Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 138 – 157
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doi:10.3906/mat-1507-61

$\mathcal{W}$-Gorenstein objects in triangulated categories

Chaoling HUANG[1,2,*], **Kaituo LIU**[3]
[1]College of Mathematics and Statistics, Hubei Normal University, Huangshi, P.R. China
[2]School of Science, Hubei University of Automotive Technology, Shiyan, P.R. China
[3]School of Mathematics and Statistics, Central South University, Changsha, P.R. China

Received: 17.07.2015 • **Accepted/Published Online:** 23.03.2016 • **Final Version:** 16.01.2017

Abstract: We fix a proper class of triangles ξ in a triangulated category $\mathcal{C}$. Let $\mathcal{W}$ be a class of objects in $\mathcal{C}$ such that $\xi xt^i_\xi(W,\ W')=0$ for all $W,W'\in\mathcal{W}$ and all $i\geq 1$. In this paper, we introduce the notion of $\mathcal{W}$-Gorenstein objects and $\mathcal{G}(\mathcal{W})$-(co)resolution dimensions of any object in $\mathcal{C}$ and study the properties of $\mathcal{W}$-Gorenstein objects and characterize the finite $\mathcal{G}(\mathcal{W})$-(co)resolution dimensions of any object. Some applications are given.

Key words: Triangulated category, proper class of triangles, $\mathcal{W}$-Gorenstein object

1. Introduction

Triangulated categories were first introduced by Grothendieck and Verdier [20] in the 1960s for doing homological algebra in abelian categories. From then on, they have been useful in algebraic geometry and homological algebra. For this, one can reference [3, 11, 17]. Beligiannis in [4] first developed a homological algebra in a triangulated category. Let ξ be a proper class of triangles in a triangulated category $\mathcal{C}$. He introduced ξ-projective objects, ξ-projective resolution, ξ-projective dimension, and their duals. Asadollahi and Salarian in [1] introduced and studied ξ-Gorenstein projective objects in triangulated categories. Using the class $\mathcal{G}(\mathcal{P})$ of the full subcategory of ξ-Gorenstein projective objects of $\mathcal{C}$, they related an invariant called ξ-Gorenstein projective dimension to any object A of $\mathcal{C}$ and then investigated some properties on the ξ-Gorenstein projective dimension. Motivated by the classical structure of Tate cohomology, Asadollahi and Salarian in [2] developed and studied a Tate cohomology theory in a triangulated category $\mathcal{C}$. Based on Asadollahi and Salarian's work, the authors in [18] further studied Gorenstein homological dimensions for triangulated categories. More importantly, they proved the equality $sup\{\xi\text{-}\mathcal{G}pdM|for\ any\ M\in\mathcal{C}\}=sup\{\xi\text{-}\mathcal{G}idM|for\ any\ M\in\mathcal{C}\}$.

It is well known that the idea of relative homological algebra was introduced by Eilenberg and Moore [9], and was reinvigorated by Enochs, Jenda, and Torrecillas [6–8]. An R-module M is said to be Gorenstein projective (for short G-projective; see [6]) if there is an exact complex

$$\mathbf{P}=\cdots\to P_1\to P_0\to P^0\to P^1\to\cdots$$

of projective modules with $M=Ker(P^0\to P^1)$ such that $Hom(\mathbf{P},\ Q)$ is exact for each projective R-module Q. To date, many authors have studied the related subjects; see [5, 10, 12–15, 19, 21, 22]. Let $\mathcal{W}$

*Correspondence: clhuang1978@live.cn
2010 *AMS Mathematics Subject Classification:* 18G25, 18G30, 55N20.

be a self-orthogonal class of left R-modules. Geng and Ding in [10] introduced the notion of $\mathcal{W}$-Gorenstein modules, which is a common generalization of some modules such as Gorenstein projective (injective) and V-Gorenstein projective (injective) modules. Let $\mathcal{A}$ be an abelian category. In [23], the author introduced the so-called resolving subcategory $\mathcal{X}$ of $\mathcal{A}$ and researched the $\mathcal{X}$-resolution dimensions and special $\mathcal{X}$-precovers for resolving subcategory $\mathcal{X}$ of $\mathcal{A}$.

Motivated by the above-mentioned, our aim in this paper is to contribute in developing the relative homological algebra in triangulated categories. Precisely speaking, for a fixed proper class of triangles ξ and a fixed class of objects $\mathcal{W}$ such that $\xi xt^i_\xi(W,\ W')=0$ for all $W,W'\in\mathcal{W}$ and all $i\geq 1$ in a triangulated category $\mathcal{C}$, we introduce the notion of $\mathcal{W}$-Gorenstein objects and $\mathcal{G}(\mathcal{W})$-(co)resolution dimensions of any object in the category $\mathcal{C}$, where the symbol $\mathcal{G}(\mathcal{W})$ denotes the full subcategory of $\mathcal{W}$-Gorenstein objects in $\mathcal{C}$. The paper is organized as follows. In the second section, we recall some definitions and collect some fundamental results about triangulated categories that will be used throughout the paper. In Section 3, using the notion of completely $\mathcal{W}$-exact complexes, we introduce the notion of $\mathcal{W}$-Gorenstein objects. More precisely, let $X^{\cdot}$ be a completely $\mathcal{W}$-exact complex. For any integer n, there exists a $\mathcal{C}(\mathcal{W},\ -)$-exact and $\mathcal{C}(-,\ \mathcal{W})$-exact triangle $K_{n+1}\stackrel{g_{n+1}}{\longrightarrow}X_{n+1}\stackrel{f_{n+1}}{\longrightarrow}K_n\longrightarrow\Sigma K_{n+1}$ in ξ. The object K_n for any integer n is called a $\mathcal{W}$-Gorenstein object. We also introduce the notion of $\mathcal{G}(\mathcal{W})$-(co)resolution dimensions of any object in $\mathcal{C}$ and then consider their properties. In Section 4, we use the properties developed in the earlier section to characterize the finite $\mathcal{G}(\mathcal{W})$-(co)resolution dimensions of any object in the triangulated category $\mathcal{C}$. In the last section, we give some applications.

2. Preliminaries

In this section we recall some definitions and elementary properties about triangulated categories that are used throughout the paper. First of all, for the definition of triangulated categories and some basic properties, one can refer to Neeman's book [17]. The following result is crucial in this paper.

Proposition 2.1 *(See [4, 2.1] and [1, Proposition 2.2]). Let $\mathcal{C}$ be an additive category and $\Sigma:\mathcal{C}\to\mathcal{C}$ be an autoequivalent functor and $\Delta\subseteq Diag(\mathcal{C},\ \Sigma)$. Suppose that the triple $(C,\ \Sigma,\ \Delta)$ satisfies all the axioms of a triangulated category except possibly of the octahedral axiom. Then the following are equivalent:*

(1) (Base change). For any triangle $A\stackrel{f}{\to}B\stackrel{g}{\to}C\stackrel{h}{\to}\Sigma A\in\Delta$ and morphism $\epsilon:E\to C$, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & M \arrow[r, equal] \arrow[d] & M \arrow[r] \arrow[d] & 0 \arrow[d] \\
A \arrow[r, "f"] \arrow[d, equal] & G \arrow[r, "g"] \arrow[d] & E \arrow[r, "h"] \arrow[d, "\varepsilon"'] & \Sigma A \arrow[d, equal] \\
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & \Sigma A \arrow[d] \\
0 \arrow[r] & \Sigma M \arrow[r, equal] & \Sigma M \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and the vertical diagrams are triangles in Δ.

(2) (Cobase change). For any triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \Delta$ *and morphism* $\alpha: A \to D$*, there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & N \arrow[r, equal] \arrow[d] & N \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma^{-1}C \arrow[r, "-\Sigma^{-1}h"] \arrow[d, equal] & A \arrow[r, "g"] \arrow[d, "\alpha"'] & B \arrow[r, "h"] \arrow[d] & C \arrow[d, equal] \\
\Sigma^{-1}C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & C \arrow[d] \\
0 \arrow[r] & \Sigma N \arrow[r, equal] & \Sigma N \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and the vertical diagrams are triangles in Δ*.*

(3) (Octahedral axiom). Given two morphisms $f_1 : A \to B$ *and* $f_2 : B \to C$*, there is a commutative diagram:*

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r] \arrow[d, "f_2"'] & X \arrow[r] \arrow[d] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f_2f_1"] \arrow[d, "f_1"'] & C \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & \Sigma A \arrow[d, "\Sigma f_1"'] \\
B \arrow[r, "f_2"] \arrow[d] & C \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma B \arrow[d] \\
0 \arrow[r] & \Sigma X \arrow[r, equal] & \Sigma X \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and the third vertical diagrams are triangles in Δ*.*

A triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is called split if it is isomorphic to the triangle $A \xrightarrow{(0\ 1)} A \bigoplus C \xrightarrow{\binom{1}{0}} C \xrightarrow{0} \Sigma A$. It is easy to see that it is split if and only if f is a section or g is a retraction or $h = 0$. The full subcategory of the split triangles is denoted by Δ_0. A class of triangles ξ in $\mathcal{C}$ is closed under base change if for any triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \xi$ as in above proposition (1), the triangle $A \to G \to E \to \Sigma A$ is in ξ. Dually, one can define the class ξ is closed under cobase change. The class ξ is closed under suspension if for any triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \xi$ and any integer $i \in \mathbb{Z}$, the triangle $\Sigma^i A \xrightarrow{(-1)^i\Sigma^i f} \Sigma^i B \xrightarrow{(-1)^i\Sigma^i g} \Sigma^i C \xrightarrow{(-1)^i\Sigma^i h} \Sigma^{i+1}A$ is in ξ. The class ξ is closed under saturation if in the situation of base change in the above proposition, whenever the third vertical and the second horizontal triangle is in ξ, then the triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is in ξ. Recall that a full subcategory ξ is called a proper class if (1) ξ is closed under isomorphisms, finite coproducts, and $\Delta_0 \subseteq \xi \subseteq \Delta$; (2) ξ is closed under suspensions and is saturated; (3) ξ is closed under base and cobase change.

