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Quasi-metric trees and q-hyperconvex hulls

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Abstract: The investigation of metric trees began with J. Tits in 1977. Recently we studied a more general notion of quasi-metric tree. In the current article we prove, among other facts, that the q-hyperconvex hull of a q-hyperconvex T_0-quasi-metric tree is itself a T_0-quasi-metric tree. This is achieved without using the four-point property, a geometric concept used by Aksoy and Maurizi to show that every complete metric tree is hyperconvex.

Key words: Metric interval, metric tree, T_0-quasi-metric, quasi-metric interval, quasi-metric tree

1. Introduction

Metric trees have been studied since 1977 (see [11] J. Tits, *A theorem of Lie-Kolchin for trees, Contributions to Algebra: a collection of papers dedicated to Ellis Kolchin*, Academic Press, New York, 1977). A number of results on metric trees have found applications in many fields of mathematics like geometry, topology, and group theory. The importance of metric trees is not limited to mathematics, as for instance the study of phylogenetic trees in biology and medicine also employs metric trees (see [10] C. Semple, M. Steel, *Phylogenetics*, Oxford University Lecture Series in Mathematics and its Applications, 24 2003).

The applications of metric trees in information science, particularly in computer science [3, 6], are great motivations for the generalization of results about metric trees from a symmetric setting to an asymmetric framework. Thus this project is a part of that broad mission.

In [4], Dress proved that any metric tree is median. Here, we establish that this result still holds when the hypothesis is relaxed by dropping the symmetry condition of the distance function. Moreover, Aksoy and Maurizi in [3] studied the relationship between a metric tree and its hyperconvex hull by using the four-point property of a metric tree.

In [9], we started investigating the concept of a metric tree in T_0-quasi-metric spaces, which we called quasi-metric tree. In this article, we continue our study of this concept by generalizing some well-known results about metric trees from metric setting to the quasi-metric point of view. In particular, we prove that the q-hyperconvex hull of a T_0-quasi-metric tree is a T_0-quasi-metric tree. Among other results, we also extend the result of Agyingi et al., which says that when a T_0-quasi-metric space is joincompact then each of its endpoints is an endpoint of its q-hyperconvex hull. Several examples are also provided in this paper to illustrate the concepts involved in the study.

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Surprisingly, our investigations reveal that many classical results about metric trees do not require the use of symmetry of distance function. Hence they hold in a quasi-metric setting, although sometimes in a slightly different form.

2. Preliminaries

This section recalls some important definitions that we shall use in the rest of the paper.

Definition 1 *([8]) Let X be a set and $d : X \times X \to [0, \infty)$ a function into the set of all nonnegative reals. Then d is called a* quasi-pseudometric *on X if*

(a) $d(x, x) = 0$ whenever $x \in X$,

(b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

We shall say that d is a T_0-quasi-metric *provided that d also satisfies the following condition*

(c) For each $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$.

Remark 1 *Let d be a quasi-pseudometric on a set X; then the mapping $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric, called the* conjugate quasi-pseudometric of *d. As usual, a quasi-pseudometric d on X such that $d = d^{-1}$ is called a* pseudometric*. Note that for any (T_0-)quasi-pseudometric d, $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ is a pseudometric (metric). Moreover, if d is a quasi-pseudometric on X, then (X, d) is called a* quasi-pseudometric space*.*

For more details about the theory of quasi-metric spaces, we refer the reader to [5, 8].

Remark 2 *We note that for a quasi-pseudometric space (X, d), we have the following:*

1. *For each $x \in X$ and $\epsilon > 0$, $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ denotes the* open ϵ-ball *at x.*
2. *The collection of all open balls yields a base for a topology $\tau(d)$. It is called the* topology induced by d *on X.*
3. *Similarly we set for each $x \in X$ and $\epsilon \geq 0$, $C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$. Note that $C_d(x, \epsilon)$ is $\tau(d^{-1})$-closed, but not $\tau(d)$-closed in general.*

The following T_0-quasi-metric will be useful in the sequel. If $a, b \in \mathbb{R}$, we shall put $a \dot{-} b = \max\{a - b, 0\}$. Note that $r(a, b) = a \dot{-} b$ with $a, b \in \mathbb{R}$ defines a T_0-quasi-metric on $\mathbb{R}$.

