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On a family of saturated numerical semigroups with multiplicity four

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Abstract: In this study, we will give some results on Arf numerical semigroups of multiplicity four generated by $\{4, k, k+1, k+2\}$ where k is an integer not less than 5 and $k \equiv 1 (\text{mod } 4)$.

Key words: Numerical semigroups, saturated numerical semigroups, Arf numerical semigroups, Frobenius number, genus, Apery set

1. Introduction

A subset S of $\mathbb{N}$ is called a numerical semigroup if it is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is finite, where $\mathbb{N}$ denotes the set of nonnegative integers [1].

Let $A = \{u_1, u_2, ..., u_p\}$ be a subset of $\mathbb{N}$, and assume that $u_1 < u_2 < ... < u_p$. We denote $\langle A \rangle = \{\sum_{i=1}^{p} a_i u_i : a_1, ..., a_p \in \mathbb{N}\}$. Clearly $\langle A \rangle$ is a submonoid of $\mathbb{N}$, which is generated by A. The monoid $\langle A \rangle$ is a numerical semigroup if and only if the greatest common divisor $\gcd(A)$ of elements of A is 1. If S is a numerical semigroup and A is a subset of $\mathbb{N}$ such that $S = \langle A \rangle$, then we say that A is a system of generators of S. If, in addition, no proper subset of A is a system of generators of S, then we say A is a minimal system of generators of S. If $A = \{u_1 < u_2 < ... < u_p\}$ is a minimal system of generators of S, then u_1 is called the multiplicity of S, denoted by $m(S)$, and p is called the embedding dimension of S, denoted by $e(S)$. It is known that $m(S) = \min(S \setminus \{0\})$ and $e(S) \leq m(S)$. We say that S has maximal embedding dimension if $e(S) = m(S)$ [7].

The Frobenius number of S is the largest integer that does not belong to S and it is denoted by $F(S)$. We define $n(S) = \sharp(\{0, 1, ..., F(S)\} \cap S)$, where $\sharp(A)$ denotes the cardinality of the set A [7]. It is customary to write $S = \langle u_1, u_2, ..., u_p \rangle = \{0, s_1, s_2, s_3, ..., s_n, F(S)+1, \rightarrow ...\}$, where "$\rightarrow$" means that every integer greater than $F(S)+1$ belongs to S, $n = n(S)$ and $s_i < s_{i+1}$ for $i = 1, 2, ...n$. We say that an integer x is a pseudo-Frobenius number if $x \in \mathbb{Z} \setminus S$ and $x + s \in S$ for $s \in S \setminus \{0\}$. We will denote by $PF(S)$ the set of pseudo-Frobenius numbers of S. There is an order relation on S defined by $a \leq_S b$ if $b - a \in S$ [6].

The elements of $\mathbb{N} \setminus S$ are called gaps of S. The set of gaps of S is denoted by $H(S)$. Its cardinality is called the genus of S, denoted by $g(S)$. A gap x of a numerical semigroup S is said to be fundamental if $\{2x, 3x\} \subset S$. The set of all fundamental gaps of S is denoted by $FH(S)$. We have $g(S) + n(S) = F(S) + 1$ [7].

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Let S be a numerical semigroup and $m \in S \diagdown \{0\}$. The Apery set of the element m in S is $Ap(S,m) = \{s \in S : s - m \notin S\}$ (see, for instance, [3]). It is known that $PF(S) = \{w - m : w \in Maximals \leq_S Ap(S,m)\}$. Thus, $F(S) = Max(Ap(S,m)) - m$.

A numerical semigroup S is an Arf numerical semigroup if the element $x + y - z$ belongs to S for every $x, y, z \in S$ such that $x \geq y \geq z$ [4]. Barucci and his colleagues proved that an Arf numerical semigroup has maximal embedding dimension (see, for instance, [1]).

For any numerical semigroup S, define the sets $S_i = \{x \in S : x \geq s_i\}$ and $S(i) = (S - S_i) = \{x \in \mathbb{N} : x + S_i \subseteq S\}$ for $i \geq 0$. It is obvious that every $S(i)$ is a numerical semigroup and we obtain the following chain:

$$S_n \subset S_{n-1} \subset ... \subset S = S_0 = S(0) \subset S(1) \subset ... \subset S(n-1) \subset S(n) \subset \mathbb{N}.$$

The number $t(S) = \sharp(S(1) \diagdown S)$ is called the type of S. Similarly, we put $t_i = t_i(S) = \sharp(S(i) \diagdown S(i-1))$. Clearly, $t_1(S) = t(S)$, but in the general case $t_i(S) \neq t(S(i))$. It is known that S is an Arf numerical semigroup if $t_i = s_i - s_{i-1} - 1$, for every $1 \leq i \leq n = n(S)$ [2].