3. $\mathcal{W}$-Gorenstein objects

In this paper, we fix a proper class of triangles ξ in a triangulated category $\mathcal{C}$. Recall that for any object of C in $\mathcal{C}$ and any integer $n \geq 0$, the ξ-extension functor $\xi xt^n_\xi(-,\ C)$ is defined to be the nth right ξ-derived functor of the functor $\mathcal{C}(-,\ C)$; see [4]. Let $\mathcal{W}$ be a class of objects in a triangulated category $\mathcal{C}$ such that $\xi xt^i_\xi(W,\ W') = 0$ for all $W, W' \in \mathcal{W}$ and all $i \geq 1$. Let $\mathcal{H}$ be a subcategory of $\mathcal{C}$. For an object $M \in \mathcal{C}$, write $\mathcal{H} \perp M$ if $\xi xt^{\geq 1}_\xi(X,\ M) = 0$ for any object $X \in \mathcal{H}$. $\mathcal{H} \perp \mathcal{W}$ denotes that $\xi xt^{\geq 1}_\xi(X,\ W) = 0$ for any object $X \in \mathcal{H}$ and for any object $W \in \mathcal{W}$. We set $\mathcal{H}^\perp = \{M \in \mathcal{C} | \xi xt^{\geq 1}_\xi(X,\ M) = 0\ for\ any\ X \in \mathcal{H}\}$. ${}^\perp\mathcal{H} = \{M \in \mathcal{C} | \xi xt^{\geq 1}_\xi(M,\ X) = 0\ for\ any\ X \in \mathcal{H}\}$. In this section, the notion of $\mathcal{W}$-Gorenstein objects is introduced and studied.

Definition 3.1 *A ξ-exact complex $X^\cdot$ is a diagram*

$$X^\cdot = \cdots \to X_1 \overset{d_1}{\to} X_0 \overset{d_0}{\to} X_{-1} \to \cdots$$

in $\mathcal{C}$, such that there exists a triangle $K_{n+1} \overset{g_{n+1}}{\longrightarrow} X_{n+1} \overset{f_{n+1}}{\longrightarrow} K_n \longrightarrow \Sigma K_{n+1}$ in ξ with $d_{n+1} = g_n f_{n+1}$ for any $n \in \mathbb{Z}$.

Definition 3.2 *A triangle $A \to B \to C \to \Sigma A$ in ξ is called $\mathcal{C}(-,\ \mathcal{W})$-exact if for any $W \in \mathcal{W}$, the induced complex*

$$0 \to \mathcal{C}(C,\ W) \to \mathcal{C}(B,\ W) \to \mathcal{C}(A,\ W) \to 0$$

is exact in the category $\mathcal{A}b$ of abelian groups.

Definition 3.3 *A ξ-exact complex $X^\cdot$ is called $\mathcal{C}(-,\ \mathcal{W})$-exact if for any $n \in \mathbb{Z}$, there exists a $\mathcal{C}(-,\ \mathcal{W})$-exact triangle $K_{n+1} \overset{g_{n+1}}{\to} X_{n+1} \overset{f_{n+1}}{\to} K_n \to \Sigma K_{n+1}$ in ξ.*

The $\mathcal{C}(\mathcal{W},\ -)$-exact triangle and the $\mathcal{C}(\mathcal{W},\ -)$-exact complex can be defined dually.

Definition 3.4 *A ξ-exact complex $X^\cdot$ is called completely $\mathcal{W}$-exact if it is both $\mathcal{C}(\mathcal{W},\ -)$-exact and $\mathcal{C}(-,\ \mathcal{W})$-exact and for each integer n, $X_n \in \mathcal{W}$.*

Remark 3.5 *Let $X^\cdot$ be a completely $\mathcal{W}$-exact complex. For any integer n, there exists a $\mathcal{C}(\mathcal{W},\ -)$-exact and $\mathcal{C}(-,\ \mathcal{W})$-exact triangle $K_{n+1} \overset{g_{n+1}}{\longrightarrow} X_{n+1} \overset{f_{n+1}}{\longrightarrow} K_n \longrightarrow \Sigma K_{n+1}$ in ξ. Thus for any $W \in \mathcal{W}$ there are short exact sequences*

$$0 \to \mathcal{C}(K_n,\ W) \to \mathcal{C}(X_{n+1},\ W) \to \mathcal{C}(K_{n+1},\ W) \to 0$$

and

$$0 \to \mathcal{C}(W,\ K_{n+1}) \to \mathcal{C}(W,\ X_{n+1}) \to \mathcal{C}(W,\ K_n) \to 0$$

in $\mathcal{A}b$. We paste them together and can obtain two exact sequences

$$\cdots \to \mathcal{C}(X_{-1},\ W) \to \mathcal{C}(X_0,\ W) \to \mathcal{C}(X_1,\ W) \to \cdots$$

and

$$\cdots \to \mathcal{C}(W,\ X_1) \to \mathcal{C}(W,\ X_0) \to \mathcal{C}(W,\ X_{-1}) \to \cdots$$

in $\mathcal{A}b$.

Definition 3.6 *Let $X^{\cdot}$ be a completely $\mathcal{W}$-exact complex. For any integer n, there exists a $\mathcal{C}(\mathcal{W},\ -)$-exact and $\mathcal{C}(-,\ \mathcal{W})$-exact triangle $K_{n+1} \xrightarrow{g_{n+1}} X_{n+1} \xrightarrow{f_{n+1}} K_n \longrightarrow \Sigma K_{n+1}$ in ξ. The object K_n for any integer n is called a $\mathcal{W}$-Gorenstein object. We use $\mathcal{G}(\mathcal{W})$ to denote the full subcategory of $\mathcal{W}$-Gorenstein objects in $\mathcal{C}$.*

Remark 3.7 *Recall that an object $P \in \mathcal{C}$ (respectively, $I \in \mathcal{C}$) is called ξ-projective (respectively, ξ-injecitve) if for any triangle $A \to B \to C \to \Sigma A$ in ξ, the induced complex $0 \to \mathcal{C}(P,\ A) \to \mathcal{C}(P,\ B) \to \mathcal{C}(P,\ C) \to 0$ (respectively, $0 \to \mathcal{C}(C,\ I) \to \mathcal{C}(B,\ I) \to \mathcal{C}(A,\ I) \to 0$) is exact in $\mathcal{A}b$. The symbol $\mathcal{P}(\xi)$ (respectively, $\mathcal{I}(\xi)$) denotes the full subcategory of ξ-projective (respectively, ξ-injective) objects of $\mathcal{C}$. If we use $\mathcal{P}(\xi)$ (respectively, $\mathcal{I}(\xi)$) to replace $\mathcal{W}$, $\mathcal{W}$-Gorenstein objects are just ξ-$\mathcal{G}$projective (ξ-$\mathcal{G}$injective) objects in [1].*

Definition 3.8 *Let $\mathcal{X}$ be a subcategory of $\mathcal{C}$. For $M \in \mathcal{C}$, an $\mathcal{X}$-resolution of M is a ξ-exact complex*

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$$

where $X_i \in \mathcal{X}$, $i = 1,\ 2,\ \cdots$. If M admits an $\mathcal{X}$-resolution, the $\mathcal{X}$-resolution dimension of M, denoted by $resdim_{\mathcal{X}}(M)$, is defined as the infimum of the set of n such that there exists a ξ-exact complex $0 \to X_n \to \cdots \to X_0 \to M \to 0$ where all X_i are in $\mathcal{X}$, $i = 1,\ 2,\ \cdots n$. If no such n exists, set $resdim_{\mathcal{X}}(M) = \infty$. We use $res\widehat{\mathcal{X}}$ to denote the subcategory of objects in $\mathcal{C}$ with $resdim_{\mathcal{X}}(M) < \infty$. A $\mathcal{X}$-resolution of M is called proper if, for any $H \in \mathcal{X}$, the following complex

$$\cdots \to \mathcal{C}(H,\ X_n) \to \mathcal{C}(H,\ X_{n-1}) \to \cdots \to \mathcal{C}(H,\ X_1) \to \mathcal{C}(H',\ X_0) \to 0$$

is exact. We use $res\widetilde{\mathcal{X}}$ to denote the subcategory of objects of $\mathcal{C}$ admitting a proper $\mathcal{X}$-resolution.

Dually, one can define the (proper) $\mathcal{X}$-coresolution and $\mathcal{X}$-coresolution dimension of M. We use $coresdim_{\mathcal{X}}(M)$ to denote the $\mathcal{X}$-coresolution dimension of M, use $cores\widehat{\mathcal{X}}$ to denote the subcategory of objects in $\mathcal{C}$ with $coresdim_{\mathcal{X}}(M) < \infty$, and use $cores\widetilde{\mathcal{X}}$ to denote the subcategory of objects of $\mathcal{C}$ admitting a proper $\mathcal{X}$-coresolution.

Lemma 3.9 *(1) Let $M' \xrightarrow{f} M \xrightarrow{g} M'' \xrightarrow{h} \Sigma M'$ be a $\mathcal{C}(\mathcal{W},\ -)$-exact triangle in ξ. If M' and M'' are in $res\widetilde{\mathcal{W}}$, then so is M.*

(2) Let $M' \to M \to M'' \to \Sigma M'$ be a $\mathcal{C}(-,\ \mathcal{W})$-exact triangle in ξ. If M' and M'' are in $cores\widetilde{\mathcal{W}}$, then so is M.

