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Lie symmetry analysis and exact solutions of the Sawada–Kotera equation

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Abstract: In the present paper, the Sawada–Kotera equation is considered by Lie symmetry analysis. All of the geometric vector fields to the Sawada–Kotera equation are obtained, and then the symmetry reductions and exact solutions of the Sawada–Kotera equation are investigated. Our results show that symmetry analysis is a very efficient and powerful technique in finding the solution of the proposed equation.

Key words: Sawada–Kotera equation, Lie symmetry analysis, power series solution, hyperbolic function method, trial equation method, exact solution

1. Introduction

Recently, the mathematics and physics fields have devoted considerable effort to the study of solutions to ordinary and partial differential equations (ODEs and PDEs). Among many powerful methods for solving equations, Lie symmetry analysis provides an effective procedure for integrability and conservation laws, reducing equations and exact solutions of a wide and general class of differential systems representing real physical problems [14, 18]. Sinkala et al. [17] performed the group classification of a bond-pricing PDE of mathematical finance to discover the combinations of arbitrary parameters that allow the PDE to admit a nontrivial symmetry Lie algebra, and they computed the admitted Lie point symmetries, identified the corresponding symmetry Lie algebra, and solved the PDE. Under the condition that the symmetry group of the PDE is nontrivial, it contains a standard integral transform of the fundamental solution for PDEs, and a fundamental solution can be reduced to inverting a Laplace transform or some other classical transform in [3]. In recent years, conserved vectors associated with Lie point symmetries have been widely used to investigate the existence, uniqueness, and stability of solutions of nonlinear PDEs. In [9], by the direct construction method, all of the first-order multipliers of the generalized nonlinear second-order equation are obtained, and the corresponding complete conservation laws of such equations are provided. Muatjetjeja and Khalique [12] constructed the conservation laws for the Benjamin–Bona–Mahony equation with variable coefficients equations and obtained an exact solution for the equation using the double reduction theory. Muatjetjeja and Adem [11] computed conservation laws for the 2D Zakharov–Kuznetsov equation using Noether's approach through an interesting method of increasing the order of the equation, and they obtained exact solitary, cnoidal, and snoidal wave solutions for the 2D Zakharov–Kuznetsov equation by using the Kudryashov method and Jacobi elliptic function method. Furthermore, Lie symmetry analysis helps to study their group theoretical properties and effectively assists to derive several mathematical characteristics related to their complete integrability [10]. Lie symmetry analysis and the

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dynamical system method is a feasible approach to dealing with exact explicit solutions to nonlinear PDEs and systems (see, e.g., [5–7, 13]). Liu et al. derived the symmetries, bifurcations, and exact explicit solutions to the KdV equation by using Lie symmetry analysis and the dynamical system method [8]. In the present paper, we will investigate the vector fields, symmetry reductions, and exact solutions to the Sawada–Kotera equation [2, 4, 16]

$$u_t + 5uu_{3x} + 5u_x u_{2x} + 5u^2 u_x + u_{5x} = 0, \tag{1.1}$$

where $u = u(x,t)$ is the unknown function, x is the spatial coordinate in the propagation direction, and t is the temporal coordinates, which occur in different contexts in mathematical physics.

The rest of this paper is organized as follows: in Section 2, the vector fields of Eq. (1.1) are presented by using the Lie symmetry analysis method. Based on the optimal dynamical system method, all the similarity reductions to Eq. (1.1) are obtained. In Section 3, the exact analytic solutions to the equation are investigated by means of the power series method, hyperbolic function method, and trial equation method, respectively. Finally, the conclusions are given in Section 4.

2. Lie symmetry analysis and similarity reductions

Recall that the geometric vector field of a PDE equation is as follows:

$$V = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u, \tag{2.1}$$

where the coefficient functions $\xi(x,t,u)$, $\tau(x,t,u)$, $\eta(x,t,u)$ of the vector field are to be determined later.

