$$\max_{y \in \mathbf{U}} \langle x^*, y\rangle = \langle x^*, h\rangle = 1,$$

where h is the unique maximum point according to Proposition 2.3.

Take any $\epsilon > 0$ and denote $x^*_1 = (f^{'}(\bar{x}; h) - \epsilon - \alpha_1)\, x^*$ where $\alpha_1 \in (-\infty, \infty)$.

We show that there exists sufficiently small α_1 such that $(\delta\, x^*_1, \delta\, \alpha_1) \in \partial^w_\Omega f(\bar{x})$.

First we show that the relation

$$f^{'}(\bar{x}; z) \geq \langle x^*_1, z\rangle + \alpha_1\|z\| = (f^{'}(\bar{x}; h) - \epsilon)\langle x^*, z\rangle - \alpha_1(\langle x^*, z\rangle - 1), \;\; \forall z \in \mathbf{K} \cap \mathbf{U}, \tag{3.6}$$

is satisfied for some sufficiently small α_1.

Assume to the contrary that this is not true. Then given any sequence $\alpha_n \to -\infty$, there exists $z_n \in \mathbf{K} \cap \mathbf{U}$ such that

$$f^{'}(\bar{x}; z_n) < (f^{'}(\bar{x}; h) - \epsilon)\langle x^*, z_n\rangle - \alpha_n(\langle x^*, z_n\rangle - 1), \;\; \forall \; n \in N. \tag{3.7}$$

By Proposition 2.4, there is a weakly convergent subsequence of $\{z_n\}_{n \in N}$. Without loss of generality assume that z_n converges weakly to $\tilde{z} \in \mathbf{U}$.

Let $\tilde{z} \neq h$. As h is a unique maximum point of $\langle x^*, \cdot\rangle$ over the unit ball, the inequality $\langle x^*, \tilde{z}\rangle - 1 < 0$ holds. Then, letting α_n approach to $-\infty$ in (3.7), we have $f^{'}(\bar{x}; \tilde{z}) = -\infty$, which contradicts (3.4).

Let $\tilde{z} = h$ and consequently $\langle x^*, \tilde{z}\rangle - 1 = 0$. Then by taking the limit in (3.7) and using the lower semicontinuity of the directional derivative $f^{'}(\bar{x}; \cdot)$, as well as the inequality $\langle x^*, z_n\rangle - 1 \leq 0, \; \forall n$, we have

$$f^{'}(\bar{x}; h) \leq \liminf_{n \to -\infty} f^{'}(\bar{x}; z_n) \leq (f^{'}(\bar{x}; h) - \epsilon)\langle x^*, h\rangle = f^{'}(\bar{x}; h) - \epsilon.$$

Since $\epsilon > 0,$ this is again a contradiction.

Therefore, (3.6) holds for some sufficiently small α_1. Take any $x \in \Omega,\ x \neq \bar{x}$. Then $\frac{x-\bar{x}}{\|x-\bar{x}\|} \in \mathbf{K} \cap \mathbf{U}$ and from (3.6) we obtain

$$f^{'}(\bar{x}; x-\bar{x}) \geq \langle x_1^*, x-\bar{x}\rangle + \alpha_1\|x-\bar{x}\|, \quad \forall x \in \Omega, x \neq \bar{x}.$$

This relation also holds for $x = \bar{x}$. Then from (3.3) it follows that

$$f(x) - f(\bar{x}) \geq \delta\, f^{'}(\bar{x}; x-\bar{x}) \geq \langle \delta\, x_1^*, x-\bar{x}\rangle + \delta\, \alpha_1\|x-\bar{x}\|, \quad \forall x \in \Omega.$$

Thus, $(\delta\, x_1^*, \delta\, \alpha_1) \in \partial_\Omega^w f(\bar{x})$; that is, the set of weak subdifferentials is not empty.

