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A Mehrotra predictor-corrector interior-point algorithm for semidefinite optimization

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Abstract: This paper proposes a second-order Mehrotra-type predictor-corrector feasible interior-point algorithm for semidefinite optimization problems. In each iteration, the algorithm computes the Newton search directions through a new form of combination of the predictor and corrector directions. Using the Ai–Zhang wide neighborhood for linear complementarity problems, it is shown that the complexity bound of the algorithm is $O\left(\sqrt{n}\log\varepsilon^{-1}\right)$ for the Nesterov–Todd search direction and $O\left(n\log\varepsilon^{-1}\right)$ for the Helmberg–Kojima–Monteiro search directions.

Key words: Semidefinite optimization, Mehrotra-type predictor-corrector algorithm, polynomial complexity

1. Introduction

The semidefinite optimization (SDO) problem is an important class of optimization problems in which a linear function of a matrix variable X is minimized or maximized over an affine subspace of symmetric matrices.

The SDO problem arises in many scientific and engineering fields. For applications in system and control theory, we refer to [3, 4], and for applications in statistics and combinatorial optimization to [2, 8, 10].

Among various approximations for solving SDO problems, interior-point methods (IPMs) are one of the most efficient and applicable classes of iterative algorithms that solve SDO problems in polynomial time complexity.

Due to the importance of this class of optimization problems, several authors have discussed and generalized some IPMs for linear optimization (LO) problems to the context of SDO problems. Pioneering research for this generalization was done by Nesterov and Nemirovskii [15], in which the SDO problems were solved by the primal-dual interior-point algorithms with polynomial time complexity. After that, various primal-dual interior-point algorithms were proposed for solving SDO problems by several authors such as Alizadeh [2], Vandenberghe and Boyd [21], Helmberg et al. [6], Wolkowicz et al. [23], Klerk [7], Wang and Bai [22], Mansouri and Ross [13], and Mansouri [12].

Although some of the above-mentioned interior-point algorithms theoretically have optimal iteration complexity, they are not as efficient in practice as the predictor-corrector interior-point algorithms. Among various types of predictor-corrector interior-point algorithms, the Mehrotra-type predictor-corrector (MPC) algorithm is one of the most efficient interior-point algorithms and it is a more practical primal-dual method, whose variants are the backbone of IPM software packages, such as SeDuMi [19] and SDPT3 [20]. However,

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the practical importance and the theoretical analysis of MPC algorithms motivated us to study and investigate theoretically and algorithmically a special type of this algorithm for solving and finding an optimal solution of SDO problems.

Although Zhang and Zhang [28] established convergence theory and complexity bounds of the MPC algorithm, in spite of extensive use of this method, not much about its complexity was known before the paper presented by Salahi et al. [18].

The small and wide neighborhoods are two popular neighborhoods that are frequently used in IPMs. In theory, the iteration bound for wide-neighborhood IPMs (large-update IPMs) is worse than that proved for small-neighborhood IPMs (small-update IPMs). In 2005, Ai and Zhang [1] introduced a new wide neighborhood around the central path of linear complementarity problems (LCPs) and proposed the first wide-neighborhood interior-point algorithm for LCPs [1], in which their algorithm enjoys the low iteration bound $O\left(\sqrt{n}L\right)$. Later, Li and Terlaky [9] generalized Ai and Zhang's algorithm [1] for LCPs to SDO problems and proved that the iteration complexity of their algorithm is the same as that of Ai and Zhang [1] for LCPs. Feng and Fang [5], using Ai and Zhang's wide neighborhood [1], suggested a wide-neighborhood interior-point algorithm for SDO problems.

Liu and Liu [11] proposed the first wide-neighborhood second-order corrector interior-point algorithm with the same complexity as small-neighborhood IPMs for SDO problems. Zhang [26] proposed a second-order MPC interior-point algorithm for SDO problems, in which his algorithm is an extension of the second-order MPC algorithm that was proposed by Salahi and Amiri [17] for LO. Yang et al. [24], based on an important inequality and a new wide neighborhood, suggested a second-order MPC algorithm for SDO problems. Recently, Pirhaji et al. [16] proposed a feasible interior-point algorithm for SDO problems in which their algorithm uses the Ai–Zhang wide neighborhood [1] and terminates in at most $O\left(\sqrt{n}L\right)$ iterations.