The invariants $F(S)$, $g(S)$, $n(S)$ have been studied in the literature, and several results were obtained by imposing some conditions on the elements of S. For example, Rosales (see [5]) provided a method to compute the sets of numerical semigroups with multiplicity four and given Frobenius number and genus. However, there is no closed formula for these invariants, even for numerical semigroups with multiplicity four, and the Arf and saturated condition has not been studied in this setting. In this work, we find formulas for these invariants, for numerical semigroups of the form $S = \langle 4, k, k+1, k+2 \rangle$, with $k \equiv 1 \pmod 4$, $k \geq 5$.

A numerical semigroup S is saturated if the following condition holds: $s, s_1, s_2, ..., s_k \in S$ are such that $s_i \leq s$ for all $1 \leq i \leq k$ and $c_1, c_2, ..., c_k \in \mathbb{Z}$ are such that $c_1 s_1 + c_2 s_2 + ... + c_k s_k \geq 0$, and then $s + c_1 s_1 + c_2 s_2 + ... + c_k s_k \in S$. It is known that every saturated numerical semigroup is Arf (see [7], Lemma 3.31). However, an Arf numerical semigroup need not be saturated. For example, the numerical semigroup $S = \langle 5, 8, 11, 12, 14 \rangle$ is Arf, but it is not saturated because, for $5, 8 \in S$, $9 = 8 + 2 \cdot 8 - 3 \cdot 5 \notin S$.

In this paper, we consider numerical semigroups of the form $S = \langle 4, k, k+1, k+2 \rangle$ where $k \in \mathbb{N}$, $k \geq 5$ and $k \equiv 1 (\mathrm{mod}\ 4)$. We prove that all these semigroups are Arf semigroups and we calculate the Frobenius number, the genus, and the set of gaps of each of these numerical semigroups.

2. Main results

In this section, we will give some results for the numerical semigroup $S = \langle 4, k, k+1, k+2 \rangle$, where k is an integer with $k \equiv 1 (\mathrm{mod}\ 4)$ and $k \geq 5$.

Proposition 2.1 *Let S be a numerical semigroup minimally generated by $\{u_1 < ... < u_p\}$. Then S has maximal embedding dimension if and only if $Ap(S, u_1) = \{0, u_2, ..., u_p\}$ [7].*

Theorem 2.1 *Let k be an integer such that $k \equiv 1 (mod\ 4)$ and $k \geq 5$ and let $S = \langle 4, k, k+1, k+2 \rangle$. Then we have:*

a) $F(S) = k - 2$.

b) $PF(S) = \{k-4, k-3, k-2\}$.

c) $H(S) = \{1, 2, 3, 5, 6, 7, ..., k-4, k-3, k-2\}$.

d) $n(S) = \frac{k-1}{4}$.

e) $g(S) = \frac{3(k-1)}{4}$.

Proof Notice that $\gcd\{4, k, k+1, k+2\} = 1$. Hence, S is a numerical semigroup and $Ap(S,4) = \{0, k, k+1, k+2\}$.

a) $F(S) = \max(Ap(S,4)) - 4 = (k+2) - 4 = k-2$.

b) Obviously, $\max_{\leq} Ap(S,4) = \{k, k+1, k+2\}$, so $PF(S) = \{k-4, k-3, k-2\}$.

c) If we put $k = 4r+1$ for $r \geq 1$ and $r \in \mathbb{Z}$, then $S = \langle 4, k, k+1, k+2 \rangle = \langle 4, 4r+1, 4r+2, 4r+3 \rangle = \{0, 4, 8, 12, ..., k-5, k-1, \rightarrow ...\} = \{0, 4, 8, 12, ..., 4r-4, 4r, \rightarrow ...\}$. Thus, we find that

$$H(S) = \{1, 2, 3, 5, 6, 7, ..., 4r-3, 4r-2, 4r-1\} = \{1, 2, 3, 5, 6, 7, ..., k-4, k-3, k-2\}.$$

d) From the definition of $n(S)$, we obtain $n(S) = \frac{k-5-0}{4} + 1 = \frac{k-1}{4}$.

e) We obtain that $g(S) + \frac{k-1}{4} = k-1 \Rightarrow g(S) = \frac{3(k-1)}{4}$ by the equality $g(S) + n(S) = F(S) + 1$. □

Proposition 2.2 *Let S be a numerical semigroup. The following conditions are equivalent:*

1) S is a saturated numerical semigroup.