Definition 2 *(a) A map $f : (X, d) \to (Y, e)$ between two quasi-pseudometric spaces (X, d) and (Y, e) is called an* isometry *provided that $e(f(x), f(y)) = d(x, y)$ whenever $x, y \in X$.*

(b) Two quasi-pseudometric spaces (X, d) and (Y, e) will be called isometric *provided that there exists a bijective isometry $f : (X, d) \to (Y, e)$.*

Definition 3 *A map $f : (X, d) \to (Y, e)$ between two quasi-pseudometric spaces (X, d) and (Y, e) is called* nonexpansive *provided that $e(f(x), f(y)) \leq d(x, y)$ whenever $x, y \in X$.*

We next recall the construction of the q-hyperconvex hull of a T_0-quasi-metric space (see [7] for more details).

Definition 4 *([7, Definition 2]) A quasi-pseudometric space (X,d) is called q-*hyperconvex *(or Isbell-convex) provided that for each family $(x_i)_{i\in I}$ of points in X and families $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ of nonnegative real numbers satisfying $d(x_i,x_j)\le r_i+s_j$ whenever $i,j\in I$, the following condition holds:*

$$\bigcap_{i\in I}(C_d(x_i,r_i)\cap C_{d^{-1}}(x_i,s_i))\neq\emptyset.$$

Example 1 *Let $r(a,b)=\max\{0,a\dot{-}b\}$ whenever $a,b\in\mathbb{R}$. The diagonal $\triangle$ of $(\mathbb{R}^2,r\times r^{-1})$ is isometric to $(\mathbb{R},r^s)$ where we set $N=r\times r^{-1}$, that is,*

$$N(x,y)=(x_1\dot{-}y_1)\vee(y_2\dot{-}y_1)$$

whenever $x=(x_1,x_2),y=(y_1,y_2)\in\mathbb{R}^2$.

It is readily checked $(\mathbb{R}^2,r\times r^{-1})$ is a q-hyperconvex as product of q-hyperconvex spaces (see [7, Proposition 2]).

Let (X,d) be a T_0-quasi-metric space. The function pair $f=(f_1,f_2)$ where $f_i:X\to[0,\infty)(i=1,2)$ is called *ample* provided that $d(x,y)\le f_2(x)+f_1(y)$ whenever $x,y\in X$. We say that a pair $f=(f_1,f_2)$ is *minimal* or *extremal* (among the ample pairs) if it is ample and whenever $g=(g_1,g_2)$ is ample on (X,d) and for each $x\in X$, $g_1(x)\le f_1(x)$ and $g_2(x)\le f_2(x)$, then $g_1(x)=f_1(x)$ and $g_2(x)=f_2(x)$.

Let $\mathcal{A}(X,d)$ denote the class of all ample function pairs on (X,d). For each $f=(f_1,f_2),g=(g_1,g_2)\in\mathcal{A}(X,d)$ we set

$$D(f,g)=\sup_{x\in X}(f_1(x)\dot{-}g_1(x))\vee\sup_{x\in X}(g_2(x)\dot{-}f_2(x)).$$

Then D is a T_0-quasi-metric on $\mathcal{A}(X,d)$. Note that if D maps to $[0,\infty]$, then D is called an extended T_0-quasi-metric on $\mathcal{A}(X,d)$.

By $\epsilon_q(X,d)$ we shall denote the class of all extremal ample function pairs on (X,d) equipped with the restriction of D to $\epsilon_q(X,d)\times\epsilon_q(X,d)$, which we shall denote by N. Therefore, this N is a (real-valued) T_0-quasi-metric on $\epsilon_q(X,d)\times\epsilon_q(X,d)$ (see [7, Remark 6]).

The necessary and sufficient condition for a function pair $f=(f_1,f_2)$ to be extremal is that f must satisfy the following equalities:

$$f_1(x)=\sup_{y\in X}(d^{-1}(x,y)\dot{-}f_2(y))$$

and

$$f_2(x)=\sup_{y\in X}(d(x,y)\dot{-}f_1(y))$$

whenever $x\in X$ (see [7, Lemma 6]).