The main goal of this paper is to present a second-order MPC interior-point algorithm for SDO problems in which a new scheme is used to obtain the search directions. More precisely, at each iteration of the algorithm the search direction was obtained by a new form of combination of the predictor and corrector directions. Our derived iteration-complexity bound is $O\left(\sqrt{n}\log\varepsilon^{-1}\right)$ for the Nesterov–Todd (NT) search direction and $O\left(n\log\varepsilon^{-1}\right)$ for the Helmberg–Kojima–Monteiro (HKM) search directions that coincide with the currently best iteration bound for this class of optimization problems.

The paper is organized as follows: in section 2, we introduce the SDO problem and review some basic tools in IPMs that are required in solving the SDO problem. Section 3 presents the MPC interior-point algorithm for SDO problems and describes the steps of the proposed algorithm in more detail. Some technical lemmas and results are presented in Section 4.1. Then, in Section 4.3, we prove the polynomial complexity bound of the proposed MPC algorithm for SDO problems. Finally, the paper ends with some concluding remarks in Section 5.

We will use the following notations in the paper. $\mathbb{R}^n$ denotes the space of vectors with n components. The set of all $m\times n$ matrices with real entries is denoted by $\mathbb{R}^{m\times n}$. Moreover, S^n denotes the set of $n\times n$ real symmetric matrices. $S^n_{++}\left(S^n_{+}\right)$ denotes the set of all matrices in S^n that are positive definite (positive semidefinite). For $Q\in S^n$, we write $Q\succ 0\,(Q\succeq 0)$ if Q is positive definite (positive semidefinite). The Frobenius and the spectral norms are denoted respectively by $\|\cdot\|_F$ and $\|\cdot\|$. For any matrix A; $\|A\| = \left(\rho\left(A^TA\right)\right)^{\frac{1}{2}}$, $\lambda_i(A)$ denotes the eigenvalues of A with $\lambda_{\min}(A)$ $(\lambda_{\max}(A))$ as the smallest (largest) eigenvalues and $\mathbf{det}(A)$ denotes its determinant whereas $\mathbf{Tr}(A)=\sum_{i=1}^n\lambda_i(A)$ denotes its trace.

The symmetric positive definite square root matrix of any symmetric positive definite matrix X is denoted by $X^{\frac{1}{2}}$. If $g(x) \geq 0$ is a real valued function of a real nonnegative variable, the notation $g(x) = O(x)$ means that $g(x) \leq \bar{c}x$ for some positive constant $\bar{c}$. The notation $A \sim B \Longleftrightarrow A = SBS^{-1}$ for some invertible matrix S means the similarity between A and B, and the identity matrix is denoted by I. For any $p \times q$ matrix A, $\mathbf{vec}(A)$ denotes the pq-vector obtained by stacking the columns of A one by one from the first to the last column. Assuming the matrix $Q \in S^n$, Q^+ and Q^- denote the positive and negative parts of Q as follows:

$$Q^+ := U\mathbf{Diag}\left((\lambda_1)^+, ..., (\lambda_n)^+\right)U^T, \quad Q^- := U\mathbf{Diag}\left((\lambda_1)^-, ..., (\lambda_n)^-\right)U^T,$$

where $(\lambda_i)^+ = \max\{\lambda_i, 0\}$ and $(\lambda_i)^- = \min\{\lambda_i, 0\}$. Finally, the Kronecker product of two matrices A and B is denoted by $A \otimes B$ (see [6] for the more details of the Kronecker product).

2. Semidefinte optimization and preliminaries

In this paper, we are concerned with the primal-dual interior-point algorithms for solving the primal SDO problem

$$\min\Big\{\langle C, X\rangle \quad s.t. \quad \langle A_i, X\rangle = b_i, \quad i = 1, 2, ..., m, \quad X \succeq 0\Big\}, \tag{1}$$

and its associated dual SDO problem

$$\max\Big\{b^T y \quad s.t. \quad \sum_{i=1}^{m} y_i A_i + S = C, \quad S \succeq 0\Big\}, \tag{2}$$

where $C, X, A_i \in S^n$ for $i = 1, 2, ..., m$ and $y \in \mathbb{R}^m$. We denote the feasible and interior feasible sets of problems (1) and (2) respectively by

$$\mathcal{F} := \Big\{(X, y, S) \in S_+^n \times \mathbb{R}^m \times S_+^n : \quad \langle A_i, X\rangle = b_i, \quad \sum_{i=1}^{m} y_i A_i + S = C, \quad i = 1, 2, ..., m\Big\},$$

and

$$\mathcal{F}^0 := \Big\{(X, y, S) \in S_{++}^n \times \mathbb{R}^m \times S_{++}^n : (X, y, S) \in \mathcal{F}\Big\}.$$