Now consider the case $\delta = 1$. From $(x_1^*, \alpha_1) \in \partial_\Omega^w f(\bar{x})$, we have

$$\sup\{\langle x^*, h\rangle + \alpha\|h\| :\ (x^*, \alpha) \in \partial_\Omega^w f(\bar{x})\} \geq \langle x_1^*, h\rangle + \alpha_1\|h\| =$$

$$(f^{'}(\bar{x}; h) - \epsilon)\langle x^*, h\rangle - \alpha_1(\langle x^*, h\rangle - 1) = f^{'}(\bar{x}; h) - \epsilon.$$

Since this relation holds for any $\epsilon > 0,$ we obtain

$$\sup\{\langle x^*, h\rangle + \alpha\|h\| :\ (x^*, \alpha) \in \partial_\Omega^w f(\bar{x})\} \geq f^{'}(\bar{x}; h). \tag{3.8}$$

On the other hand, it is not difficult to show that, for any $(x^*, \alpha) \in \partial_\Omega^w f(\bar{x})$, the inequality

$$f^{'}(\bar{x}; h) \geq \langle x^*, h\rangle + \alpha\|h\|$$

and consequently

$$f^{'}(\bar{x}; h) \geq \sup\{\langle x^*, h\rangle + \alpha\|h\| :\ (x^*, \alpha) \in \partial_\Omega^w f(\bar{x})\} \tag{3.9}$$

hold. Then, for given $h \in \mathbf{K} \cap \mathbf{U}$, the required relation (3.5) follows from (3.8) and (3.9). Since both sides in (3.5) are superlinear in h, it is also true for all $h \in \mathbf{K}$.

□

4. Weak subdifferentials and optimality condition

In this section we consider the necessary and sufficient conditions for a class of nonconvex and nonsmooth optimization problems in reflexive Banach spaces by applying weak subdifferentials, augmented normal cones, and the function $\sigma_\Omega(x^*; \bar{x})$. Similar optimality conditions are considered in [14] and [7] for the Euclidean space and any finite normed space, respectively.

We will use the following so-called separation property introduced in [12].

Definition 4.1 *[12] Let $\mathbf{C}$ and $\mathbf{K}$ be closed cones of a normed space $\mathbf{X}$. Let $\tilde{\mathbf{C}}$ and $\tilde{K}^{\partial}$ be the closure of the sets $co(\mathbf{C} \cap \mathbf{U})$ and $co((bd(\mathbf{K}) \cap \mathbf{U}) \cup \{0_{\mathbf{X}}\})$, respectively. The cones $\mathbf{C}$ and $\mathbf{K}$ are said to have the separation property with respect to the norm $\|\cdot\|$ if*

$$\tilde{\mathbf{C}} \cap \tilde{K}^{\partial} = \emptyset. \tag{4.1}$$

Take any positive number $\beta < 1$ and $x^* \in \mathbf{U}^*$. Consider the cone

$$\mathbf{C} = \text{cone}\{x \in \mathbf{U} : \langle x^*, x\rangle \geq \beta\}. \tag{4.2}$$

In the following theorem we show that under some conditions on the σ-supporting cone, the cones $\mathbf{C}$ and $\mathbf{K}$ satisfy the separation property.

Theorem 4.2 *Let $\mathbf{X}$ be a reflexive Banach space and let there exist $x^* \in \mathbf{U}^*$ such that $\sigma_\Omega(x^*;\bar{x}) < 1$. Then given any positive number $\beta \in (\sigma_\Omega(x^*;\bar{x}), 1)$, cones $\mathbf{C}$ and $\mathbf{K}$ satisfy the separation property.*

Proof: By the assumption of the theorem

$$\sup_{y\in\mathbf{K}\cap\mathbf{U}} \langle x^*, y\rangle = \sigma_\Omega(x^*;\bar{x}) < 1 = \|x^*\|_* = \max_{x\in\mathbf{U}} \langle x^*, x\rangle. \tag{4.3}$$

Denote $\alpha = \sigma_\Omega(x^*;\bar{x})$ and take any $\beta > 0$ such that

$$\alpha = \sup_{y\in\mathbf{K}\cap\mathbf{U}} \langle x^*, y\rangle = \sigma_\Omega(x^*;\bar{x}) < \beta < 1. \tag{4.4}$$

Since $\mathbf{X}$ is reflexive, there exists $a \in \mathbf{U}$ such that $\langle x^*, a\rangle = \|x^*\|_* = 1$. Then $a \in \mathbf{C}$ and $\mathbf{C} \neq \emptyset$.

Denote $\tilde{\mathbf{C}} = \text{cl}(\text{co}(\mathbf{C}\cap\mathbf{U}))$ and $\tilde{K}^\partial = \text{cl}(\text{co}((\text{bd}(\mathbf{K})\cap\mathbf{U})\cup\{0_\mathbf{X}\}))$. We need to prove $\tilde{\mathbf{C}}\cap\tilde{K}^\partial = \emptyset$.