For each $x\in X$, the function pair $f_x(y)=(d(x,y),d(y,x))$ whenever $y\in X$ is a minimal function pair on (X,d). The map e_X defined by $x\mapsto f_x$ whenever $x\in X$ defines an isometric embedding of (X,d) into $(\epsilon_q(X,d),N)$ (see [7, Lemma 1]). The couple $(\epsilon_q(X,d),N)$ is called the q*-hyperconvex hull* of (X,d). Note that the q-hyperconvex hull of a T_0-quasi-metric space is q-hyperconvex and it is unique up to isometry.

Note that $N(f_x,f_y)=d(x,y)$ whenever $x,y\in X$. Moreover, $N(f,f_x)=f_1(x)$ and $N(f_x,f)=f_2(x)$ whenever $x\in X$ and $f\in\epsilon_q(X,d)$.

Example 2 *(compare [1, Example 5]) Let* $a, b \in (0, \infty)$ *and* $Y = [0, a] \times [0, b]$*. We set*

$$D((\alpha_1, \alpha_2), (\beta_1, \beta_1)) = (\alpha_1 \dot{-} \beta_1) \vee (\alpha_2 \dot{-} \beta_2)$$

whenever $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in Y$*. Then* Y *is identified with the* q*-hyperconvex hull of the subspace* $X = \{(a, 0), (0, b)\}$ *of* Y*.*

The following definition can be compared with [3, Definition 1.2] and the definition in [4, p.322].

Definition 5 *([9, Definition 2]) Let* (X, d) *be a quasi-pseudometric space. The subset* $\langle x, y \rangle_d$ *of* X *is defined by*

$$\langle x, y \rangle_d = \{z \in X : d(x, z) + d(z, y) = d(x, y)\}.$$

Moreover $\langle x, y \rangle_d$ *is called a* quasi-pseudometric interval *of* (X, d)*.*

Lemma 1 *Let* (X, d) *be a quasi-pseudometric space and* $x, y \in X$*. If* $z \in \langle x, y \rangle_d$*, then* $\langle x, z \rangle_d \subseteq \langle x, y \rangle_d$*. Furthermore, if* $z \in \langle x, y \rangle_d$*, then* $\langle z, y \rangle_d \subseteq \langle x, y \rangle_d$*.*

Proof Suppose that $z \in \langle x, y \rangle_d$, then $d(x, y) = d(x, z) + d(z, y)$. If $t \in \langle x, z \rangle_d$ then we have that $d(x, t) + d(t, z) = d(x, z)$.

Thus,

$$\begin{aligned} d(x, y) &\leq d(x, t) + d(t, y) \leq d(x, t) + d(t, z) + d(z, y) \\ &\leq d(x, z) + d(z, y) = d(x, y). \end{aligned}$$

Therefore, $d(x, y) = d(x, t) + d(t, y)$. Hence $t \in \langle x, y \rangle_d$. The last statement follows by a similar argument. □

Example 3 *Consider the four point set* $X = \{1, 2, 3, 4\}$*. Let the* T_0*-quasi-metric* q *be defined by the distance metric*

$$M = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

that is, $q_{i,j} = q(i, j)$ *whenever* $i, j \in X$*. One can check easily that* q *is a* T_0*-quasi-metric on* X*.*

Furthermore, it is readily checked that $2 \in \langle 1, 3 \rangle_q$*,* $2 \in \langle 3, 1 \rangle_q$*, and* $2 \in \langle 4, 1 \rangle_q$*. Observe also that* $2 \in \langle 3, 4 \rangle_q$ *but* $q(1, 2) + q(2, 3) + q(3, 4) \neq q(1, 4)$*, since* $3 \notin \langle 1, 4 \rangle_q$*.* □

Definition 6 *We say that a quasi-pseudometric* (X, d) *is* thready *if for all* $v, w \in \langle x, y \rangle_d$ *one has*

$$d(x, v) + d(v, w) + d(w, y) = d(x, w) + d(w, v) + d(v, y) = d(x, y)$$

whenever $x, y \in X$*.*

3. Quasi-metric tree

The following definition was given in [9] (compare [2–4]).