We also assume, without loss of generality, that the relative interior set $\mathcal{F}^0$ is nonempty and all of the matrices A_i are linearly independent. Under these assumptions both primal and dual problems are solvable and the optimality conditions for problems (1) and (2) can be written as follows:

$$\begin{aligned} \langle A_i, X\rangle &= b_i, \qquad i = 1, 2, ..., m, \\ \sum_{i=1}^{m} y_i A_i + S &= C, \\ XS &= 0, \end{aligned} \tag{3}$$

where the last equality is called the complementarity equation. The standard IPM replaces the complementarity equation $XS = 0$ by the perturbed one $XS = \mu I$ and tends μ to zero to obtain an ϵ-optimal solution of the

SDO problem. However, the function defined by the left-hand side of system (3) is a map from $S^n \times \mathbb{R}^m \times S^n$ into $\mathbb{R}^{n\times n} \times \mathbb{R}^m \times S^n$ and therefore the Newton method cannot be straightforwardly applied. To remedy this, Zhang [27] introduced a general symmetrization scheme based on using the operator $H_P : \mathbb{R}^{n\times n} \longrightarrow S^n$ defined as

$$H_P(M) := \frac{1}{2}\Big[PMP^{-1} + \big(PMP^{-1}\big)^T\Big], \quad \forall M \in \mathbb{R}^{n\times n},$$

where $P \in \mathbb{R}^{n\times n}$ is a nonsingular matrix belonging to the specific class

$$\mathcal{C}(X,S) := \{P \in S^n_{++} |\ PXSP^{-1} \in S^n\}. \tag{4}$$

Thus, for any given matrix $P \in \mathcal{C}(X,S)$, system (3) can be written equivalently as the following nonlinear system:

$$\begin{aligned} \langle A_i, X\rangle &= b_i, \quad i = 1,2,...,m, \\ \sum_{i=1}^m y_i A_i + S &= C, \\ H_P(XS) &= 0. \end{aligned} \tag{5}$$

Applying the perturbed Newton method, system (5) leads to the following linear system for search direction $(\Delta X, \Delta y, \Delta S) \in S^n \times \mathbb{R}^m \times S^n$:

$$\begin{aligned} \langle A_i, \Delta X\rangle &= 0, \quad i = 1,2,...,m, \\ \sum_{i=1}^m \Delta y_i A_i + \Delta S &= 0, \\ H_P(X\Delta S + \Delta XS) &= \tau\mu I - H_P(XS), \end{aligned} \tag{6}$$

where $\tau \in [0,1]$ is the target parameter and $\mu = \frac{\mathbf{Tr}(XS)}{n}$ is the normalized duality gap corresponding to (X,y,S). Different choices of the matrix $P \in \mathcal{C}(X,S)$ lead to the different search directions. For instance, the choice $P := W^{\frac{1}{2}}$, where

$$W := X^{-\frac{1}{2}}\left(X^{\frac{1}{2}}SX^{\frac{1}{2}}\right)^{\frac{1}{2}}X^{-\frac{1}{2}} = S^{\frac{1}{2}}\left(S^{\frac{1}{2}}XS^{\frac{1}{2}}\right)^{-\frac{1}{2}}S^{\frac{1}{2}}, \tag{7}$$

leads to the NT search direction while the choice of $P := X^{-\frac{1}{2}}$ and $P := S^{\frac{1}{2}}$ conclude HKM search directions.