First we show that for any $x \in \tilde{\mathbf{C}}$ the inequality $\langle x^*, x\rangle \geq \beta$ holds. Let $x \in \text{co}(\mathbf{C}\cap\mathbf{U})$. Then the following representation is true $x = \sum_{i=1}^{n(x)} \alpha_i \tilde{x}_i$ for some $n(x) \in \mathbf{N}$, where $\tilde{x}_i \in \mathbf{C}\cap\mathbf{U}$ and $\sum_{i=1}^{n(x)} \alpha_i = 1$. As $\tilde{x}_i \in \mathbf{C}\cap\mathbf{U}$, from (4.2) we have

$$\langle x^*, x\rangle = \sum_{i=1}^{n(x)} \alpha_i \langle x^*, \tilde{x}_i\rangle \geq \beta. \tag{4.5}$$

Let $x \in \text{cl}(\text{co}(\mathbf{C}\cap\mathbf{U}))$, which means there exists sequence x_n in $\mathbf{C}\cap\mathbf{U}$ such that x_n is convergent to x weakly and consequently $\langle x^*, x_n\rangle \to \langle x^*, x\rangle$. Then by (4.5), we have $\langle x^*, x\rangle \geq \beta$.

It is clear from (4.4) that for any $y \in \mathbf{K}\cap\mathbf{U}$, the relation $\langle x^*, y\rangle \leq \alpha < \beta$ holds. Since $\beta > 0$, we have $\langle x^*, 0\rangle = 0 < \beta$. Thus, $\langle x^*, y\rangle \leq \max\{\alpha, 0\} < \beta$ for any $y \in \tilde{K}^\partial$. Therefore, $\tilde{\mathbf{C}}\cap\tilde{K}^\partial = \emptyset$. □

The condition of reflexivity of $\mathbf{X}$ is important in the proof of Theorem 4.2, although it is our opinion that it can be relaxed. We provide an example below where the space $\mathbf{X}$ is not reflexive but the separation property is still valid.

Example 4.3 *Consider the Banach space $\mathbf{X} = C^0([0,1], R)$ with the norm $\|f\|_\infty = \max\{f(x) : x \in [0,1]\}$. Clearly $\mathbf{X}$ is not reflexive [4].*

Let the linear continuous function x^* be defined as follows:

$$\langle x^*, f\rangle := \int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt \text{ where } f \in \mathbf{X}.$$

We show that $x^* \in \mathbf{U}^*$. Clearly, $\langle x^*, f\rangle \leq 1$ for any $f \in \mathbf{U}$ and hence $\|x^*\|_* \leq 1$. Consider a sequence of functions $f_n(x)$ defined by

$$f_n(x) = \begin{cases} 1 & if \quad x \in [0, \frac{1}{2} - \frac{1}{n}] \\ -nx + \frac{n}{2} & if \quad x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}] \\ -1 & if \quad x \in [\frac{1}{2} + \frac{1}{n}, 1]. \end{cases}$$

It is easy to check that $\|f_n\|_\infty = 1$ *and* $\langle x^*, f_n\rangle = 1 - \frac{2}{n}$. *Therefore,* $\|x^*\|_* \geq \sup_n(1 - \frac{2}{n}) = 1$; *that is,* $\|x^*\|_* \in \mathbf{U}^*$.

Now we consider any set Ω *satisfying* $\sigma_\Omega(x^*; \bar{x}) < 1$. *Take an arbitrary number* $\beta \in (\sigma_\Omega(x^*; \bar{x}), 1)$. *First we show that* $\mathbf{C} \neq \emptyset$.

Clearly there is n *such that* $\langle x^*, f_n\rangle = 1 - 2/n \in (\beta, 1)$; *hence,* $\mathbf{C} \neq \emptyset$. *Now we show that for any* $x \in \tilde{\mathbf{C}}$, *the inequality* $\langle x^*, x\rangle \geq \beta$ *holds. By following the proof of Theorem 4.2, for any* $x \in co(\mathbf{C} \cap \mathbf{U})$, *the inequality* $\langle x^*, x\rangle \geq \beta$ *holds. Let* $x \in cl(co(\mathbf{C} \cap \mathbf{U}))$, *which means there exists sequence* x_n *in* $co(\mathbf{C} \cap \mathbf{U}))$ *such that* x_n *is convergent to* x, *i.e.* $\|x_n - x\| \to 0$. *We have*

$$\langle x^*, x_n - x\rangle \leq \|x_n - x\| \to 0;$$

that means $\langle x^*, x_n\rangle \to \langle x^*, x\rangle$ *and consequently,* $\langle x^*, x\rangle \geq \beta$. *Hence, for any* $x \in \tilde{\mathbf{C}}$ *the inequality* $\langle x^*, x\rangle \geq \beta$ *holds.*

Again, by following the proof of Theorem 4.2, it is easy to show that $\langle x^*, y\rangle \leq \max\{\alpha, 0\} < \beta$ *for any* $y \in \tilde{K}^\partial$. *Therefore,* $\tilde{K}^\partial \cap \tilde{\mathbf{C}} = \emptyset$.