If the vector field (2.1) generates a symmetry of the equation (1.1), then V must satisfy the Lie symmetry condition

$$\mathrm{Pr}V(\Delta)|_{\Delta=0} = 0,$$

where $\mathrm{Pr}V$ denotes the 5th prolongation of V, and $\Delta = u_t + 5uu_{3x} + 5u_x u_{2x} + 5u^2 u_x + u_{5x}$. Moreover, the prolongation $\mathrm{Pr}V$ depends on the equation

$$\mathrm{Pr}V = \eta\partial_u + \eta^x\partial_{u_x} + \eta^{2x}\partial_{u_{2x}} + \eta^{3x}\partial_{u_{3x}} + \eta^{4x}\partial_{u_{4x}} + \eta^{5x}\partial_{u_{5x}},$$

where the coefficient functions $\eta^{kx}(k = 1,\ldots,5)$ are given as

$$\eta^{kx} = D_x^k(\eta - \tau u_t - \xi u_x) + \tau u_{kxt} + \xi u_{(k+1)x}, \quad k = 1,\ldots,5.$$

Here symbol D_x denotes the total differentiation operator and is defined as

$$D_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{2x}\partial_{u_x} + \ldots.$$

Then, in terms of the Lie symmetry analysis method, we obtain that all of the geometric vector fields of Eq. (1.1) are as follows:

$$V_1 = x\partial_x + 5t\partial_t - 2u\partial_u, \quad V_2 = \partial_x, \quad V_3 = \partial_t.$$

Moreover, it is necessary to show that the vector fields of Eq. (1.1) are closed under the Lie bracket, and we have

$$[V_1,V_1] = [V_2,V_2] = [V_3,V_3] = 0,$$
$$[V_1,V_2] = -[V_2,V_1] = V_2, \quad [V_1,V_3] = -[V_3,V_1] = 5V_3, \quad [V_2,V_3] = -[V_3,V_2] = 0.$$

Based on the adjoint representations of the vector fields, we obtain the optimal systems of the Sawada–Kotera equation as follows:

$$\{V_1, V_2, V_3, V_3 + vV_2\},$$

where v is an arbitrary constant.

In the preceding section, we obtained the vector fields and the optimal systems of Eq. (1.1). Now we deal with the symmetry reductions and exact solutions to the equation. We will consider the following similarity reductions and group-invariant solutions based on the optimal dynamical system method. From an optimal system of group-invariant solutions to an equation, every other such solution to the equation can be derived.

(1) For the generator V_1, it yields

$$u = t^{-\frac{2}{5}} f(z), \tag{2.2}$$

where $z = xt^{-\frac{1}{5}}$. Substituting (2.2) into Eq. (1.1), we reduce it to the following ODE:

$$-\frac{2}{5}f - \frac{1}{5}zf' + 5ff''' + 5f'f'' + 5f^2f' + f''''' = 0, \tag{2.3}$$

where $f' = \frac{df}{dz}$.

(2) For the generator V_2, we get that the trivial solution to Eq. (1.1) is $u(x,t) = c$, where c is an arbitrary constant.

(3) For the generator V_3, we have

$$u = f(z), \tag{2.4}$$

where $z = x$. Substituting (2.4) into Eq. (1.1), we obtain the following ODE:

$$5ff''' + 5f'f'' + 5f^2f' + f''''' = 0, \tag{2.5}$$

where $f' = \frac{df}{dz}$.

(4) For the generator $V_3 + vV_2$, we have

$$u = f(z), \tag{2.6}$$

where $z = x - vt$. Substituting (2.6) into Eq. (1.1), we have

$$-vf' + 5ff''' + 5f'f'' + 5f^2f' + f''''' = 0, \tag{2.7}$$

where $f' = \frac{df}{dz}$, and v is an arbitrary constant.

3. Exact solutions

By seeking exact solutions of the PDEs, we mean those that can be obtained from some ODEs or, in general, from PDEs of lower order than the original PDE. In terms of this definition, the exact solutions to Eq. (1.1) are obtained actually in the preceding Section 2. In spite of this, we still want to detect the explicit solutions expressed in terms of elementary or at least known functions of mathematical physics, in terms of quadratures, and so on.

3.1. Exact power series solution to Eq. (2.3)

We know that the power series can be used to solve differential equations, including many complicated differential equations [1, 15], and so we consider the exact analytic solutions to the reduced equation by using the power series method. Once we get the exact analytic solutions of the reduced ODEs, the exact power series solutions to Eq. (1.1) are obtained. In view of (2.3), we seek a solution in a power series of the form

$$f(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{3.1}$$

Substituting (3.1) into (2.3) and comparing coefficients, we obtain the following recursion formula:

$$\begin{aligned} c_{n+5} = -\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)}\Big(&5\sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_k c_{n-k+3} \\ &+5\sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1)c_{k+1}c_{n-k+2} + 5\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1)c_i c_{k-i} c_{n-k+1} \\ &-\frac{2}{5}c_n - \frac{1}{5}nc_n\Big), \end{aligned} \tag{3.2}$$

for all $n = 0, 1, 2, \ldots$.