Definition 7 *Let* (X, d) *be a* T_0*-quasi-metric space. Then* (X, d) *is a* (quasi-metric) tree *or* directed tree*, if it satisfies the following two conditions:*

(QMT1) For any $x, y \in X$*, there exists a unique function pair*

$$\varphi = \varphi_{xy} = ((\varphi_{xy})_1, (\varphi_{xy})_2)$$

where

$$(\varphi_{xy})_1 : ([0, d(x, y)], r^{-1}) \longrightarrow (X, d)$$

is an isometric embedding such that $(\varphi_{xy})_1(0) = x$ *and* $(\varphi_{xy})_1(d(x, y)) = y$*, and*

$$(\varphi_{xy})_2 : ([0, d^{-1}(x, y)], r) \longrightarrow (X, d^{-1})$$

is an isometric embedding such that $(\varphi_{xy})_2(0) = y$ *and* $(\varphi_{xy})_2(d^{-1}(x, y)) = x$*. With* $r(x, y) = \max\{x - y, 0\}$ *whenever* $x, y \in \mathbb{R}$*.*

(QMT2) For any pair $\varphi = (\varphi_1, \varphi_2)$*, where* φ_i *is an injective continuous function from* $[0, 1]$ *into* X*,* $(i = 1, 2)$ *such that*

$$\varphi_i : [0, 1] \longrightarrow X : t \mapsto x_t$$

one has $d(x_0, x_t) + d(x_t, x_1) = d(x_0, x_1)$*.*

Lemma 2 *If* (X, d) *is a quasi-metric tree and* $x, y, z \in X$*, then there exists a unique* $w \in X$ *such that*

$$\langle x, y\rangle_d \cap \langle z, x\rangle_d = \langle x, w\rangle_d.$$

Proof Assume that (X, d) is a quasi-metric tree and $x, y, z \in X$. Let $t_0 = d^{-1}(z, x)$ and

$$(\varphi_{zx})_2 : ([0, d^{-1}(z, x)], r) \longrightarrow (X, d^{-1})$$

be the unique isometric embedding such that $(\varphi_{zx})_2(0) = x$ and $(\varphi_{zx})_2(t_0) = z$. Consider

$$u_0 = \sup\{s \in [0, t_0] : (\varphi_{zx})_2(s) \in \langle x, y\rangle_d\}$$

and by letting $w = (\varphi_{zx})_2(u_0)$, we have $\langle x, w\rangle_d = (\varphi_{zx})_2([0, u_0])$ and u_0 is unique since $(\varphi_{zx})_2$ is bijective as (X, d) is a quasi-metric tree.

Hence $\langle x, w\rangle_d = \langle x, y\rangle_d \cap \langle z, x\rangle_d$. □

Corollary 1 *If* (X, d) *is a quasi-metric tree and* $x, y, z \in X$*, then there exists a unique* $v \in X$ *such that*

$$\langle y, x\rangle_d \cap \langle x, z\rangle_d = \langle v, x\rangle_d.$$

Example 4 *Let* $X = \{0, 1\}$ *be equipped with its usual order* $\leq$ *and with its natural* T_0*-quasi-metric* d. *Then* $d(x, y) = 0$ *if* $x \leq y$ *and* $d(x, y) = 1$ *if* $x > y$.

For each $x, y \in X$, *we define* $\varphi_{xy} = ((\varphi_{xy})_1, (\varphi_{xy})_2)$ *as follows:* $(\varphi_{00})_1(0) = 0, (\varphi_{01})_1(0) = 0, (\varphi_{10})_1(0) = 1, (\varphi_{11})_1(0) = 1, (\varphi_{10})_1(1) = 0$ *and* $(\varphi_{00})_2(0) = 0, (\varphi_{01})_2(0) = 1, (\varphi_{10})_2(0) = 1 = (\varphi_{11})_2(0), (\varphi_{01})_2(1) = 0.$ *It is readily checked that these are isometric embeddings and satisfy (QMT1) and (QMT2). Therefore,* (X, d) *is a* T_0*-quasi-metric tree.*

The following proposition improves [9, Lemma 2].

Proposition 1 *Let* (X, d) *be a quasi-metric tree. If* $z \in \langle x, y\rangle_d$, *then*

$$\langle x, z\rangle_d \cup \langle z, y\rangle_d = \langle x, y\rangle_d.$$

Moreover,

$$\langle x, z\rangle_d \cap \langle z, y\rangle_d = \{z\}$$

whenever $x, y, z \in X$.