2) $s + d_S(s) \in S$, for all $s \in S$, $s > 0$, where $d_S(s) = \gcd\{x \in S : x \leq s\}$.

3) $s + kd_S(s) \in S$, for all $s \in S$, $s > 0$ and $k \in \mathbb{N}$ [7].

Theorem 2.2 *Let $S = \langle 4, k, k+1, k+2 \rangle$ be a numerical semigroup, where k is an integer with $k \equiv 1(mod\ 4)$ and $k \geq 5$. Then S is a saturated numerical semigroup.*

Proof Let k be an integer with $k \equiv 1 (\text{mod } 4)$ and $k \geq 5$. If we put $k = 4r+1$, $r \geq 1$ and $r \in \mathbb{Z}$, then we write that $S = \langle 4, k, k+1, k+2 \rangle = \{0, 4, 8, ..., k-5, k-1, \rightarrow ...\} = \{0, 4, 8, ..., 4r-4, 4r, \rightarrow ...\}$

a) If $s > 4r$, then $d_S(s) = 1$ and we find that $s + d_S(s) = s+1 > 4r+1 \Rightarrow s + d_S(s) \in S$, for all $s \in S$, $s > 0$.

b) If $s \leq 4r$, then $d_S(s) = 4$ and we obtain that $s + d_S(s) = s + 4 \in S$, for all $s \in S$, $s > 0$. Thus, we obtain that S is a saturated numerical semigroup by Proposition 2.2. □

Corollary 2.1 *Let $S = \langle 4, k, k+1, k+2 \rangle$ where k is an integer with $k \equiv 1(mod\ 4)$ and $k \geq 5$. Then S is an Arf numerical semigroup.*

Proof Let k be an integer with $k \equiv 1(\text{mod } 4)$ and $k \geq 5$. Then

$$S = \langle 4, k, k+1, k+2 \rangle = \{0, 4, 8, 12, ..., k-5, k-1, \rightarrow ...\} = \left\{0 = s_0, s_1, ..., s_{n=n(S)}, \rightarrow ...\right\}.$$

If we show that S is satisfying property $t_i = s_i - s_{i-1} - 1$, for every $1 \leq i \leq n = n(S)$, then S is an Arf numerical semigroup. Clearly, $s_i = 4i$ for $0 \leq i \leq n = n(S)$. Thus, $s_i - s_{i-1} - 1 = 4i - 4(i-1) - 1 = 3$.

Moreover, for $0 \leq i \leq\ n = n(S)$, we write that

$$\begin{aligned} S_1 &= \{x \in S : x \geq 4\} = \{4, 8, ..., k-1, \rightarrow ...\} = \{4, 8, ..., 4r, \rightarrow ...\} \\ &\vdots \\ S_i &= \{4i, 4i+4, ..., k-1, \rightarrow ...\} = \{4i, 4i+4, ..., 4r, \rightarrow ...\} \\ &\vdots \\ S_n &= \{k-1, \rightarrow ...\} = \{4r, \rightarrow ...\} \end{aligned}$$

and

$$\begin{aligned} S(1) &= S - S_1 = \{0, 4, 8, ..., k-9, k-5 \rightarrow ...\} \\ &\vdots \\ S(i) &= \{0, 4, 8, ..., k-1-4i, \rightarrow ...\} \\ &\vdots \\ S(n-1) &= \{0, 4, \rightarrow ...\} \\ S(n) &= \mathbb{N}. \end{aligned}$$

Hence, $t_i = t_i(S) = \sharp(S(i) \diagdown S(i-1)) = \sharp(\{x : 4(n-i) < x < 4(n-i-1), x \in \mathbb{N}\}) = 3$. □

Theorem 2.3 *Consider the numerical semigroups* $S = \langle 4, k, k+1, k+2 \rangle$*, where* k *is an integer with* $k \equiv 1(mod\ 4)$ *and* $k \geq 9$*. Then* $PF(S) \subseteq FH(S)$*.*