□

The next theorem describes the necessary condition of optimality that generalizes Theorem 4 in [14] to any reflexive spaces by applying the function $\sigma_\Omega(x^*; \bar{x})$.

Theorem 4.4 *Let* $\mathbf{X}$ *be a reflexive Banach space,* $\Omega \subset \mathbf{X}$ *and* $f : \Omega \to R$ *be a given function. Assume that* $\bar{x}$ *is a minimizer of* f *over* Ω *and there exists* $x^* \in U^*$ *such that* $\sigma_\Omega(x^*; \bar{x}) < 1$. *Letting* $\Omega \setminus \{\bar{x}\} \neq \emptyset$, *Assumption 1 holds and*

$$\bar{\beta} := \inf\{f^{'}(\bar{x}; h) : \; h \in \mathbf{K} \cap \mathbf{U}\} > 0. \tag{4.6}$$

Then there exists $(z^*, \alpha) \in \partial^w_\Omega f(\bar{x})$ *with* $z^* \neq 0$, $\alpha \geq 0$ *such that*

$$\langle z^*, x - \bar{x}\rangle + \alpha\|x - \bar{x}\| \geq 0, \quad \forall x \in \Omega, \tag{4.7}$$

$$\langle z^*, z - \bar{x}\rangle + \alpha\|z - \bar{x}\| < 0, \quad \textit{for some } z \notin \Omega. \tag{4.8}$$

Proof: Let $\sigma_\Omega(x^*; \bar{x}) < 1$ for $x^* \in U^*$. By Theorem 4.2, there exists cone $\mathbf{C}$ such that $\mathbf{C}$ and $\mathbf{K}$ are separable in the sense of Definition 4.1. Therefore, by [12, Theorem 4.3], there exists $(y^*, \gamma) \in \partial^w_\Omega f(\bar{x})$ with $y^* \neq 0$ and $\gamma \geq 0$ such that it separates the sets $\mathbf{C}$ and $\mathbf{K}$ in the following sense:

$$\langle y^*, y\rangle + \gamma\|y\| < 0 \leq \langle y^*, x\rangle + \gamma\|x\|, \; \forall\, y \in \mathbf{C} \setminus \{0\} \text{ and } \forall\, x \in \mathbf{K}.$$

The rest of the proof is the same as in Theorem 4 in [14].

□

The following theorem presents sufficient conditions guaranteeing the existence of nontrivial solutions to

$$(0, 0) \in \partial^w_\Omega f(\bar{x}) + N^A(\bar{x}; \Omega). \tag{4.9}$$

Theorem 4.5 *Let all the conditions of Theorem 4.4 hold. Then there exists a nontrivial solution to 4.9.*

Proof: All conditions of Theorem 4.4 hold. Therefore, there exists $(z^*, \alpha) \in \partial^w_\Omega f(\bar{x})$ such that $z^* \neq 0$, $\alpha \geq 0$ and (4.7),(4.8) hold. Multiplying both sides of (4.7) by -1, we obtain

$$\langle -z^*, x - \bar{x} \rangle - \alpha \|x - \bar{x}\| \leq 0 \ \ \forall x \in \Omega,$$

which means $(-z^*, -\alpha) \in N^A(\bar{x}; \Omega)$. Thus, (4.9) is satisfied.

Now we show that (z^*, α) is a nontrivial solution; that is, $-\alpha > -\|-z^*\|_*$ or $\alpha < \|z^*\|_*$. By contradiction let $\alpha \geq \|z^*\|_*$. Then from the Cauchy–Schwarz inequality it follows that

$$\langle z^*, x - \bar{x} \rangle + \alpha \|x - \bar{x}\| \geq \langle z^*, x - \bar{x} \rangle + \|z^*\|_* \cdot \|x - \bar{x}\| \geq 0, \ \ \forall x.$$

This contradicts (4.8). □

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