Defining

$$\hat{X} := PXP, \quad \hat{S} := P^{-1}SP^{-1}, \quad \hat{A}_i := P^{-1}A_iP^{-1}, \tag{8}$$

and applying the scaled search directions

$$\Delta\hat{X} := P\Delta XP, \quad \Delta\hat{S} := P^{-1}\Delta SP^{-1}, \tag{9}$$

the Newton search direction system (6) can be written as follows:

$$\begin{aligned} \langle \hat{A}_i, \Delta \hat{X} \rangle &= 0, \quad i = 1, 2, ..., m, \\ \sum_{i=1}^{m} \Delta y_i \hat{A}_i + \Delta \hat{S} &= 0, \\ H(\hat{X}\Delta \hat{S} + \Delta \hat{X} \hat{S}) &= \tau\mu I - H(\hat{X}\hat{S}), \end{aligned} \tag{10}$$

where $H(\cdot) = H_I(\cdot)$. Due to (8) and the fact that $PXSP^{-1} \in S^n$, it readily follows that $\hat{X}\hat{S} = \hat{S}\hat{X}$ and $H(\hat{X}\hat{S}) = H(\hat{S}\hat{X}) = \hat{X}\hat{S}$.

3. The Mehrotra-type predictor-corrector algorithm

In this section, we describe a second-order MPC interior-point algorithm for SDO problems, which is the subject of our study in this paper. Most of the interior-point algorithms for SDO problems are based on the following so-called small and negative infinity neighborhoods:

$$\begin{aligned} \mathcal{N}_F(\beta) &:= \left\{ (X, y, S) \in \mathcal{F}^0 : \left\| \tau\mu I - X^{\frac{1}{2}} S X^{\frac{1}{2}} \right\|_F \leq \beta\tau\mu \right\}, \\ \mathcal{N}_\infty^-(\gamma) &:= \left\{ (X, y, S) \in \mathcal{F}^0 : \ \lambda_{\min}(XS) \geq \gamma\mu \right\}, \end{aligned}$$

where $\beta, \gamma \in (0, 1)$. Motivated by Ai and Zhang [1], in this paper, our algorithm will restrict the iterates to the wide neighborhood

$$\mathcal{N}(\tau, \beta) := \left\{ (X, y, S) \in \mathcal{F}^0 : \quad \left\| \left(\tau\mu I - X^{\frac{1}{2}} S X^{\frac{1}{2}} \right)^+ \right\|_F \leq \beta\tau\mu \right\}, \tag{11}$$

where the parameters τ and β are chosen appropriately such that all the iterates reside in the neighborhood $\mathcal{N}(\tau, \beta)$. Due to the definition of $\mathcal{N}(\tau, \beta)$, if $(X, y, S) \in \mathcal{N}(\tau, \beta)$, then $\lambda_i \left(X^{\frac{1}{2}} S X^{\frac{1}{2}} \right) \geq (1 - \beta)\tau\mu$.

We define $\hat{R}_c := \left(\tau\mu I - H\left(\hat{X}\hat{S} \right) \right) = \left(\tau\mu I - \hat{X}\hat{S} \right)$ and decompose it to the positive and negative parts as $\hat{R}_c = \hat{R}_c^+ + \hat{R}_c^-$, where

$$\hat{R}_c^+ := \left(\tau\mu I - H\left(\hat{X}\hat{S} \right) \right)^+ = \left(\tau\mu I - \hat{X}\hat{S} \right)^+, \tag{12}$$

$$\hat{R}_c^- := \left(\tau\mu I - H\left(\hat{X}\hat{S} \right) \right)^- = \left(\tau\mu I - \hat{X}\hat{S} \right)^-. \tag{13}$$

The proposed MPC algorithm for SDO problems, in the predictor step, computes the affine scaling search direction $(\Delta \hat{X}^a, \Delta y^a, \Delta \hat{S}^a)$ by

$$\begin{aligned} \langle \hat{A}_i, \Delta \hat{X}^a \rangle &= 0, \quad i = 1, 2, ..., m, \\ \sum_{i=1}^{m} \Delta y_i^a \hat{A}_i + \Delta \hat{S}^a &= 0, \\ H(\hat{X}\Delta \hat{S}^a + \Delta \hat{X}^a \hat{S}) &= \hat{R}_c^- + \sqrt{n}\hat{R}_c^+, \end{aligned} \tag{14}$$

while the algorithm computes the scaled corrector search direction $(\Delta\hat{X}^c, \Delta y^c, \Delta\hat{S}^c)$ by solving the following system:

$$\begin{aligned} \langle \hat{A}_i, \Delta\hat{X}^c\rangle &= 0, \quad i = 1, 2, ..., m, \\ \sum_{i=1}^{m} \Delta y_i^c \hat{A}_i + \Delta\hat{S}^c &= 0, \\ H(\hat{X}\Delta\hat{S}^c + \Delta\hat{X}^c\hat{S}) &= -H(\Delta\hat{X}^a\Delta\hat{S}^a). \end{aligned} \tag{15}$$