Proof We just prove (1). Dually, (2) can be proved. Since M' and M'' are in $res\widetilde{\mathcal{W}}$, there are triangles $K_0' \to W_0' \xrightarrow{\partial_0'} M' \to \Sigma K_0'$ and $K_0'' \to W_0'' \xrightarrow{\partial_0''} M'' \to \Sigma K_0''$ in ξ with $W_0',\ W_0'' \in \mathcal{W}$ and $K_0',\ K_0'' \in res\widetilde{\mathcal{W}}$. Since $M' \xrightarrow{f} M \xrightarrow{g} M'' \xrightarrow{h} \Sigma M'$ is $\mathcal{C}(\mathcal{W},\ -)$-exact, there is a morphism $\eta \in \mathcal{C}(W_0'',\ M)$ such that $g\eta = \partial_0''$. Thus we have the following diagram where the first two squares are commutative.

$$\begin{tikzcd}
W_0' \arrow[r, "\binom{1}{0}"'] \arrow[d, "\partial_0'"'] & W_0' \oplus W_0'' \arrow[r, "(0\ 1)"'] \arrow[d, "(f\partial_0'\ \eta)"'] & W_0'' \arrow[r] \arrow[d, "\partial_0''"'] & \Sigma W_0' \arrow[d, "\Sigma\partial_0'"'] \\
M' \arrow[r, "f"'] & M \arrow[r, "g"'] & M'' \arrow[r, "h"'] & \Sigma M'
\end{tikzcd}$$

Since $(0,\ h\partial_0'') = h\partial_0''(0,\ 1) = hg(f\partial_0'\eta) = 0$, so $h\partial_0'' = 0$, which shows that the third square is also commutative. By (TR2), we have the following diagram in which the two top rows and the two left columns are triangles with the top first square commutative.

$$\begin{array}{ccccccc}
\Sigma^{-1}W_0'' & \longrightarrow & W_0' & \longrightarrow & W_0'\oplus W_0'' & \longrightarrow & W_0'' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}M'' & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \\
\downarrow & & \downarrow & & & & \\
K_0'' & & \Sigma K_0' & & & & \\
\downarrow & & \downarrow & & & & \\
W_0'' & \longrightarrow & \Sigma W_0' & & & &
\end{array}$$

Appying [16, Lemma 2.6], there is an object ΣK_0 and there are arrow maps such that the following diagram is commutative except for its bottom right square, which commutes up to sign -1, and all four rows and columns are triangles.

$$\begin{array}{ccccccc}
\Sigma^{-1}W_0'' & \longrightarrow & W_0' & \longrightarrow & W_0'\oplus W_0'' & \longrightarrow & W_0'' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}M'' & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_0'' & \longrightarrow & \Sigma K_0' & \longrightarrow & \Sigma K_0 & \longrightarrow & \Sigma K_0'' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
W_0'' & \longrightarrow & \Sigma W_0' & \longrightarrow & \Sigma(W_0'\oplus W_0'') & \longrightarrow & \Sigma W_0''
\end{array}$$

Using (TR2), we have the following diagram, in which all the rows and columns are triangles and all squares are commutative except for the second square on the top, which is commutative up to sign -1.

$$\begin{array}{ccccccc}
K_0' & \xrightarrow{a} & K_0 & \xrightarrow{b} & K_0'' & \xrightarrow{c} & \Sigma K_0' \\
{\scriptstyle d}\downarrow & & {\scriptstyle e}\downarrow & & {\scriptstyle o}\downarrow & & {\scriptstyle p}\downarrow \\
W_0' & \xrightarrow{q} & W_0'\oplus W_0'' & \xrightarrow{i} & W_0'' & \xrightarrow{j} & \Sigma W_0' \\
{\scriptstyle k}\downarrow & & {\scriptstyle l}\downarrow & & {\scriptstyle m}\downarrow & & {\scriptstyle n}\downarrow \\
M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \xrightarrow{h} & \Sigma M' \\
{\scriptstyle r}\downarrow & & {\scriptstyle s}\downarrow & & {\scriptstyle t}\downarrow & & {\scriptstyle u}\downarrow \\
\Sigma K_0' & \xrightarrow{v} & \Sigma K_0 & \xrightarrow{w} & \Sigma K_0'' & \xrightarrow{x} & \Sigma^2 K_0'
\end{array}$$

Using sign criteria, we have the following diagram, in which all the rows and columns are triangles and all squares are commutative except for the second square on the bottom, which is commutative up to sign -1.

$$\begin{tikzcd}
K_0' \arrow[r, "a"] \arrow[d, "-d"'] & K_0 \arrow[r, "b"] \arrow[d, "-e"'] & K_0'' \arrow[r, "c"] \arrow[d, "o"'] & \Sigma K_0' \arrow[d, "p"'] \\
W_0' \arrow[r, "q"] \arrow[d, "-k"'] & W_0' \oplus W_0'' \arrow[r, "i"] \arrow[d, "-l"'] & W_0'' \arrow[r, "j"] \arrow[d, "-m"'] & \Sigma W_0' \arrow[d, "-n"'] \\
M' \arrow[r, "f"] \arrow[d, "r"'] & M \arrow[r, "g"] \arrow[d, "s"'] & M'' \arrow[r, "h"] \arrow[d, "-t"'] & \Sigma M' \arrow[d, "-u"'] \\
\Sigma K_0' \arrow[r, "v"] & \Sigma K_0 \arrow[r, "w"] & \Sigma K_0'' \arrow[r, "x"] & \Sigma^2 K_0'
\end{tikzcd}$$

Using $\mathcal{C}(W,\ -)$, we have the following commutative diagram except for the second square on the top, in which all rows and columns are exact

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & \mathcal{C}(W,\ K_0') \arrow[r, "\kappa"] \arrow[d, "\mu"'] & \mathcal{C}(W,\ K_0) \arrow[r, "\beta"] \arrow[d, "\nu"'] & \mathcal{C}(W,\ K_0'') \arrow[d] & \\
0 \arrow[r] & \mathcal{C}(W,\ W_0') \arrow[r, "\lambda"] \arrow[d] & \mathcal{C}(W,\ W_0' \oplus W_0'') \arrow[r] \arrow[d, "\alpha"'] & \mathcal{C}(W,\ W_0'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{C}(W,\ M') \arrow[r] \arrow[d] & \mathcal{C}(W,\ M) \arrow[r] & \mathcal{C}(W,\ M'') \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

By snake lemma, α is epic. Since $\mathcal{C}(W,\ -)$ is a cohomological functor, $\mathcal{C}(W,\ \Sigma K_0) \to \mathcal{C}(W,\ \Sigma W_0' \oplus W_0'')$ is monic. Replacing W with ΣW, we have that $\mathcal{C}(\Sigma W,\ \Sigma K_0) \to \mathcal{C}(\Sigma W,\ \Sigma w_0' \oplus w_0'')$ is monic. Therefore, $\mathcal{C}(W,\ K_0) \to \mathcal{C}(W,\ W_0' \oplus W_0'')$ is monic. Since λ, μ, and ν are monic, κ is monic. Using the dual method above, one can prove that β is epic. Continuing the above procedure, we have $M \in res\widetilde{\mathcal{W}}$. □

In the rest of this paper, let $\mathcal{W}$ be a class of objects in a triangulated category $\mathcal{C}$ such that $\xi xt^i_\xi(W,\ W') = 0$, $\xi xt^0_\xi(-,\ W) \cong C(-\ ,W)$, and $\xi xt^0_\xi(W,\ -) \cong C(W\ ,-)$ for all $W, W' \in \mathcal{W}$ and all $i \geq 1$.

Lemma 3.10 *Let $A \to B \to C \to \Sigma A$ be a triangle in ξ.*

(1) If $C \perp \mathcal{W}$, then $A \perp \mathcal{W}$ if and only if $B \perp \mathcal{W}$. If $A \perp \mathcal{W}$ and $B \perp \mathcal{W}$, then $C \perp \mathcal{W}$ if and only if $A \to B \to C \to \Sigma A$ is a $\mathcal{C}(-,\ \mathcal{W})$-exact triangle.

(2) If $\mathcal{W} \perp A$, then $\mathcal{W} \perp B$ if and only if $\mathcal{W} \perp C$. If $\mathcal{W} \perp B$ and $\mathcal{W} \perp C$, then $\mathcal{W} \perp A$ if and only if $A \to B \to C \to \Sigma A$ is a $\mathcal{C}(\mathcal{W},\ -)$-exact triangle.

Proof (1) It is a consequence of [1, Proposition 3.8]. Dually, one can prove (2). □

The following proposition provides a criterion for a given object of C to be $\mathcal{W}$-Gorenstein.

Proposition 3.11 *An object M in $\mathcal{C}$ is a $\mathcal{W}$-Gorenstein object if and only if $M \in {}^{\perp}\mathcal{W} \bigcap \mathcal{W}^{\perp}$ and M has a proper $\mathcal{W}$-resolution and a proper $\mathcal{W}$-coresolution.*

Proof $\Rightarrow$: Since M is a $\mathcal{W}$-Gorenstein object, there is a completely $\mathcal{W}$-exact complex

$$X^{\cdot} = \cdots \to X_1 \overset{d_1}{\to} X_0 \overset{d_0}{\to} X_{-1} \to \cdots$$

in $\mathcal{C}$, such that for each integer n, $X_n \in \mathcal{W}$ and that there exists a both $\mathcal{C}(\mathcal{W},\ -)$-exact and $\mathcal{C}(-,\ \mathcal{W})$-exact triangle $K_{n+1} \overset{g_{n+1}}{\longrightarrow} X_{n+1} \overset{f_{n+1}}{\longrightarrow} K_n \longrightarrow \Sigma K_{n+1}$ in ξ with $d_{n+1} = g_n f_{n+1}$ for any $n \in \mathbb{Z}$ and $M = K_{-1}$. Thus M has a proper $\mathcal{W}$-resolution and a proper $\mathcal{W}$-coresolution. Moreover, by [1, Proposition 3.8], M belongs to ${}^{\perp}\mathcal{W} \bigcap \mathcal{W}^{\perp}$.

$\Leftarrow$: Let $\cdots \to X_1 \to X_0 \to M \to 0$ and $0 \to M \to X_{-1} \to X_{-2} \to \cdots$ be a proper $\mathcal{W}$-resolution and a proper $\mathcal{W}$-coresolution of M, respectively. Pasting them together, by [1, Proposition 3.8], one can check it is both $\mathcal{C}(\mathcal{W},\ -)$-exact and $\mathcal{C}(-,\ \mathcal{W})$-exact, since $M \in {}^{\perp}\mathcal{W} \bigcap \mathcal{W}^{\perp}$. □

Recall that a class $\mathcal{H}$ in abelian categories is closed under extensions if for every short exact sequence $0 \to H_1 \to H_2 \to H_3 \to 0$ with $H_1 \in \mathcal{H}$ and $H_3 \in \mathcal{H}$, then $H_2 \in \mathcal{H}$. In a triangulated category $\mathcal{C}$, we give a similar definition.