Thus, for arbitrarily chosen constants c_i $(i = 0, 1, \ldots, 4)$, we obtain

$$c_5 = -\frac{1}{120}\big(5c_0^2 c_1 + 30c_0 c_3 + 10c_1 c_2 - \frac{2}{5}c_0\big).$$

Furthermore, from (3.2), it yields

$$c_6 = -\frac{1}{720}\big(10c_0 c_1^2 + 10c_0^2 c_2 + 120c_0 c_4 + 60c_1 c_3 + 20c_2^2 - \frac{3}{5}c_1\big), \tag{3.3}$$

$$c_7 = -\frac{1}{2520}\big(30c_0 c_1 c_2 + 15c_0^2 c_3 + 300c_0 c_5 + 5c_1^3 + 180c_1 c_4 + 120c_2 c_3 - \frac{4}{5}c_2\big), \tag{3.4}$$

and so on.

Thus, for arbitrary chosen constant numbers c_i $(i = 0, 1, \ldots, 4)$, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (3.3) and (3.4) in a unique manner. This implies that for Eq. (2.3), there exists a power series solution (3.1) with the coefficients given by (3.3) and (3.4). Furthermore, it is easy to prove the convergence of the power series (3.1) with the coefficients given by (3.3) and (3.4). Therefore, this power series solution (3.1) to Eq. (2.3) is an exact analytic solution.

Hence, the power series solution of Eq. (2.3) can be written as

$$
\begin{aligned}
f(z) &= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + \sum_{n=1}^{\infty} c_{n+5} z^{n+5} \\
&= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 - \frac{1}{120}\big(5c_0^2 c_1 + 30 c_0 c_3 + 10 c_1 c_2 - \frac{2}{5}c_0\big) z^5 \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} \Bigg(5\sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3) c_k c_{n-k+3} \\
&\quad + 5\sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1) c_{k+1} c_{n-k+2} + 5\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1) c_i c_{k-i} c_{n-k+1} \\
&\quad - \frac{2}{5}c_n - \frac{1}{5} n c_n \Bigg) z^{n+5}.
\end{aligned}
$$

Thus, the exact power series solution of Eq. (1.1) is

$$
\begin{aligned}
u(x,t) &= c_0 t^{-\frac{2}{5}} + c_1 x t^{-\frac{3}{5}} + c_2 x^2 t^{-\frac{4}{5}} + c_3 x^3 t^{-1} + c_4 x^4 t^{-\frac{6}{5}} - \frac{1}{120}\big(5c_0^2 c_1 + 30 c_0 c_3 + 10 c_1 c_2 - \frac{2}{5}c_0\big) x^5 t^{-\frac{7}{5}} \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} \Bigg(5\sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3) c_k c_{n-k+3} \\
&\quad + 5\sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1) c_{k+1} c_{n-k+2} + 5\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1) c_i c_{k-i} c_{n-k+1} \\
&\quad - \frac{2}{5}c_n - \frac{1}{5} n c_n \Bigg) x^{n+5} t^{-\frac{n+7}{5}}.
\end{aligned} \tag{3.5}
$$

Remark 1 *We would like to reiterate that the power series solutions that have been obtained in this section are exact analytic solutions to the equation. Moreover, we can see that these power series solutions converge for the given chosen constants* c_i $(i = 0, 1, \ldots, 4)$ *of (3.5); it is an actual value for mathematical and physical applications.*

3.2. Exact analytical solutions to Eq. (2.5)

We will consider the exact analytic solutions to the reduced equation by using the hyperbolic function method, and the exact analytical solutions to Eq. (1.1) are obtained. Integrating Eq. (2.5), we have

$$5ff'' + \frac{5}{3}f^3 + f'''' + c = 0, \tag{3.6}$$

where c is an integration constant.