Proof $PF(S) = \{k-4, k-3, k-2\}$ where $k-2 = F(S)$ by Theorem 2.1. By our assumption $k \geq 9$ and we have $k - 2i \geq 9 - 8 = 1$ for each $i = 2, 3, 4$. Thus,

$$2(k-i) = 2k - 2i = k + (k-2i) \geqslant k+1 > F(S),$$

$$3(k-i) > 2(k-i) > F(S)$$

for each $i = 2, 3, 4$. This proves that

$$\{2(k-i), 3(k-i)\} \subseteq S$$

for each $i = 2, 3, 4$. In other words, each of $k-2$, $k-3$, and $k-4$ is a fundamental gap of S. This completes the proof. □

Proposition 2.3 *Let* $S = \langle 4, k, k+1, k+2 \rangle = \{0 = s_0, 4 = s_1, 8 = s_2, ..., 4(n-1) = s_{n-1}, 4n = s_{n=n(S)} \rightarrow ...\}$*, where* k *is an integer,* $k \equiv 1(mod\ 4)$ *and* $k \geq 5$*. Then we have:*

(i) $s_n - s_{n-1} = 4$,

(ii) $s_{n-1} = 4(n(S) - 1)$,

(iii) $t_i = 3$ for all $i = 1, 2, ..., n = n(S)$,

(iv) $g(S) = \sum_{i=1}^{n(S)} t_i$.

Proof It is clear that:

(i) $s_n - s_{n-1} = 4n - 4(n-1) = 4$.

(ii) $s_{n-1} = k - 5 = (k-1) - 4 = \frac{4((k-1)-4)}{4} = 4\left(\frac{k-1}{4} - 1\right) = 4(n(S) - 1)$.

(iii) $t_i = s_i - s_{i-1} - 1 = 4i - 4(i-1) - 1 = 3$ for every $1 \le i \le n(S) = n$.

(iv) $g(S) = \frac{3(k-1)}{4} = 3n(S) = \sum_{i=1}^{n(S)} 3 = \sum_{i=1}^{n(S)} t_i$. □

Example 2.1 *Take* $k = 25$*. Then* $S = \langle 4, 25, 26, 27 \rangle = \{0, 4, 8, 12, 16, 20, 24, \to ...\}$ *is a numerical semigroup with the Arf property. Thus,*

$$F(S) = k - 2 = 25 - 2 = 23$$

and

$$n(S) = \frac{k-1}{4} = \frac{25-1}{4} = 6.$$

Also

$$H(S) = \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23\},$$

$$PF(S) = \{21, 22, 23\},\ t(S) = 3 \text{ and } g(S) = \frac{3(k-1)}{4} = \frac{3(25-1)}{4} = 18,$$

$$FH(S) = \{10, 13, 14, 15, 17, 18, 19, 21, 22, 23\} \text{ and } PF(S) \subseteq FH(S),$$

$$s_{n-1} = 4(n(S) - 1) = 4 \cdot (6-1) = 20,\ g(S) = \sum_{i=1}^{n(S)=6} (t_i = 3) = 3 + 3 + 3 + 3 + 3 + 3 = 18.$$

Conclusion 2.1 *In the numerical semigroup theory, it is well known that if* $S = \langle u_1, u_2 \rangle$*, then* $F(S) = u_1 \cdot u_2 - u_1 - u_2$*,* $n(S) = \frac{(u_1-1)(u_2-1)}{2} = g(S)$*, and* $t(S) = 1$ *[1]. However, it is very difficult to compute* $n(S)$*, Frobenius number, genus, and type of* S *when* $m(S) = 4$ *and it is hard to decide whether* S *is an Arf numerical semigroup. There are not any formulas for these. However, sometimes these invariants can be calculated if the elements of* S *satisfy some conditions. For example, Rosales gave a method to compute the set of numerical semigroups with multiplicity four and given Frobenius number and genus [5], but no formulas are given to compute invariants of these numerical semigroups, and there is no comment for their being an Arf numerical semigroup.*

However, in this paper, we can compute these invariants rather quickly for the numerical semigroup $S = \langle 4, k, k+1, k+2 \rangle$ *where* k *is an integer with* $k \equiv 1 (mod\ 4)$ *and* $k \ge 5$.

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