Definition 3.12 *Let $\mathcal{H}$ be a class of objects. It is said to be closed under extensions if for any triangle $A \to B \to C \to \Sigma A$ in ξ, A and C are in $\mathcal{H}$, then so is B.*

Lemma 3.13 *$\mathcal{G}(\mathcal{W})$ is closed under extensions.*

Proof Let $A \to B \to C \to \Sigma A$ be a triangle in ξ with $A \in \mathcal{G}(\mathcal{W})$ and $C \in \mathcal{G}(\mathcal{W})$. By Proposition 3.11, $A,\ C \in {}^{\perp}\mathcal{W} \bigcap \mathcal{W}^{\perp}$, and then $A \to B \to C \to \Sigma A$ is both $\mathcal{C}(\mathcal{W},\ -)$-exact and $\mathcal{C}(-,\ \mathcal{W})$-exact following Lemma 3.10. By Lemma 3.9, B has a proper $\mathcal{W}$-resolution and a proper $\mathcal{W}$-coresolution. Since A and C are in ${}^{\perp}\mathcal{W} \bigcap \mathcal{W}^{\perp}$, by Lemma 3.10, so is B. By Proposition 3.11, B is included in $\mathcal{G}(\mathcal{W})$. □

Proposition 3.14 *(1) If $M \in res\widetilde{\mathcal{G}(\mathcal{W})}$, then $M \in res\widetilde{\mathcal{W}}$.*

(2) If $M \in cores\widetilde{\mathcal{G}(\mathcal{W})}$, then $M \in cores\widetilde{\mathcal{W}}$.

Proof We prove part (1); the proof of part (2) is dual. Since $M \in res\widetilde{\mathcal{G}(\mathcal{W})}$, there is a $\mathcal{C}(\mathcal{G}(\mathcal{W}),\ -)$-exact triangle $N \to G_0 \to M \to \Sigma N$ in ξ with $G_0 \in \mathcal{G}(\mathcal{W})$ and $N \in res\widetilde{\mathcal{G}(\mathcal{W})}$. Thus there is a $\mathcal{C}(\mathcal{W},\ -)$-exact triangle $G_0' \to W_0 \to G_0 \to \Sigma G_0'$ in ξ with $G_0' \in \mathcal{G}(\mathcal{W})$ and $W_0 \in \mathcal{W}$. For the triangle $\Sigma^{-1}M \to N \to G_0 \to N$ and the morphism $W_0 \overset{\varepsilon}{\to} G_0$, by [1, Proposition 2.2 (a)], there exists a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & G_0' & = & G_0' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}M & \longrightarrow & H & \longrightarrow & W_0 & \longrightarrow & M \\
\| & & \downarrow & & \varepsilon\downarrow & & \| \\
\Sigma^{-1}M & \longrightarrow & N & \longrightarrow & G_0 & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma G_0' & = & \Sigma G_0' & \longrightarrow & 0
\end{array}$$

in which all horizontal and vertical diagrams are triangles in ξ. Since $N \to G_0 \to M \to \Sigma N$ is $\mathcal{C}(\mathcal{W},\ -)$-exact, by [1, Proposition 3.8], $\xi xt^1(\mathcal{W},\ N) = 0$. On the other hand, $G_0' \in \mathcal{W}^{\perp}$, so $\xi xt^1(\mathcal{W},\ H) = 0$. Therefore, the triangle $H \to W_0 \to M \to \Sigma H$ is $\mathcal{C}(\mathcal{W},\ -)$-exact. Since $N \in res\widetilde{\mathcal{G}(\mathcal{W})}$, there is a $\mathcal{C}(\mathcal{G}(\mathcal{W}),\ -)$-exact triangle $K \to G_1 \to N \to \Sigma K$ in ξ with $G_1 \in \mathcal{G}(\mathcal{W})$, $K \in res\widetilde{\mathcal{G}(\mathcal{W})}$, and $\xi xt^1_\xi(\mathcal{W},\ K) = 0$. There exists a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K & = & K & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G_0' & \longrightarrow & L & \longrightarrow & G_1 & \longrightarrow & \Sigma G_0' \\
\| & & \downarrow & & \varepsilon\downarrow & & \| \\
G_0' & \longrightarrow & H & \longrightarrow & N & \longrightarrow & G_0' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma K & = & \Sigma K & \longrightarrow & 0
\end{array}$$

in which all horizontal and vertical diagrams are triangles in ξ. Since $G_0' \in \mathcal{G}(\mathcal{W})$ and $G_1 \in \mathcal{G}(\mathcal{W})$, then $L \in \mathcal{G}(\mathcal{W})$. Since $\xi xt^1_\xi(\mathcal{W},\ K) = 0$, the triangle $K \to L \to H \to \Sigma K$ is $\mathcal{C}(\mathcal{W},\ -)$-exact. Therefore, $H \in res\widetilde{\mathcal{G}(\mathcal{W})}$. Continuing the above process, one can prove that $M \in res\widetilde{\mathcal{W}}$. □

Lemma 3.15 $\mathcal{G}(\mathcal{W})$ *is closed under direct summands.*

Proof Let $G \cong G' \bigoplus G'' \in \mathcal{G}(\mathcal{W})$. Consider the following split triangles $G' \xrightarrow{\binom{1}{0}} G \xrightarrow{(0\ 1)} G'' \longrightarrow \Sigma G'$ and $G'' \xrightarrow{\binom{0}{1}} G \xrightarrow{(1\ 0)} G' \longrightarrow \Sigma G''$, which are $\mathcal{C}(\mathcal{G}(\mathcal{W}),\ -)$-exact and $\mathcal{C}(-,\ \mathcal{G}(\mathcal{W}))$-exact. Therefore, G' and G'' admit both a proper $\mathcal{G}(\mathcal{W})$-resolution and a proper $\mathcal{G}(\mathcal{W})$-coresolution. By Proposition 3.14, G' and G'' admit both a proper $\mathcal{W}$-resolution and a proper $\mathcal{W}$-coresolution. Since

$$\xi xt^i_\xi(G' \bigoplus G'',\ \mathcal{W}) \cong \xi xt^i_\xi(G',\ \mathcal{W}) \bigoplus \xi xt^i_\xi(G'',\ \mathcal{W})$$

and

$$\xi xt^i_\xi(\mathcal{W},\ G' \bigoplus G'') \cong \xi xt^i_\xi(\mathcal{W}\ G') \bigoplus \xi xt^i_\xi(\mathcal{W},\ G'').$$

Both G' and G'' are in ${}^{\perp}\mathcal{W}\bigcap\mathcal{W}^{\perp}$. By Proposition 3.11, G' and G'' are included in $\mathcal{G}(\mathcal{W})$, which shows that $\mathcal{G}(\mathcal{W})$ is closed under direct summands. □

Proposition 3.16 *(1) $\xi xt_{\xi}^{i\geq 1}(G,\ M)=0$ for any $G\in\mathcal{G}(\mathcal{W})$ and any object $M\in\mathcal{C}$ with $resdim_{\mathcal{W}}(M)<\infty$.*

(2) $\xi xt_{\xi}^{i\geq 1}(N,\ G)=0$ for any $G\in\mathcal{G}(\mathcal{W})$ and any object $N\in\mathcal{C}$ with $coresdim_{\mathcal{W}}(N)<\infty$.

Proof We just need to prove (1). First we assume $Q\in\mathcal{W}$. Since $G\in\mathcal{G}(\mathcal{W})$, there is a $\mathcal{C}(-,\ \mathcal{W})$-exact triangle $K\to P\to G\to\Sigma K$ in ξ with $P\in\mathcal{W}$ and $K\in\mathcal{G}(\mathcal{W})$. By [1, Proposition 3.8], $\xi xt_{\xi}^{1}(G,\ Q)=0$ and $\xi xt_{\xi}^{2}(G,\ Q)\cong\xi xt_{\xi}^{1}(K,\ Q)=0$ for $K\in\mathcal{G}(\mathcal{W})$. Therefore, $\xi xt_{\xi}^{i\geq 1}(G,\ Q)=0$. Now one can prove it completely by induction on $resdim_{\mathcal{W}}(M)$. □

Proposition 3.17 *(1) For each $M\in\mathcal{C}$ with $resdim_{\mathcal{G}(\mathcal{W})}(M)=n<\infty$, there are two ξ-exact sequences, $0\to K\to X\to M\to 0$ and $0\to M\to A\to X'\to 0$, where $X,\ X'\in\mathcal{G}(\mathcal{W})$, $resdim_{\mathcal{W}}(K)\leq n-1$, and $resdim_{\mathcal{W}}(A)\leq n$. If $n=0$, this should be interpreted as $K=0$.*

(2) For each $M\in\mathcal{C}$ with $coresdim_{\mathcal{G}(\mathcal{W})}(M)=n<\infty$, there are two ξ-exact sequences, $0\to M\to Y\to N\to 0$ and $0\to Y'\to B\to M\to 0$, where $Y,\ Y'\in\mathcal{G}(\mathcal{W})$, $coresdim_{\mathcal{W}}(N)\leq n-1$, and $coresdim_{\mathcal{W}}(B)\leq n$. If $n=0$, this should be interpreted as $N=0$.

Proof We just prove (1) by induction on n, since one can prove (2) dually. If $n=1$, there is $\mathcal{G}(\mathcal{W})$-resolution $0\to X_1\to X_0\to M\to 0$ of M with $X_1\in\mathcal{G}(\mathcal{W})$ and $X_0\in\mathcal{G}(\mathcal{W})$. There is a triangle $X_1\to W\to X_1'\to\Sigma X_1$ in ξ with $W\in\mathcal{W}$ and $X_1'\in\mathcal{G}(\mathcal{W})$. For the triangle $X_1\to X_0\to M\to\Sigma X_1$ and the morphism $\alpha:X_1\to W$, by [1, Proposition 2.2 (b)], there exists a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1}X_1 & = & \Sigma^{-1}X_1 & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\Sigma^{-1}M & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & M\\
\| & & {\scriptstyle\alpha}\downarrow & & \downarrow & & \|\\
\Sigma^{-1}M & \longrightarrow & W & \longrightarrow & X & \longrightarrow & M\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & X_1' & = & X_1' & \longrightarrow & 0
\end{array}$$

in which all horizontal and vertical diagrams are triangles in ξ. Since $X_0\in\mathcal{G}(\mathcal{W})$ and $X_1'\in\mathcal{G}(\mathcal{W})$, then $X\in\mathcal{G}(\mathcal{W})$ by Lemma 3.13. Then we get the first required ξ-exact sequence $0\to W\to X\to M\to 0$ with $W\in\mathcal{W}$ and $X\in\mathcal{G}(\mathcal{W})$ from the third row of the above diagram. For X, there is a triangle $X\to W_0\to X'\to\Sigma X$ in ξ with $W_0\in\mathcal{W}$ and $X'\in\mathcal{G}(\mathcal{W})$. For morphisms $f_1:W\to X$ and $f_2:X\to W_0$, by [1, Proposition 2.2 (C)], there is a commutative diagram:

$$\begin{tikzcd}
W \arrow[r, "f_1"] \arrow[d, equal] & X \arrow[r] \arrow[d, "f_2"'] & M \arrow[r] \arrow[d] & \Sigma W \arrow[d, equal] \\
W \arrow[r, "f_2 f_1"] \arrow[d, "f_1"'] & W_0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & \Sigma W \arrow[d, "\Sigma f_1"'] \\
X \arrow[r, "f_2"] \arrow[d] & W_0 \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
0 \arrow[r] & \Sigma M \arrow[r, equal] & \Sigma M \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and the third vertical diagrams are triangles in ξ. The third column of the above diagram yields the second required ξ-exact complex $0 \to M \to A \to X' \to 0$ with $X' \in \mathcal{G}(\mathcal{W})$ and $resdim_{\mathcal{W}}(A) \leq 1$, since W_0 and W_0 are in $\mathcal{W}$. Assume that the result holds for $n-1 (n \geq 2)$. Since $resdim_{\mathcal{G}(\mathcal{W})}(M) = n$, there is a ξ-exact triangle $K \to V_0 \to M \to \Sigma K$ in ξ with $resdim_{\mathcal{G}(\mathcal{W})}(K) = n-1$ and $V_0 \in \mathcal{G}(\mathcal{W})$. For K, by induction hypothesis, there is a ξ-exact complex $0 \to K \stackrel{\alpha}{\to} A_K \to X'_K \to 0$, where $X'_K \in \mathcal{G}(\mathcal{W})$, $resdim_{\mathcal{W}}(A_K) \leq n-1$. By [1, Proposition 2.2 (b)], there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Sigma^{-1} X'_K \arrow[r, equal] \arrow[d] & \Sigma^{-1} X'_K \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma^{-1} M \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d, "\alpha"'] & V_0 \arrow[r] \arrow[d] & M \arrow[d, equal] \\
\Sigma^{-1} M \arrow[r] \arrow[d] & A_K \arrow[r] \arrow[d] & X_M \arrow[r] \arrow[d] & M \arrow[d] \\
0 \arrow[r] & X'_K \arrow[r, equal] & X'_K \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and vertical diagrams are triangles in ξ. Since V_0 and X'_K are in $\mathcal{G}(\mathcal{W})$, then by Lemma 3.13 X_M is in $\mathcal{G}(\mathcal{W})$. Then we get the first needed $\mathcal{W}$-exact sequence $0 \to A_K \to X_M \to M \to 0$ with $resdim_{\mathcal{W}}(A_K) \leq n-1$ and $X_M \in \mathcal{G}(\mathcal{W})$ from the third row of the above diagram. Since $X_M \in \mathcal{G}(\mathcal{W})$, there is a triangle $X_M \to W_1 \to X'_M \to \Sigma X_M$ in ξ with $W_1 \in \mathcal{W}$ and $X'_M \in \mathcal{G}(\mathcal{W})$. For morphisms $g_1 : A_K \to X_M$ and $g_2 : X_M \to W_1$, by [1, Proposition 2.2 (C)], there is a commutative diagram:

$$\begin{tikzcd}
A_K \arrow[r, "g_1"] \arrow[d, equal] & X_M \arrow[r] \arrow[d, "g_2"'] & M \arrow[r] \arrow[d] & \Sigma A_K \arrow[d, equal] \\
A_K \arrow[r, "g_2 g_1"] \arrow[d, "g_1"'] & W_1 \arrow[r] \arrow[d, equal] & A_M \arrow[r] \arrow[d] & \Sigma A_K \arrow[d, "\Sigma g_1"'] \\
X_M \arrow[r, "g_2"] \arrow[d] & W_1 \arrow[r] \arrow[d] & X'_M \arrow[r] \arrow[d] & \Sigma X_M \arrow[d] \\
0 \arrow[r] & \Sigma X_M \arrow[r, equal] & \Sigma X_M \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and the third vertical diagrams are triangles in ξ. The third column of the above diagram yields the second needed ξ-exact complex $0 \to M \to A_M \to X'_M \to 0$ with $X'_M \in \mathcal{G}(\mathcal{W})$ and $resdim_{\mathcal{W}}(A_M) \le n$ for $W_1 \in \mathcal{W}$ and $resdim_{\mathcal{W}}(A_K) \le n-1$. □

Definition 3.18 *A morphism $\varphi: G \to M$ of $\mathcal{C}$, where $G \in \mathcal{G}(\mathcal{W})$, is called a $\mathcal{G}(\mathcal{W})$-precover of M if it can be completed to a $\mathcal{C}(\mathcal{G}(\mathcal{W}),\ -)$-exact triangle $K \to G \to M \to \Sigma K$. M is said to have a special $\mathcal{G}(\mathcal{W})$-precover if there is a triangle $K \to G \to M \to \Sigma K$ with $G \in \mathcal{G}(\mathcal{W})$ and $\xi xt^1_{\xi}(\mathcal{G}(\mathcal{W}),\ K) = 0$. M is said to have a $\mathcal{G}(\mathcal{W})$-approximation if there is a triangle $K \to G \to M \to \Sigma K$ with $G \in \mathcal{G}(\mathcal{W})$ and $resdim_{\mathcal{W}}(K) < \infty$.*

It is clear that M has a $\mathcal{G}(\mathcal{W})$-precover if it has a special $\mathcal{G}(\mathcal{W})$-precover and M has a special $\mathcal{G}(\mathcal{W})$-precover if it has a $\mathcal{G}(\mathcal{W})$-approximation. Dually, one can define the $\mathcal{G}(\mathcal{W})$-preenvelope, special $\mathcal{G}(\mathcal{W})$-preenvelope, and $\mathcal{G}(\mathcal{W})$-coapproximation of M. Following Proposition 3.17 we have

Corollary 3.19 *(1) Every object A of $\mathcal{C}$ with $resdim_{\mathcal{G}(\mathcal{W})}(A) < \infty$ has a $\mathcal{G}(\mathcal{W})$-approximation and a special $\mathcal{G}(\mathcal{W})$-precover.*

(2) Every object B of $\mathcal{C}$ with $coresdim_{\mathcal{G}(\mathcal{W})}(B) < \infty$ has a $\mathcal{G}(\mathcal{W})$-coapproximation and a special $\mathcal{G}(\mathcal{W})$-preenvelope.

Before we end this section, we consider the so-called stability of $\mathcal{W}$-Gorenstein objects. More precisely, now we use $\mathcal{G}(\mathcal{W})$ to replace $\mathcal{W}$ in Definition 3.6. The objects "K_n" for all $n \in \mathbb{Z}$ are called $\mathcal{G}(\mathcal{W})$-Gorenstein objects. We use $\mathcal{G}^2(\mathcal{W})$ to denote the full subcategory of $\mathcal{G}(\mathcal{W})$-Gorenstein objects in $\mathcal{C}$. We claim that

Theorem 3.20 $\mathcal{G}^2(\mathcal{W}) = \mathcal{G}(\mathcal{W})$.

Proof Let $K \in \mathcal{G}(\mathcal{W})$. Consider the diagram

$$K^{\cdot}: \cdots \to 0 \to K \to K \to 0 \to \cdots$$

in $\mathcal{C}$. It is clear that $K \to K \to 0 \to \Sigma K$ and $0 \to K \to K \to 0$ are triangles in ξ that are both $\mathcal{C}(\mathcal{G}(\mathcal{W}),\ -)$-exact and $\mathcal{C}(-,\ \mathcal{G}(\mathcal{W}))$-exact. Thus $K^{\cdot}$ is a complete $\mathcal{G}(\mathcal{W})$-exact complex and $K \in \mathcal{G}^2(\mathcal{W})$.

Let $G \in \mathcal{G}^2(\mathcal{W})$. Now we use Proposition 3.11 to check that $G \in \mathcal{G}(\mathcal{W})$. For any integer n, there is a $\mathcal{C}(\mathcal{W},\ -)$-exact and $\mathcal{C}(-,\ \mathcal{W})$-exact triangle $G_{n+1} \to K_{n+1} \to G_n \to \Sigma G_{n+1}$ with $K_{n+1} \in \mathcal{G}(\mathcal{W})$ and $G = G_{-1}$. By Proposition 3.11 $K_n \in {}^{\perp}\mathcal{W} \bigcap \mathcal{W}^{\perp}$ for any integer n. By [1, Proposition 3.8] and its duality, we have that $G \in {}^{\perp}\mathcal{W} \bigcap \mathcal{W}^{\perp}$. Now we need to construct a proper $\mathcal{W}$-coresolution of G. A proper $\mathcal{W}$-resolution of G can be constructed dually. Since $K_{-1} \in \mathcal{G}(\mathcal{W})$, there is a $\mathcal{C}(-,\ \mathcal{W})$-triangle $K_{-1} \overset{g_2}{\to} W_0 \to V_0 \to \Sigma K_{-1}$ in ξ such that $W_0 \in \mathcal{W}$ and $V_0 \in \mathcal{G}(\mathcal{W})$, and then $V_0 \in cores\widetilde{\mathcal{W}}$. By [1, Proposition 2.2 (C)], there is a commutative diagram:

$$\begin{tikzcd}
G \arrow[r, "g_1"] \arrow[d, equal] & K_{-1} \arrow[r] \arrow[d, "g_2"'] & G_{-2} \arrow[r] \arrow[d] & \Sigma G \arrow[d, equal] \\
G \arrow[r, "g_2g_1"] \arrow[d, "g_1"'] & W_0 \arrow[r] \arrow[d, equal] & U_0 \arrow[r] \arrow[d] & \Sigma G \arrow[d, "\Sigma g_1"'] \\
K_{-1} \arrow[r, "g_2"] \arrow[d] & W_0 \arrow[r] \arrow[d] & V_0 \arrow[r] \arrow[d] & \Sigma K_{-1} \arrow[d] \\
0 \arrow[r] & \Sigma G_{-2} \arrow[r, equal] & \Sigma G_{-2} \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and the third vertical diagrams are triangles in ξ. Since $V_0 \in^{\perp} \mathcal{W}$ and $G_{-2} \in^{\perp} \mathcal{W}$, $U_0 \in^{\perp} \mathcal{W}$. We have a $\mathcal{C}(-,\ \mathcal{W})$-exact triangle $G \to W_0 \to U_0 \to \Sigma G$. By [1, Proposition 2.2 (b)], there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Sigma^{-1}V_0 \arrow[r, equal] \arrow[d] & \Sigma^{-1}V_0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma^{-1}G_{-3} \arrow[r] \arrow[d, equal] & G_{-2} \arrow[r] \arrow[d, "\alpha"'] & K_{-2} \arrow[r] \arrow[d] & G_{-3} \arrow[d, equal] \\
\Sigma^{-1}G_{-3} \arrow[r] \arrow[d] & U_0 \arrow[r] \arrow[d] & Z_0 \arrow[r] \arrow[d] & G_{-3} \arrow[d] \\
0 \arrow[r] & V_0 \arrow[r, equal] & V_0 \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and vertical diagrams are triangles in ξ. Since $K_{-2} \in cores\widetilde{\mathcal{W}}$ and $V_0 \in cores\widetilde{\mathcal{W}}$, $Z_0 \in cores\widetilde{\mathcal{W}}$. Since $K_{-2} \in^{\perp} \mathcal{W}$ and $V_0 \in^{\perp} \mathcal{W}$, $Z_0 \in^{\perp} \mathcal{W}$. Therefore, there is a $\mathcal{C}(-,\ \mathcal{W})$-exact triangle $Z_0 \to W_{-1} \to V_{-1} \to \Sigma Z_0$ with $V_{-1} \in cores\widetilde{\mathcal{W}}$ and $W_{-1} \in \mathcal{W}$. Since $Z_0 \in^{\perp} \mathcal{W}$ and $W_{-1} \in^{\perp} \mathcal{W}$, $V_{-1} \in^{\perp} \mathcal{W}$. By [1, Proposition 2.2 (b)], there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Sigma^{-1}V_{-1} \arrow[r, equal] \arrow[d] & \Sigma^{-1}V_{-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
U_0 \arrow[r] \arrow[d, equal] & Z_0 \arrow[r] \arrow[d, "\alpha"'] & G_{-3} \arrow[r] \arrow[d] & \Sigma U_0 \arrow[d, equal] \\
U_0 \arrow[r] \arrow[d] & W_{-1} \arrow[r] \arrow[d] & U_{-1} \arrow[r] \arrow[d] & \Sigma U_0 \arrow[d] \\
0 \arrow[r] & V_{-1} \arrow[r, equal] & V_{-1} \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and the vertical diagrams are triangles in ξ. Since $G_{-3} \in^{\perp} \mathcal{W}$ and $V_{-1} \in^{\perp} \mathcal{W}$, $U_{-1} \in^{\perp} \mathcal{W}$. Therefore, the third row is a $\mathcal{C}(-,\ \mathcal{W})$-exact triangle. Continuing the above procedure, one can get a proper $\mathcal{W}$-coresolution of G. $\square$

4. $\mathcal{W}$-Gorenstein (co)resolution dimensions

In this section, we give some characterizations of the finite $\mathcal{G}(\mathcal{W})$-(co)resolution dimensions. For doing this, we first give the following lemma.

Lemma 4.1 *Let $0 \to N \to G_1 \to G_0 \to M \to 0$ be a ξ-exact complex with $G_0 \in \mathcal{G}(\mathcal{W})$ and $G_1 \in \mathcal{G}(\mathcal{W})$. Then there are two ξ-exact complexes $0 \to N \to P \to G \to M \to 0$ with $P \in \mathcal{W}$ and $G \in \mathcal{G}(\mathcal{W})$ and $0 \to N \to H \to Q \to M \to 0$ with $Q \in \mathcal{W}$ and $H \in \mathcal{G}(\mathcal{W})$.*

Proof By hypothesis, there are two triangles $N \to G_1 \to K \to \Sigma N$ and $K \to G_0 \to M \to \Sigma K$ in ξ. Since $G_1 \in \mathcal{G}(\mathcal{W})$, there is a triangle with $G_1 \to P \to G_1' \to \Sigma G_1$ in ξ with $P \in \mathcal{W}$ and $G_1' \in \mathcal{G}(\mathcal{W})$. For morphisms $f_1 : N \to G_1$ and $f_2 : G_1 \to P$, by [1, Proposition 2.2 (C)], there is a commutative diagram:

$$\begin{tikzcd}
N \arrow[r, "f_1"] \arrow[d, equal] & G_1 \arrow[r] \arrow[d, "f_2"'] & K \arrow[r] \arrow[d] & \Sigma N \arrow[d, equal] \\
N \arrow[r, "f_2 f_1"] \arrow[d, "f_1"'] & P \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & \Sigma N \arrow[d, "\Sigma f_1"'] \\
G_1 \arrow[r, "f_2"] \arrow[d] & P \arrow[r] \arrow[d] & G_1' \arrow[r] \arrow[d] & \Sigma G_1 \arrow[d] \\
0 \arrow[r] & \Sigma K \arrow[r, equal] & \Sigma K \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and the third vertical diagrams are triangles in ξ. For the triangle $K \to G_0 \to M \to \Sigma K$ and the morphism $\alpha : K \to X$, by [1, Proposition 2.2 (b)], there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Sigma^{-1} G_1' \arrow[r, equal] \arrow[d] & \Sigma^{-1} G_1' \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma^{-1} M \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d, "\alpha"'] & G_0 \arrow[r] \arrow[d] & M \arrow[d, equal] \\
\Sigma^{-1} M \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & M \arrow[d] \\
0 \arrow[r] & G_1' \arrow[r, equal] & G_1' \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and vertical diagrams are triangles in ξ. Since $G_0 \in \mathcal{G}(\mathcal{W})$ and $G_1' \in \mathcal{G}(\mathcal{W})$, then $G \in \mathcal{G}(\mathcal{W})$. Then we get the ξ-exact complex $0 \to N \to P \to G \to M \to 0$ with $P \in \mathcal{W}$ and $G \in \mathcal{G}(\mathcal{W})$. Similarly, we use base change and octahedral axiom and can obtain the other required ξ-exact complex. $\square$

Proposition 4.2 *For any object M in $\mathcal{C}$ and any positive integer n, the following are equivalent,*

(1) $resdim_{\mathcal{G}(\mathcal{W})}(M) \leq n$;

(2) For some integer k with $1 \leq k \leq n$, there is a ξ-exact complex $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ such that $P_i \in \mathcal{G}(\mathcal{W})$ if $0 \leq i < k$ and $P_j \in \mathcal{W}$ if $j \geq k$.

(3) For any integer k with $1 \leq k \leq n$, there is a ξ-exact complex $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ such that $P_i \in \mathcal{G}(\mathcal{W})$ if $0 \leq i < k$ and $P_j \in \mathcal{W}$ if $j \geq k$.

(2') For some integer k with $0 \leq k \leq n$, there is a ξ-exact complex $0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0$ such that $A_k \in \mathcal{G}(\mathcal{W})$ and other $A_i \in \mathcal{W}$.

(3') For any integer k with $0 \leq k \leq n$, there is a ξ-exact complex $0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0$ such that $A_k \in \mathcal{G}(\mathcal{W})$ and other $A_i \in \mathcal{W}$.

Proof $(3) \Rightarrow (2)$ and $(2) \Rightarrow (1)$: It is clear.

$(1) \Rightarrow (3)$: Let $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ be a ξ-exact complex with all $G_i \in \mathcal{G}(\mathcal{W})$. We prove (3) by induction on n. Let $n = 1$. Since $G_1 \in \mathcal{G}(\mathcal{W})$, there is a triangle $G_1 \overset{\alpha}{\to} P_1 \to N \to \Sigma G_1$ in ξ with $P_1 \in \mathcal{W}$ and $N \in \mathcal{G}(\mathcal{W})$. By [1, Proposition 2.2 (b)], there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Sigma^{-1}N \arrow[r, equal] \arrow[d] & \Sigma^{-1}N \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma^{-1}M \arrow[r] \arrow[d, equal] & G_1 \arrow[r] \arrow[d, "\alpha"'] & G_0 \arrow[r] \arrow[d] & M \arrow[d, equal] \\
\Sigma^{-1}M \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & D_0 \arrow[r] \arrow[d] & M \arrow[d] \\
0 \arrow[r] & N \arrow[r, equal] & N \arrow[r] & 0
\end{tikzcd}$$

in which all horizontal and vertical diagrams are triangles in ξ. Since $G_0 \in \mathcal{G}(\mathcal{W})$ and $N \in \mathcal{G}(\mathcal{W})$, $D_0 \in \mathcal{G}(\mathcal{W})$. Then we get the $\mathcal{W}$-exact sequence $0 \to P_1 \to D_0 \to M \to 0$ with $P_1 \in \mathcal{W}$ and $D_0 \in \mathcal{G}(\mathcal{W})$. Now assume that $n > 1$. There is a triangle $A \to G_0 \to M \to \Sigma A$ in ξ with $resdim_{\mathcal{G}(\mathcal{W})}(A) \leq n-1$. By the induction hypothesis, for any integer k with $2 \leq k \leq n$, there is a ξ-exact complex $0 \to P_n \to \cdots \to P_1 \to A \to 0$ such that $P_i \in \mathcal{G}(\mathcal{W})$ if $1 \leq i < k$ and $P_j \in \mathcal{W}$ if $j \geq k$. Therefore, there is a ξ-exact complex $0 \to P_n \to \cdots \to P_1 \to G_0 \to M \to 0$. There is a triangle $B \to P_1 \to A \to \Sigma B$ in ξ. For the ξ-exact complex $0 \to B \to P_1 \to G_0 \to M \to 0$, by Lemma 4.1, there is a ξ-exact complex $0 \to B \to P_1' \to G_0' \to M \to 0$ with $P_1' \in \mathcal{W}$ and $G_0' \in \mathcal{G}(\mathcal{W})$. Therefore, we get the desired exact sequence $0 \to P_n \to \cdots \to P_2 \to P_1' \to G_0' \to M \to 0$.

$(3') \Rightarrow (2')$ and $(2') \Rightarrow (1)$: It is clear.