Proof Suppose that $z \in \langle x, y\rangle_d$. Let $a \in \langle x, z\rangle_d \cup \langle z, y\rangle_d$. We are going to prove that $a \in \langle x, y\rangle_d$. We have two cases.

Case 1. If $a \in \langle x, z\rangle_d$, we have that $d(x, z) = d(x, a) + d(a, z)$. Thus

$$d(x, y) \leq d(x, a) + d(a, y) \leq d(x, a) + d(a, z) + d(z, y)$$

$$\leq d(x, z) - d(a, z) + d(a, z) + d(z, y).$$

Furthermore, $d(x, y) \leq d(x, a) + d(a, y) \leq d(x, z) + d(z, y) = d(x, y)$, since $z \in \langle x, z\rangle_d$. Thus $d(x, y) = d(x, a) + d(a, y)$. Hence $a \in \langle x, y\rangle_d$.

Case 2. If $a \in \langle z, y\rangle_d$, then $d(z, y) = d(z, a) + d(a, y)$. We have that

$$d(x, y) \leq d(x, a) + d(a, y) \leq d(x, z) + d(z, a) + d(a, y)$$

$$\leq d(x, z) + d(z, y) - d(a, y) + d(a, y).$$

Moreover, we have $d(x, y) \leq d(x, a) + d(a, y) \leq d(x, z) + d(z, y) = d(x, y)$. Thus $a \in \langle x, y\rangle_d$.

We now prove that if $w \in \langle x, y\rangle_d$, then $w \in \langle x, z\rangle_d \cup \langle z, y\rangle_d$. It is sufficient to prove that $w \in \langle x, z\rangle_d$. We have that $d(x, w) + d(w, y) = d(x, y)$ since $w \in \langle x, y\rangle_d$; hence $0 \leq d(x, w) \leq d(x, y)$.

Since (X, d) is a quasi-metric tree, let $\varphi = ((\varphi_{xz})_1, (\varphi_{xz})_2)$ be the unique function pair whose existence is guaranteed by the definition of a quasi-metric tree.

From [9, Proposition 4], we have that $(\varphi_{xz})_1([0, d(x, w)]) \subseteq \langle x, z\rangle_d$. We have that $(\varphi_{xz})_1(d(x, w)) = w \in \langle x, z\rangle_d$ by taking $w = z$. One can prove that $w \in \langle z, y\rangle_d$ analogously by using $(\varphi_{xz})_2$.

The second assertion follows from [9, Lemma 2 (a)]. □

Definition 8 *We say that a quasi-pseudometric* (X, d) *is* median *if it satisfies*

$$\langle x, y\rangle_d \cap \langle y, z\rangle_d \cap \langle z, x\rangle_d \neq \emptyset$$

whenever $x, y, z \in X$.

Proposition 2 *Any quasi-metric tree is median.*

Proof Consider a quasi-metric tree (X,d) and let $x,y,z \in X$. We have to show that

$$\langle x,y\rangle_d \cap \langle y,z\rangle_d \cap \langle z,x\rangle_d \neq \emptyset.$$

Suppose that $y \notin \langle z,x\rangle_d$ and $z \notin \langle x,y\rangle_d$. Then by Lemma 2 there exists $u^z_{x,y} = u$ such that $\langle x,y\rangle_d \cap \langle z,x\rangle_d = \langle x,u\rangle_d$. Consider the function pair $\varphi = (\varphi_1, \varphi_2)$ with $\varphi_i : ([0,1], r^{-1}) \to (X,d)$ $(i=1,2)$ defined by

$$\varphi_1(t) = \begin{cases} (\varphi_{xy})_1(d(x,y) - 2td(u,y)) & if \quad 0 \leq t \leq \frac{1}{2} \\ (\varphi_{zx})_1(d(z,u) + (1-2t)d(z,u)) & if \quad \frac{1}{2} \leq t \leq 1, \end{cases}$$

and

$$\varphi_2(t) = \begin{cases} (\varphi_{xy})_2(d(y,u) - 2td(y,x)) & if \quad 0 \leq t \leq \frac{1}{2} \\ (\varphi_{xz})_2(d(z,u) + (2t-1)d(u,x)) & if \quad \frac{1}{2} \leq t \leq 1. \end{cases}$$