Inspired by [25], after calculating the predictor and corrector search directions, the new iterate is given by

$$\left(\hat{X}(\alpha), y(\alpha), \hat{S}(\alpha)\right) := \left(\hat{X}, y, \hat{S}\right) + \alpha\left(\Delta\hat{X}^a, \Delta y^a, \Delta\hat{S}^a\right) + 2g(\alpha)\left(\Delta\hat{X}^c, \Delta y^c, \Delta\hat{S}^c\right), \tag{16}$$

where $g(\alpha) := 1 - \sqrt{1-\alpha^2}$ and $\alpha \in (0, 1]$ is the step size that gives sufficient reduction of the duality gap and ensures $\left(\hat{X}(\alpha), y(\alpha), \hat{S}(\alpha)\right) \in \mathcal{N}(\tau, \beta)$. From (16), we have

$$\begin{aligned} \hat{X}(\alpha)\hat{S}(\alpha) &= \hat{X}\hat{S} + \alpha\left(\hat{X}\Delta\hat{S}^a + \Delta\hat{X}^a\hat{S}\right) + 2g(\alpha)\left(\hat{X}\Delta\hat{S}^c + \Delta\hat{X}^c\hat{S}\right) \\ &\quad + \alpha^2\Delta\hat{X}^a\Delta\hat{S}^a + 2\alpha g(\alpha)\left(\Delta\hat{X}^a\Delta\hat{S}^c + \Delta\hat{X}^c\Delta\hat{S}^a\right) + 4g^2(\alpha)\Delta\hat{X}^c\Delta\hat{S}^c. \end{aligned} \tag{17}$$

This expression, together with the linearity of operator $H(\cdot)$ and the complementarity equations in systems (14) and (15), implies that

$$H\left(\hat{X}(\alpha)\hat{S}(\alpha)\right) = \hat{X}\hat{S} + \alpha\left(\hat{R}_c^- + \sqrt{n}\hat{R}_c^+\right) + H\left(\Delta\hat{X}(\alpha)\Delta\hat{S}(\alpha)\right), \tag{18}$$

where

$$\begin{aligned} H\left(\Delta\hat{X}(\alpha)\Delta\hat{S}(\alpha)\right) &= -g^2(\alpha)H\left(\Delta\hat{X}^a\Delta\hat{S}^a\right) + 2\alpha g(\alpha)H\left(\Delta\hat{X}^a\Delta\hat{S}^c + \Delta\hat{X}^c\Delta\hat{S}^a\right) \\ &\quad + 4g^2(\alpha)H\left(\Delta\hat{X}^c\Delta\hat{S}^c\right). \end{aligned} \tag{19}$$

Similar to [14], the third equations in systems (14) and (15) in term of the Kronecker product respectively become

$$\hat{E}\mathbf{vec}\left(\Delta\hat{X}^a\right) + \hat{F}\mathbf{vec}\left(\Delta\hat{S}^a\right) = \mathbf{vec}\left(\hat{R}_c^-\right) + \sqrt{n}\mathbf{vec}\left(\hat{R}_c^+\right), \tag{20}$$

$$\hat{E}\mathbf{vec}\left(\Delta\hat{X}^c\right) + \hat{F}\mathbf{vec}\left(\Delta\hat{S}^c\right) = -\mathbf{vec}\left(H\left(\Delta\hat{X}^a\Delta\hat{S}^a\right)\right), \tag{21}$$

where

$$\hat{E} \equiv \frac{1}{2}\left(\hat{S}\otimes I + I\otimes\hat{S}\right), \qquad \hat{F} \equiv \frac{1}{2}\left(\hat{X}\otimes I + I\otimes\hat{X}\right). \tag{22}$$

Below, we describe our algorithm in more detail.