$(1) \Rightarrow (3')$: Let $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ be a ξ-exact complex with all $G_i \in \mathcal{G}(\mathcal{W})$. We prove (3) by induction on n. If $n = 1$, by Lemma 4.1, the assertion is true. Now we assume that $n \geq 2$. There are two triangles $K \to G_1 \to K_0 \to \Sigma K$ and $K_0 \to G_0 \to M \to \Sigma K_0$ in ξ. For the ξ-exact complex $0 \to K \to G_1 \to G_0 \to M \to 0$, by Lemma 4.1, we get two exact sequences $0 \to K \to G_1' \to P_0 \to M \to 0$ with $G_1' \in \mathcal{G}(\mathcal{W})$ and $P_0 \in \mathcal{W}$ and $0 \to G_n \to \cdots \to G_2 \to G_1' \to P_0 \to M \to 0$. There is a triangle $N \to P_0 \to M \to \Sigma N$ in ξ with $resdim_{\mathcal{G}(\mathcal{W})}(N) \leq n-1$. By the induction hypothesis, for any integer k with $1 \leq k \leq n$, there is a ξ-exact complex $0 \to A_n \to \cdots \to A_1 \to N \to 0$ such that $A_k \in \mathcal{G}(\mathcal{W})$ and other $A_i \in \mathcal{W}$. Therefore, we get the wanted exact sequence $0 \to A_n \to \cdots \to A_1 \to P_0 \to M \to 0$. Now we prove the case $k = 0$. There is a triangle $A \to G_0 \to M \to \Sigma A$ in ξ with $resdim_{\mathcal{G}(\mathcal{W})}(A) \leq n-1$. By the induction hypothesis, there is a ξ-exact complex $0 \to B_n \to \cdots \to B_1 \to A \to 0$ such that $B_1 \in \mathcal{G}(\mathcal{W})$ and other $B_i \in \mathcal{W}$. So we have a ξ-exact complex $0 \to B_n \to \cdots \to B_1 \to G_0 \to M \to 0$. There is a triangle $B \to B_1 \to B' \to \Sigma B$ in ξ. For the ξ-exact complex $0 \to B \to B_1 \to G_0 \to M \to 0$, by Lemma 4.1, we get a ξ-exact complex $0 \to B \to P'' \to G \to M \to 0$ with $G \in \mathcal{G}(\mathcal{W})$ and $P'' \in \mathcal{W}$. Hence the exact sequence $0 \to B_n \to \cdots \to B_2 \to P'' \to G \to M \to 0$ is desired. □

Dually, we have the following result.

Proposition 4.3 *For any object M in $\mathcal{C}$ and any positive integer n, the following are equivalent,*

(1) $coresdim_{\mathcal{G}(\mathcal{W})}(M) \leq n$;

(2) For some integer k with $1 \leq k \leq n$, there is a ξ-exact complex $0 \to M \to P_0 \to P_1 \to \cdots \to P_n \to 0$ such that $P_i \in \mathcal{G}(\mathcal{W})$ if $0 \leq i < k$ and $P_j \in \mathcal{W}$ if $j \geq k$.

(3) For any integer k with $1 \leq k \leq n$, there is a ξ-exact complex $0 \to M \to P_0 \to P_1 \to \cdots \to P_n \to 0$ such that $P_i \in \mathcal{G}(\mathcal{W})$ if $0 \leq i < k$ and $P_j \in \mathcal{W}$ if $j \geq k$.

(2') For some integer k with $0 \leq k \leq n$, there is a ξ-exact complex $0 \to M \to P_0 \to P_1 \to \cdots \to P_n \to 0$ such that $P_k \in \mathcal{G}(\mathcal{W})$ and other $P_i \in \mathcal{W}$.

(3') For any integer k with $0 \leq k \leq n$, there is a ξ-exact complex $0 \to M \to P_0 \to P_1 \to \cdots \to P_n \to 0$ such that $P_k \in \mathcal{G}(\mathcal{W})$ and other $P_i \in \mathcal{W}$.

Proposition 4.4 *Assume that $\mathcal{W}$ is closed under direct summands. For any object M with $resdim_{\mathcal{G}(\mathcal{W})}(M) < \infty$ in $\mathcal{C}$ and any nonnegative integer n, the following are equivalent:*

(1) $resdim_{\mathcal{G}(\mathcal{W})}(M) \leq n$.

(2) M has a proper $\mathcal{G}(\mathcal{W})$-resolution of length $\leq n$.

(3) M has a $\mathcal{G}(\mathcal{W})$-approximation, $K \to G \to M \to \Sigma K$ with $resdim_{\mathcal{W}}(K) \leq n-1$.

(4) $\xi xt_{\xi}^{n+j}(M,\ W) = 0$ for all $j \geq 1$ and all $W \in \mathcal{W}$.

(5) $\xi xt_{\xi}^{n+j}(M,\ N) = 0$ for all $j \geq 1$ and all N with $resdim_{\mathcal{W}}(N) < \infty$.

(6) $\xi xt_{\xi}^{n+1}(M,\ N) = 0$ for all N with $resdim_{\mathcal{W}}(N) < \infty$.

Proof

$(2) \Rightarrow (1)$: It is trivial.

$(1) \Rightarrow (2)$: By Proposition 3.17 (1), there is a triangle $K \to G \to M \to \Sigma K$ in ξ with $resdim_{\mathcal{W}}(K) \leq n-1$ and $G \in \mathcal{G}(\mathcal{W})$. Thus, by Proposition 3.16 (1), $\xi xt_{\xi}^{j}(A,\ K) = 0$ for all $A \in \mathcal{G}(\mathcal{W})$ and $j \geq 1$. Therefore, the triangle $K \to G \to M \to \Sigma K$ is $\mathcal{C}(\mathcal{G}(\mathcal{W}),\ -)$-exact. Replacing M with K, one can get that M has a proper $\mathcal{G}(\mathcal{W})$-resolution of length $\leq n$.

$(1) \Leftrightarrow (3)$: It follows from Corollary 3.19 (1).

$(1) \Rightarrow (4)$: There is a $\mathcal{G}(\mathcal{W})$-exact complex $0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ such that $G_i \in \mathcal{G}(\mathcal{W})$ for $0 \leq i \leq n$. $\xi xt_{\xi}^{n+j}(M,\ W) \cong \xi xt_{\xi}^{j}(M,\ W) = 0$ for all $W \in \mathcal{W}$ and all $j \geq 1$ by [1, Proposition 3.8] and Proposition 3.16 (1).

$(4) \Rightarrow (5)$: It is clear by the dimension shifting theorem for any triangle.

$(5) \Rightarrow (6)$: It is trivial.

$(6) \Rightarrow (1)$: Set $resdim_{\mathcal{G}(\mathcal{W})}(M) \leq m < \infty$. If $m \leq n$, there is nothing to do. Therefore, assume that $m > n$. By Proposition 4.2, there are $\mathcal{W}$-exact triangles in ξ $K_i \to W_i \to K_{i-1} \to \Sigma K_i$ for

$1 \le i \le m-1$ and $K_0 \to G \to M \to \Sigma K_0$ with $W_i \in \mathcal{W}$ and $G \in \mathcal{G}(\mathcal{W})$ and $resdim_{\mathcal{G}(\mathcal{W})}(M) \le m-1$. If $n = 0$, $\xi xt^1_\xi(M,\ K_0) = 0$ by Proposition 3.16 (1), $K_0 \to G \to M \to \Sigma K_0$ is split. Then by Lemma 3.15, $resdim_{\mathcal{G}(\mathcal{W})}(M) = 0 \le n$. Now set $n \ge 1$. $\xi xt^1_\xi(K_n,\ K_{n+1}) \cong \xi xt^{n+1}_\xi(M,\ K_{n+1}) = 0$ by Proposition 3.16 (1). Thus $K_{n+1} \to W_{n+1} \to K_n \to \Sigma K_{n+1}$ is split, and then $K_n \in \mathcal{W}$. Therefore, $resdim_{\mathcal{G}(\mathcal{W})}(M) \le n$. □

Dually,

Proposition 4.5 *Assume that $\mathcal{W}$ is closed under direct summands. For any object M with $coresdim_{\mathcal{G}(\mathcal{W})}(M) < \infty$ in $\mathcal{C}$ and any nonnegative integer n, the following are equivalent:*

(1) $coresdim_{\mathcal{G}(\mathcal{W})}(M) \le n$.

(2) M has a proper $\mathcal{G}(\mathcal{W})$-coresolution of length $\le n$.

(3) M has a $\mathcal{G}(\mathcal{W})$-coapproximation, $M \to G \to K \to \Sigma K$ with $coresdim_{\mathcal{W}}(K) \le n-1$.

(4) $\xi xt^{n+j}_\xi(W,\ M) = 0$ for all $j \ge 1$ and all $W \in \mathcal{W}$.

(5) $\xi xt^{n+j}_\xi(N,\ M) = 0$ for all $j \ge 1$ and all N with $coresdim_{\mathcal{W}}(N) < \infty$.

(6) $\xi xt^{n+1}_\xi(N,\ M) = 0$ for all N with $coresdim_{\mathcal{W}}(N) < \infty$.

5. Applications

Asadollahi and Salarian gave a nice theorem about the finiteness of ξ-$\mathcal{G}$projective dimensions; see [1, Theorem 4.6]. In [18], the authors used the vanishing of $\xi xt^i_\xi(-,\ -)$ to characterize the ξ-$\mathcal{G}$projective and ξ-$\mathcal{G}$injective dimensions of objects in $\mathcal{C}$; see [18, Lemma 4.4 and 4.5]. As mentioned in Remark 3.7, if we use $\mathcal{P}(\xi)$ (respectively, $\mathcal{I}(\xi)$) to replace $\mathcal{W}$, $\mathcal{W}$-Gorenstein objects are just ξ-$\mathcal{G}$projective (ξ-$\mathcal{G}$injective) objects in [1]. We denote the class of all ξ-$\mathcal{G}$projective objects of $\mathcal{C}$ by $\mathcal{GP}(\xi)$, and denote the class of all ξ-$\mathcal{G}$injective objects of $\mathcal{C}$ by $\mathcal{GI}(\xi)$. According to [1, Theorem 3.11 and Proposition 3.13], $\mathcal{GP}(\xi)$ is closed under extensions and direct summands. Dually, so is $\mathcal{GI}(\xi)$. It is clear that $\xi xt^0_\xi(N,\ M) \cong \mathcal{C}(N,\ M)$ if $N \in \mathcal{P}(\xi)$ or $M \in \mathcal{I}(\xi)$. It is well known that $\mathcal{P}(\xi)$ (respectively, $\mathcal{I}(\xi)$) is closed under direct summands. In this section, we use the preceding results to characterize the ξ-$\mathcal{G}$projective (ξ-$\mathcal{G}$injective) dimensions as well as codimensions of objects in $\mathcal{C}$. We use ξ-$\mathcal{G}pd(M)$ and ξ-$\mathcal{G}id(M)$ instead of $resdim_{\mathcal{GP}(\xi)}(M)$ and $coresdim_{\mathcal{GI}(\xi)}(M)$ to denote the ξ-$\mathcal{G}$projective and ξ-$\mathcal{G}$injective dimensions of M, respectively. Therefore, we have

Proposition 5.1 *For any object M with ξ-$\mathcal{G}pd(M) < \infty$ in $\mathcal{C}$ and any positive integer n, the following are equivalent:*

(1) ξ-$\mathcal{G}pd(M) \le n$;

(2) For some integer k with $1 \le k \le n$, there is a ξ-exact complex $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ such that $P_i \in \mathcal{GP}(\xi)$ if $0 \le i < k$ and $P_j \in \mathcal{P}(\xi)$ if $j \ge k$.