It follows that

$$\varphi_1(0) = (\varphi_{xy})_1(d(x,y)) = y,$$

$$\varphi_1(1) = (\varphi_{zx})_1(d(z,u) - d(z,u)) = (\varphi_{zx})_1(0) = z$$

and

$$\varphi_1\left(\frac{1}{2}\right) = (\varphi_{xy})_1(d(x,y) - d(u,y)) = (\varphi_{xy})_1(d(x,u)) = u$$

or

$$\varphi_1\left(\frac{1}{2}\right) = (\varphi_{zx})_1(d(z,u)) = u.$$

Thus $\varphi_1(\frac{1}{2}) = (\varphi_{zx})_1 d(x,u) = u = (\varphi_{zx})_1(d(z,u))$. Then the function φ_1 is well defined and continuous since

$$(\varphi_{xy})_1(d(x,y) - d(u,y)) = (\varphi_{xy})_1(d(x,u)) = u = (\varphi_{xz})_1(d(x,u)).$$

Similarly, one checks that the function φ_2 is well defined and continuous. Moreover, since $y \neq u \neq z$ and

$$\varphi_1\left(\left[0,\frac{1}{2}\right]\right) \cap \varphi_1\left(\left[\frac{1}{2},1\right]\right) = (\varphi_{xy})_1([d(x,u), d(x,y)]) \cap (\varphi_{xz})_1([d(z,u), d(x,u)])$$

$$\subseteq \langle u,y\rangle_d \cap \langle u,z\rangle_d = \{u\}.$$

Thus φ_1 is injective. Furthermore, we have

$$u = \varphi_1(\frac{1}{2}) \in \langle \varphi_1(0), \varphi_1(1)\rangle_d = \langle y,z\rangle_d.$$

By a similar argument, one uses φ_2 to show that $u = \varphi_2(\frac{1}{2}) \in \langle y,z\rangle_d$. Thus

$$u \in \langle x,y\rangle_d \cap \langle y,z\rangle_d \cap \langle z,x\rangle_d$$

Therefore (X,d) is median. □

4. Connections between a quasi-metric tree and its q-hyperconvex hull

In this section, we examine the relationship between a T_0-quasi-metric tree and its q-hyperconvex hull.

Theorem 1 *If (X,d) is a q-hyperconvex T_0-quasi-metric tree, then its q-hyperconvex hull $(\epsilon_q(X,d),N)$ is a T_0-quasi-metric tree too.*

Proof It is well known that the q-hyperconvex hull of a T_0-quasi-metric space (X,d) is T_0. Suppose that $(\epsilon_q(X,d),N)$ is the q-hyperconvex hull of (X,d). Let $f,g \in \epsilon_q(X,d)$. Then by [7, Corollary 4] there exist $x,y \in X$ such that $f=f_x$ and $g=f_y$.

Since (X,d) satisfies (QMT1), then there exists a unique function pair $\varphi=\varphi_{xy}=((\varphi_{xy})_1,(\varphi_{xy})_2)$ where $(\varphi_{xy})_1$ and $(\varphi_{xy})_2$ are isometric embeddings.

Then we define

$$\psi=\psi_{f_xf_y}=((\psi_{f_xf_y})_1,(\psi_{f_xf_y})_2)$$

by

$$(\psi_{f_xf_y})_1=e_X\circ(\varphi_{xy})_1 \text{ and } (\psi_{f_xf_y})_2=e_X\circ(\varphi_{xy})_2$$

where

$$(\psi_{f_xf_y})_1:([0,D(f_x,f_y)],r^{-1})\to(\epsilon_q(X,d),D)$$

and

$$(\psi_{f_xf_y})_2:([0,D(f_x,f_y)],r)\to(\epsilon_q(X,d),D^{-1}).$$

Then $(\psi_{f_xf_y})_1$ and $(\psi_{f_xf_y})_2$ are isometric embeddings since they are composites of isometric embeddings.

Furthermore,

$$(\psi_{f_xf_y})_1(0)=(e_X\circ(\varphi_{xy})_1)(0)=e_X((\varphi_{xy})_1(0))=e_X(x)=f_x$$

and

$$(\psi_{f_xf_y})_2(N(f_xf_y))=(e_X\circ(\varphi_{xy})_2)(N(f_x,f_y))=e_X((\varphi_{xy})_1(d(x,y)))=e_X(y)=f_y.$$

Thus $(\epsilon_q(X,d),N)$ satisfies (QMT1).