Considering the homogeneous balance between the highest order derivative and the nonlinear term in Eq. (3.6), we deduce that the balance number $n = 2$. Then we set

$$f = a_0 + a_1 T + a_2 T^2, \tag{3.7}$$

where $T' = 1 - T^2$, $T = \tanh(z)$. Substituting (3.7) into Eq. (3.6), collecting all the terms of powers of T, and setting each coefficient to zero, we get a system of algebraic equations:

$$\begin{aligned}
T^1: &\quad 16a_1 - 10a_0a_1 + 10a_1a_2 + 5a_0^2a_1 = 0,\\
T^2: &\quad 136a_2 - 30a_0a_2 - 10a_1^2 + 10a_2^2 - 10a_0a_2 + 5a_0a_1^2 + 5a_0^2a_2 = 0,\\
T^3: &\quad -40a_1 - 50a_1a_2 + 10a_0a_1 + \frac{5}{3}a_1^3 + 10a_0a_1a_2 = 0,\\
T^4: &\quad -240a_2 - 40a_2^2 + 30a_0a_2 + 10a_1^2 + 5a_1^2a_2 + 5a_0a_2^2 = 0,\\
T^5: &\quad 24a_1 + 40a_1a_2 + 5a_1a_2^2 = 0,\\
T^6: &\quad 120a_2 + 30a_2^2 + \frac{5}{3}a_2^3 = 0.
\end{aligned} \tag{3.8}$$

Setting $PS = \{T^1, T^2, \ldots, T^6\}$, by Wu's elimination method, we obtain the characteristic series CS as follows:

$$C_1 := 5a_0^2 - 40a_0 + 76, \quad C_2 := a_1, \quad C_3 := a_2^2 + 18a_2 + 72.$$

Since the initial formula of the series CS is a nonzero constant, then $Zero(PS) = Zero(CS)$. Calculating $Zero(CS)$, we have

$$a_0 = \frac{20 \pm 2\sqrt{5}}{5}, \quad a_1 = 0, \quad a_2 = -6 \tag{3.9}$$

or

$$a_0 = \frac{20 \pm 8\sqrt{5}}{5}, \quad a_1 = 0, \quad a_2 = -12. \tag{3.10}$$

Substituting (3.9) and (3.10) into Eq. (3.7), we obtain the solitary wave solutions of Eq. (1.1):

$$u(x,t) = \frac{20 \pm 2\sqrt{5}}{5} - 6\tanh^2(x),$$

or

$$u(x,t) = \frac{20 \pm 8\sqrt{5}}{5} - 12\tanh^2(x).$$

3.3. Exact traveling wave solutions to Eq. (2.7)

Considering the exact analytic solutions to the reduced equation by using the trial equation method, the exact traveling wave solutions to Eq. (1.1) are obtained. By integrating Eq. (2.7), it yields

$$5ff'' + \frac{5}{3}f^3 + f'''' - vf + c = 0, \tag{3.11}$$

where c is an integration constant. According to the polynomial trial equation method to the nonlinear PDEs with rank homogeneous, take the trial equation as follows:

$$f'' = a_2f^2 + a_1f + a_0, \tag{3.12}$$

where a_k $(k = 0, 1, 2)$ are constants to be determined later. Integrating Eq. (3.12), we have

$$(f')^2 = \frac{2}{3}a_2 f^3 + a_1 f^2 + 2a_0 f + d, \tag{3.13}$$

where d is an integration constant. From (3.12) and (3.13), we obtain

$$f'''' = \frac{10}{3}a_2^2 f^3 + 5a_1 a_2 f^2 + (6a_0 a_2 + a_1^2)f + 2a_2 d + a_0 a_1. \tag{3.14}$$

Substituting (3.12) and (3.14) into (3.11), we obtain the following equation:

$$b_3 f^3 + b_2 f^2 + b_1 f + b_0 = 0,$$

where $b_3 = \frac{10}{3}a_2^2 + 5a_2 + \frac{5}{3}$, $b_2 = 5a_1 a_2 + 5a_1$, $b_1 = 6a_0 a_2 + a_1^2 + 5a_0 - v$, $b_0 = 2a_2 d + a_0 a_1 + c$.

For determining the coefficients a_0, a_1, a_2, d, set $b_k = 0$ $(k = 0, 1, 2, 3)$, and we can obtain an algebraic equations system and solve it as follows:

Case 1. $a_2 = -1$, $a_1 =$ an arbitrary constant, $a_0 = a_1^2 - v$, $d = \frac{1}{2}(a_1^3 - a_1 v + c)$.

Case 2. $a_2 = -\frac{1}{2}$, $a_1 = 0$, $a_0 = \frac{v}{2}$, $d = c$.