(3) For any integer k with $1 \le k \le n$, there is a ξ-exact complex $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ such that $P_i \in \mathcal{GP}(\xi)$ if $0 \le i < k$ and $P_j \in \mathcal{P}(\xi)$ if $j \ge k$.

(4) For some integer k with $0 \le k \le n$, there is a ξ-exact complex $0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0$ such that $A_k \in \mathcal{GP}(\xi)$ and other $A_i \in \mathcal{P}(\xi)$.

(5) For any integer k with $0 \le k \le n$, there is a ξ-exact complex $0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0$ such that $A_k \in \mathcal{GP}(\xi)$ and other $A_i \in \mathcal{P}(\xi)$.

(6) M has a proper $\mathcal{GP}(\xi)$-resolution of length $\leq n$.

(7) M has a $\mathcal{GP}(\xi)$-approximation, $K \to G \to M \to \Sigma K$ with $resdim_{\mathcal{P}(\xi)}(K) \leq n-1$.

(8) $\xi xt_{\xi}^{n+j}(M,\ W) = 0$ for all $j \geq 1$ and all $W \in \mathcal{P}(\xi)$.

(9) $\xi xt_{\xi}^{n+j}(M,\ N) = 0$ for all $j \geq 1$ and all N with $resdim_{\mathcal{P}(\xi)}(N) < \infty$.

(10) $\xi xt_{\xi}^{n+1}(M,\ N) = 0$ for all N with $resdim_{\mathcal{P}(\xi)}(N) < \infty$.

Proposition 5.2 *For any object M with ξ-$\mathcal{G}id(M) < \infty$ in $\mathcal{C}$ and any positive integer n, the following are equivalent:*

(1) ξ-$\mathcal{G}id(M) \leq n$;

(2) For some integer k with $1 \leq k \leq n$, there is a ξ-exact complex $0 \to M \to P_0 \to P_1 \to \cdots \to P_n \to 0$ such that $P_i \in \mathcal{GI}(\xi)$ if $0 \leq i < k$ and $P_j \in \mathcal{I}(\xi)$ if $j \geq k$.

(3) For any integer k with $1 \leq k \leq n$, there is a ξ-exact complex $0 \to M \to P_0 \to P_1 \to \cdots \to P_n \to 0$ such that $P_i \in \mathcal{GI}(\xi)$ if $0 \leq i < k$ and $P_j \in \mathcal{I}(\xi)$ if $j \geq k$.

(4) For some integer k with $0 \leq k \leq n$, there is a ξ-exact complex $0 \to M \to P_0 \to P_1 \to \cdots \to P_n \to 0$ such that $P_k \in \mathcal{GI}(\xi)$ and other $P_i \in \mathcal{I}(\xi)$.

(5) For any integer k with $0 \leq k \leq n$, there is a ξ-exact complex $0 \to M \to P_0 \to P_1 \to \cdots \to P_n \to 0$ such that $P_k \in \mathcal{GI}(\xi)$ and other $P_i \in \mathcal{I}(\xi)$.

(6) M has a proper $\mathcal{GI}(\xi)$-resolution of length $\leq n$.

(7) M has a $\mathcal{GI}(\xi)$-approximation, $M \to G \to K \to \Sigma K$ with $coresdim_{\mathcal{I}(\xi)}(K) \leq n-1$.

(8) $\xi xt_{\xi}^{n+j}(W,\ M) = 0$ for all $j \geq 1$ and all $W \in \mathcal{I}(\xi)$.

(9) $\xi xt_{\xi}^{n+j}(N,\ M) = 0$ for all $j \geq 1$ and all N with $coresdim_{\mathcal{I}(\xi)}(N) < \infty$.

(10) $\xi xt_{\xi}^{n+1}(N,\ M) = 0$ for all N with $coresdim_{\mathcal{I}(\xi)}(N) < \infty$.

Proposition 5.3 *For any object M in $\mathcal{C}$ and any positive integer n, the following are equivalent:*

(1) $resdim_{\mathcal{GI}(\xi)}(M) \leq n$;

(2) For some integer k with $1 \leq k \leq n$, there is a ξ-exact complex $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ such that $P_i \in \mathcal{GI}(\xi)$ if $0 \leq i < k$ and $P_j \in \mathcal{I}(\xi)$ if $j \geq k$.

(3) For any integer k with $1 \leq k \leq n$, there is a ξ-exact complex $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ such that $P_i \in \mathcal{GI}(\xi)$ if $0 \leq i < k$ and $P_j \in \mathcal{I}(\xi)$ if $j \geq k$.

(4) For some integer k with $0 \leq k \leq n$, there is a ξ-exact complex $0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0$ such that $A_k \in \mathcal{GI}(\xi)$ and other $A_i \in \mathcal{I}(\xi)$.

(5) For any integer k with $0 \leq k \leq n$, there is a ξ-exact complex $0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0$ such that $A_k \in \mathcal{GI}(\xi)$ and other $A_i \in \mathcal{I}(\xi)$.

Proposition 5.4 *For any object M in $\mathcal{C}$ and any positive integer n, the following are equivalent:*

(1) $coresdim_{\mathcal{GP}(\xi)}(M) \leq n$;

(2) For some integer k with $1 \leq k \leq n$, there is a ξ-exact complex $0 \to M \to P_0 \to P_1 \to \cdots \to P_n \to 0$ such that $P_i \in \mathcal{GP}(\xi)$ if $0 \leq i < k$ and $P_j \in \mathcal{P}(\xi)$ if $j \geq k$.

(3) For any integer k with $1 \leq k \leq n$, there is a ξ-exact complex $0 \to M \to P_0 \to P_1 \to \cdots \to P_n \to 0$ such that $P_i \in \mathcal{GP}(\xi)$ if $0 \leq i < k$ and $P_j \in \mathcal{P}(\xi)$ if $j \geq k$.

(4) For some integer k with $0 \leq k \leq n$, there is a ξ-exact complex $0 \to M \to P_0 \to P_1 \to \cdots \to P_n \to 0$ such that $P_k \in \mathcal{GP}(\xi)$ and other $P_i \in \mathcal{P}(\xi)$.

(5) For any integer k with $0 \leq k \leq n$, there is a ξ-exact complex $0 \to M \to P_0 \to P_1 \to \cdots \to P_n \to 0$ such that $P_k \in \mathcal{GP}(\xi)$ and other $P_i \in \mathcal{P}(\xi)$.

Acknowledgments

The authors wish to express their gratitude to the referee for his/her careful reading and comments, which improved the presentation of this article.

References

[1] Asadollahi J, Salarian S. Gorenstein objects in triangulated categories. J Algebra 2004; 281: 264-286.

[2] Asadollahi J, Salarian S. Tate cohomology and Gorensteinness for triangulated categories. J Algebra 2006; 299: 480-502.

[3] Beilinson AA, Bernstein J, Deligne P. Perverse sheaves, in: Analysis and Topology on Singular Spaces, I, Luminy, 1981, in: Astrisque, 1982; 100: 5-171.

[4] Beligiannis A. Relative homological algebra and purity in triangulated categories. J Algebra 2000; 227: 268-361.

[5] Christensen LW. Gorenstein dimensions. Lecture Notes in Math. 1747, Berlin, Germany: Springer-Verlag, 2000.

[6] Enochs EE, Jenda OMG. Gorenstein injective and projective modules. Math Z 1995; 220: 611-633.

[7] Enochs EE, Jenda OMG. Relative homological algebra. de Gruyter Expositions in Mathematic, 30 Walter de Gruyter, 2000.

[8] Enochs EE, Jenda OMG, Torrecillas B. Gorenstein flat modules. Nanjing Daxue Xuebao Shuxue Bannian Kan 1993; 10: 1-9.

[9] Eilenberg S, Moore JC. Foundations of relative homological algebra. Amer Math Soc Mem 55, Providence, R.I., 1965.

[10] Geng Y, Ding N. $\mathcal{W}$-Gorenstein modules. J Algebra 2011; 325: 132-146.

[11] Hartshorne R. Residues and Duality, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, Lecture Notes in Math., vol. 20, Berlin, Germany: Springer-Verlag, 1966.

[12] Holm H, Jørgensen P. Semi-dualizing modules and related gorenstein homological dimensions. J Pure App Alg 2006; 205: 423-445.

[13] Holm H. Gorenstein homological dimensions. J Pure App Alg 2004; 189: 167-193.

[14] Huang C, Huang Z. Gorenstein syzygy modules. J Algebra 2010; 324: 3408-3419.

[15] Huang C. G_C-projective, injective and flat modules under change of rings. J Algebra Appl 2012; 11: 1-14.

[16] May JP. The additivity of traces in triangulated categories. Adv Math 2001; 163: 34-73.

[17] Neeman A. Triangulated Categories. Ann Math Stud. vol. 148, Princeton, NJ, USA: Princeton Univ Press, 2001.

[18] Ren W, Liu Z. Gorenstein homological dimensions for triangulated categories. J Algebra 2014; 410: 258-276.

[19] Trlifaj J. Covers, envelopes and cotorsion theories, Lecture notes for the workshop, Homolgical methods in module theory Cortona. 2000.

[20] Verdier JL. Catgories drives: tat 0, in: SGA $4\frac{1}{2}$, in: Lecture Notes in Math., vol. 569, Berlin, Germany: Springer-Verlag, 1977.

[21] White D. Gorenstein projective dimension with respect to a semidualizing module. J Commut Algebra 2010; 2: 111-137.

[22] Yang X, Liu Z. Strongly Gorenstein projective, injective and flat modules. J Algebra 2008; 320: 2659-2674.

[23] Zhu X. Resolving resolution dimensions. Algebra Represent Theor 2013; 16: 1165-1191.