Let $\psi=(\psi_1,\psi_2)$, where ψ_i is an injective continuous function from $[0,1]$ into $\epsilon_q(X,d)\,(i=1,2)$ such that $\psi_i:[0,1]\to\epsilon_q(X,d):t\mapsto f_t$.

Since (X,d) satisfies (QMT2), it suffices to take $\psi_i=e_X\circ\varphi_i$. Then for $f_t\in\epsilon_q(X,d)$, there exists $x_t\in X$ such that $f_t=f_{x_t}$. Then

$$N(f_0,f_t)+N(f_t,f_1)=N(f_{x_0},f_{x_t})+N(f_{x_t},f_{x_1})=d(x_0,x_t)+d(X_t,x_1)=d(x_0,x_1)$$

$$=N(f_{x_0},f_{x_1})=N(f_0,f_1).$$

Hence $(\epsilon_q(X,d),N)$ satisfies (QMT2). Therefore $(\epsilon_q(X,d),N)$ is a T_0-quasi-metric tree. □

In [1], Agyingi et al. proved that if a quasi-metric (X,d) is joincompact, that is, the topology $\tau(d^s)$ is compact, then each endpoint of (X,d) is an endpoint of its q-hyperconvex hull $(\epsilon_q(X,d),N)$. Our next result extends this result and it can be compared to [3, Lemma 3.2].

Corollary 2 *If (X,d) is a quasi-metric tree and $(\epsilon_q(X,d),N)$ its q-hyperconvex hull, then*

$$\langle x,y\rangle_d = \langle x,y\rangle_N,$$

where $\langle x,y\rangle_N = \{h \in \epsilon_q(X,d) : N(x,y) = N(x,h) + N(h,y)\}$.

Example 5 *([1, Example 4]) We equipped the set $X = \{0,1\}$ with the T_0-quasi-metric q defined by $q(0,1) = \alpha$ and $q(1,0) = \beta$, where $\alpha,\beta \in [0,\infty)$ such that $\alpha+\beta \neq 0$. We denote this space by $(X_{\alpha\beta},q)$. It is readily checked that the q-hyperconvex hull $(Q_{\alpha\beta},D_q)$ of $(X_{\alpha\beta},q)$ is identified with $[0,\alpha]\times[0,\beta]$, where*

$$D_q((x_1,y_1),(x_2,y_2)) = \max\{x_1 \dot{-} x_2, y_1 \dot{-} y_2\}$$

whenever $(x_1,y_1),(x_2,y_2) \in [0,\alpha]\times[0,\beta]$.

The minimal function pair (f_1,f_2) on $(X_{\alpha\beta},q)$ are obtained such that $(f_1(0),f_1(1)) = (x_1,y_1)$ and $(f_2(0),f_2(1)) = (\beta-y_1,\alpha-x_1)$, where $(x_1,y_1)\in[0,\alpha]\times[0,\beta]$. Thus we identify the points of $(Q_{\alpha\beta},D_q)$ with the point of $[0,\alpha]\times[0,\beta]$.

Via the isometric embedding map $e:(x_{\alpha\beta},q)\to(Q_{\alpha\beta},D_q)$ we identity the point 0 of $(X_{\alpha\beta},q)$ with $(f_0)_1 = (0,\beta)$ on $X_{\alpha\beta}$ and the point 1 of $(X_{\alpha\beta},q)$ with $(f_1)_1 = (\alpha,0)$ on $X_{\alpha\beta}$.

It is noted that

$$(a,b) \in \langle(\alpha,0),(0,\beta)\rangle_{D_q}$$

and

$$(a,b) \in \langle(0,\beta),(\alpha,0)\rangle_{D_q}$$

whenever $(a,b)\in[0,\alpha]\times[0,\beta]$.

Moreover, it is readily checked that if $a,b>0$, then

$$(0,0) \in \langle(x_1,y_1),(0,y)\rangle_{D_q} \text{ if } x_1 = y$$

and

$$(0,0) \in \langle(x_1,y_1),(y,0)\rangle_{D_q} \text{ if } y_1 = y.$$

Furthermore,

$$\langle(0,\beta),(\alpha,0)\rangle_q = \langle(0,\beta),(\alpha,0)\rangle_{D_q}.$$

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