For Case 1, (3.13) can be rewritten as

$$(f')^2 = -\frac{2}{3}f^3 + a_1 f^2 + 2(a_1^2 - v)f + \frac{1}{2}(a_1^3 - a_1 v + c). \tag{3.15}$$

Set $\omega = (-\frac{2}{3})^{\frac{1}{3}} f$, $d_2 = a_1(-\frac{2}{3})^{-\frac{2}{3}}$, $d_1 = 2(a_1^2 - v)(-\frac{2}{3})^{-\frac{1}{3}}$, $d_0 = \frac{1}{2}(a_1^3 - a_1 v + c)$, and then (3.15) can be read as

$$(-\frac{2}{3})^{-\frac{2}{3}}(\omega')^2 = \omega^3 + d_2\omega^2 + d_1\omega + d_0. \tag{3.16}$$

Solving Eq. (3.16), we have

$$\int \frac{d\omega}{\sqrt{F(\omega)}} = \pm(-\frac{2}{3})^{\frac{1}{3}}(z - z_0), \tag{3.17}$$

where $F(\omega) = \omega^3 + d_2\omega^2 + d_1\omega + d_0$, $z_0 =$ const..

For (3.17), we have the following results:

(i) When $-27\big(\frac{2(-\frac{2}{3})^{-2}a_1^3}{27} + \frac{a_1^3 - a_1 v + c}{2} - \frac{(-\frac{2}{3})^{-\frac{1}{3}}(a_1^2 - v)(a_1^3 - a_1 v + c)}{3}\big)^2 - 4\big(2(-\frac{2}{3})^{-\frac{1}{3}}(a_1^2 - v) - \frac{(-\frac{2}{3})^{-\frac{4}{3}}a_1^2}{3}\big)^3 = 0$ and $2(-\frac{2}{3})^{-\frac{1}{3}}(a_1^2 - v) - \frac{(-\frac{2}{3})^{-\frac{4}{3}}a_1}{3} < 0$, then $F(\omega) = (\omega - \alpha)^2(\omega - \beta)$, $\alpha, \beta \in \mathbb{R}$, and $\alpha \neq \beta$. In view of $\omega > \beta$, the traveling wave solutions of Eq. (1.1) are as follows:

$$u_1(x,t) = (-\frac{2}{3})^{-\frac{1}{3}}\Big((\alpha - \beta)\tanh^2\big((-\frac{2}{3})^{\frac{1}{3}}\frac{\sqrt{\alpha - \beta}}{2}(x - vt - z_0)\big) + \beta\Big), \quad (\alpha > \beta),$$

$$u_2(x,t) = (-\frac{2}{3})^{-\frac{1}{3}}\Big((\alpha - \beta)\coth^2\big((-\frac{2}{3})^{\frac{1}{3}}\frac{\sqrt{\alpha - \beta}}{2}(x - vt - z_0)\big) + \beta\Big), \quad (\alpha > \beta),$$

$$u_3(x,t) = (-\frac{2}{3})^{-\frac{1}{3}}\Big((-\alpha + \beta)\tan^2\big((-\frac{2}{3})^{\frac{1}{3}}\frac{\sqrt{-\alpha + \beta}}{2}(x - vt - z_0)\big) + \beta\Big), \quad (\alpha < \beta).$$

(ii) When $-27\big(\frac{2(-\frac{2}{3})^{-2}a_1^3}{27}+\frac{a_1^3-a_1v+c}{2}-\frac{(-\frac{2}{3})^{-\frac{1}{3}}(a_1^2-v)(a_1^3-a_1v+c)}{3}\big)^2-4\big(2(-\frac{2}{3})^{-\frac{1}{3}}(a_1^2-v)-\frac{(-\frac{2}{3})^{-\frac{4}{3}}a_1^2}{3}\big)^3=0$ and $2(-\frac{2}{3})^{-\frac{1}{3}}(a_1^2-v)-\frac{(-\frac{2}{3})^{-\frac{4}{3}}a_1}{3}=0$, then $F(\omega)=(\omega-\alpha)^3$, $\alpha\in\mathbb{R}$, and the traveling wave solution of Eq. (1.1) is

$$u_4(x,t)=4(-\frac{2}{3})^{-\frac{2}{3}}(x-vt-z_0)^{-2}+\alpha.$$

(iii) When $-27\big(\frac{2(-\frac{2}{3})^{-2}a_1^3}{27}+\frac{a_1^3-a_1v+c}{2}-\frac{(-\frac{2}{3})^{-\frac{1}{3}}(a_1^2-v)(a_1^3-a_1v+c)}{3}\big)^2-4\big(2(-\frac{2}{3})^{-\frac{1}{3}}(a_1^2-v)-\frac{(-\frac{2}{3})^{-\frac{4}{3}}a_1^2}{3}\big)^3>0$ and $2(-\frac{2}{3})^{-\frac{1}{3}}(a_1^2-v)-\frac{(-\frac{2}{3})^{-\frac{4}{3}}a_1}{3}<0$, then $F(\omega)=(\omega-\alpha)(\omega-\beta)(\omega-\gamma)$, $\alpha,\beta,\gamma\in\mathbb{R}$, and $\alpha<\beta<\gamma$, and the traveling wave solutions of Eq. (1.1) are as follows:

$$u_5(x,t)=(-\frac{2}{3})^{-\frac{1}{3}}\Big(\alpha+(\beta-\alpha)\mathrm{sn}^2\big((-\frac{2}{3})^{\frac{1}{3}}\frac{\sqrt{\gamma-\alpha}}{2}(x-vt-z_0),m\big)\Big),$$

$$u_6(x,t)=(-\frac{2}{3})^{-\frac{1}{3}}\Big(\frac{\gamma-\beta\mathrm{sn}^2\big((-\frac{2}{3})^{\frac{1}{3}}\frac{\sqrt{\gamma-\alpha}}{2}(x-vt-z_0),m\big)}{\mathrm{cn}^2\big((-\frac{2}{3})^{\frac{1}{3}}\frac{\sqrt{\gamma-\alpha}}{2}(z-z_0),m\big)}\Big),$$

where $m^2=\frac{\beta-\alpha}{\gamma-\alpha}$.

(iv) When $-27\big(\frac{2(-\frac{2}{3})^{-2}a_1^3}{27}+\frac{a_1^3-a_1v+c}{2}-\frac{(-\frac{2}{3})^{-\frac{1}{3}}(a_1^2-v)(a_1^3-a_1v+c)}{3}\big)^2-4\big(2(-\frac{2}{3})^{-\frac{1}{3}}(a_1^2-v)-\frac{(-\frac{2}{3})^{-\frac{4}{3}}a_1^2}{3}\big)^3<0$, then $F(\omega)=(\omega-\alpha)(\omega^2+p\omega+q)$, $\alpha,p,q\in\mathbb{R}$, and $p^2-4q<0$. In view of $\omega>\beta$, the traveling wave solution of Eq. (1.1) is

$$u_7(x,t)=(-\frac{2}{3})^{-\frac{1}{3}}\Big(\alpha+\frac{2\sqrt{\alpha^2+p\alpha+q}}{1+\mathrm{cn}\big((-\frac{2}{3})^{-\frac{1}{3}}(\alpha^2+p\alpha+q)^{\frac{1}{4}}(x-vt-z_0),m\big)}-\sqrt{\alpha^2+p\alpha+q}\Big),$$

where $m^2=\frac{1}{2}\big(1-\frac{\alpha+\frac{p}{2}}{\sqrt{\alpha^2+p\alpha+q}}\big)$.

For Case 2, (3.13) can be read as

$$(f')^2=-\frac{1}{3}f^3+vf+c. \tag{3.18}$$

Set $\omega=(-\frac{1}{3})^{\frac{1}{3}}f$, $d_1=(-\frac{1}{3})^{-\frac{1}{3}}v$, $d_0=c$, and then (3.18) yields

$$(-\frac{1}{3})^{-\frac{2}{3}}(\omega')^2=\omega^3+d_1\omega+d_0. \tag{3.19}$$

Solving Eq. (3.19), we have

$$\int\frac{d\omega}{\sqrt{F(\omega)}}=\pm(-\frac{1}{3})^{\frac{1}{3}}(z-z_0), \tag{3.20}$$

where $F(\omega)=\omega^3+d_1\omega+d_0$.

For (3.20), we have the following results:

(i) When $-27\big(c-\frac{(-\frac{1}{3})^{-\frac{1}{3}}vc}{3}\big)^2+12v^3=0$ and $v>0$, then $F(\omega)=(\omega-\alpha)^2(\omega-\beta)$, $\alpha\neq\beta$. In view of $\omega>\beta$, the traveling wave solutions of Eq. (1.1) are as follows:

$$u_1(x,t)=(-\frac{1}{3})^{-\frac{1}{3}}\Big((\alpha-\beta)\tanh^2\big((-\frac{1}{3})^{\frac{1}{3}}\frac{\sqrt{\alpha-\beta}}{2}(x-vt-z_0)\big)+\beta\Big),\quad (\alpha>\beta),$$

$$u_2(x,t)=(-\frac{1}{3})^{-\frac{1}{3}}\Big((\alpha-\beta)\coth^2\big((-\frac{1}{3})^{\frac{1}{3}}\frac{\sqrt{\alpha-\beta}}{2}(x-vt-z_0)\big)+\beta\Big),\quad (\alpha>\beta),$$

$$u_3(x,t)=(-\frac{1}{3})^{-\frac{1}{3}}\Big((-\alpha+\beta)\tan^2\big((-\frac{1}{3})^{\frac{1}{3}}\frac{\sqrt{-\alpha+\beta}}{2}(x-vt-z_0)\big)+\beta\Big),\quad (\alpha<\beta).$$

(ii) When $-27\big(c-\frac{(-\frac{1}{3})^{-\frac{1}{3}}vc}{3}\big)^2+12v^3=0$ and $v=0$, then $F(\omega)=(\omega-\alpha)^3$, and the solution of Eq. (1.1) is as follows:

$$u_4(x,t)=4(-\frac{1}{3})^{-\frac{2}{3}}(x-z_0)^{-2}+\alpha.$$

(iii) When $-27\big(c-\frac{(-\frac{1}{3})^{-\frac{1}{3}}vc}{3}\big)^2+12v^3>0$ and $v>0$, then $F(\omega)=(\omega-\alpha)(\omega-\beta)(\omega-\gamma)$, $\alpha<\beta<\gamma$, and the traveling wave solutions of Eq. (1.1) are as follows:

$$u_5(x,t)=(-\frac{1}{3})^{-\frac{1}{3}}\Big(\alpha+(\beta-\alpha)\mathrm{sn}^2\big((-\frac{1}{3})^{\frac{1}{3}}\frac{\sqrt{\gamma-\alpha}}{2}(x-vt-z_0),m\big)\Big),$$

$$u_6(x,t)=(-\frac{1}{3})^{-\frac{1}{3}}\Big(\frac{\gamma-\beta\mathrm{sn}^2\big((-\frac{1}{3})^{\frac{1}{3}}\frac{\sqrt{\gamma-\alpha}}{2}(x-vt-z_0),m\big)}{\mathrm{cn}^2\big((-\frac{1}{3})^{\frac{1}{3}}\frac{\sqrt{\gamma-\alpha}}{2}(z-z_0),m\big)}\Big),$$

where $m^2=\frac{\beta-\alpha}{\gamma-\alpha}$.

(iv) When $-27\big(c-\frac{(-\frac{1}{3})^{-\frac{1}{3}}vc}{3}\big)^2+12v^3<0$, then $F(\omega)=(\omega-\alpha)(\omega^2+p\omega+q)$, and $p^2-4q<0$, in view of $\omega>\beta$, and the traveling wave solution of Eq. (1.1) is as follows:

$$u_7(x,t)=(-\frac{1}{3})^{-\frac{1}{3}}\Big(\alpha+\frac{2\sqrt{\alpha^2+p\alpha+q}}{1+\mathrm{cn}\big((-\frac{1}{3})^{-\frac{1}{3}}(\alpha^2+p\alpha+q)^{\frac{1}{4}}(x-vt-z_0),m\big)}-\sqrt{\alpha^2+p\alpha+q}\Big),$$

where $m^2=\frac{1}{2}\big(1-\frac{\alpha+\frac{p}{2}}{\sqrt{\alpha^2+p\alpha+q}}\big)$.

Remark 2 *Under the conditions of the generator V_3 and V_3+vV_2, we can also obtain the power series form solutions to Eqs. (2.5) and (2.7), respectively; the details are omitted.*

4. Summary and discussion

In this paper, we have obtained the symmetries and similarity reductions of the Sawada–Kotera equation by using the Lie symmetry analysis method. All the group-invariant solutions to the Sawada–Kotera equation (1.1) are considered based on the optimal system method. Then the exact solutions to the equation are investigated by means of the power series method, trial equation method, and hyperbolic function method, respectively, and we can see that these power series solutions converge. The basic idea described in this paper is efficient and powerful in solving wide classes of nonlinear PDEs, e.g., Caudrey–Dodd–Gibbon equations.

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