The MPC interior-point algorithm for SDO problems

• **Input parameters:** An accuracy parameter $\varepsilon > 0$, the neighborhood parameters $\beta \in [0, \frac{1}{2}]$ and $\tau \in (0, \frac{1}{4}]$, and the initial feasible solution $\left(X^0, y^0, S^0\right) \in \mathcal{N}(\tau, \beta)$ with $\mu^0 = \frac{\mathbf{Tr}\left(X^0 S^0\right)}{n}$.

• **Step 0:** Set $k := 0$.

• **Step 1:** If $n\mu^k \leq \varepsilon$, then stop. Otherwise, go to **step 2**.

• **Step 2:** Compute the predictor and corrector search directions $(\Delta \hat{X}^a, \Delta y^a, \Delta \hat{S}^a)$ and $(\Delta \hat{X}^c, \Delta y^c, \Delta \hat{S}^c)$ respectively by systems (14) and (15).

• **Step 3:** Calculate the largest step size $\bar{\alpha}^k \in (0, 1]$ such that not only $\mu\left(\bar{\alpha}^k\right) \leq \mu(\alpha)$ for $\alpha \in [0, \bar{\alpha}^k]$ but also $\left(\hat{X}(\alpha), y(\alpha), \hat{S}(\alpha)\right) \in \mathcal{N}(\tau, \beta)$ for $\alpha \in [0, \bar{\alpha}^k]$.

• **Step 4:** Compute the new iterate $\left(\hat{X}(\bar{\alpha}^k), y(\bar{\alpha}^k), \hat{S}(\bar{\alpha}^k)\right)$ by (16) and then set $(\hat{X}^{k+1}, y^{k+1}, \hat{S}^{k+1}) = \left(\hat{X}(\bar{\alpha}^k), y(\bar{\alpha}^k), \hat{S}(\bar{\alpha}^k)\right)$. Calculate $\mu^{k+1} = \frac{\mathbf{Tr}\left(\hat{X}^{k+1}\hat{S}^{k+1}\right)}{n}$ and go to **step 1**.

4. Complexity analysis

4.1. Technical results

In this subsection, we present some technical lemmas that will be used frequently during the analysis of the proposed algorithm in the previous section. From now on, we assume that λ_i for $i = 1, 2, .., n$ are the eigenvalues of the matrix $\hat{X}\hat{S}$. It should be noticed that the matrices $\hat{X}\hat{S}, \hat{S}\hat{X}, XS, SX, X^{\frac{1}{2}}SX^{\frac{1}{2}}$, and $S^{\frac{1}{2}}XS^{\frac{1}{2}}$ have the same eigenvalues, since they are all similar to each other. The following lemma is a direct result of similarity between the matrices $X^{\frac{1}{2}}SX^{\frac{1}{2}}$ and $\hat{X}^{\frac{1}{2}}\hat{S}\hat{X}^{\frac{1}{2}}$. For proof and more details see [5].

Lemma 4.1 *If $\beta, \tau \in (0, 1)$ are given constants, then $\mathcal{N}(\tau, \beta)$ is scaling invariant. That is, (X, y, S) is in the neighborhood $\mathcal{N}(\tau, \beta)$ if and only if $\left(\hat{X}, y, \hat{S}\right)$ is.*

Lemma 4.2 (Lemma 4.1 in [14]) *Let $u, v, r \in \mathbb{R}^n$ and $E, F \in \mathbb{R}^{n\times n}$ satisfy $Eu + Fv = r$. If $FE^T \in S^n_{++}$, then*

$$\left\|\left(FE^T\right)^{-\frac{1}{2}} Eu\right\|^2 + \left\|\left(FE^T\right)^{-\frac{1}{2}} Fv\right\|^2 + 2u^T v = \left\|\left(FE^T\right)^{-\frac{1}{2}} r\right\|^2.$$

Lemma 4.3 (Lemma 4.6 in [14]) *For any $u, v \in \mathbb{R}^n$ and $G \in S^n_{++}$, we have*

$$\|u\|\,\|v\| \leq \sqrt{\boldsymbol{cond(G)}}\left\|G^{-\frac{1}{2}}u\right\|\left\|G^{\frac{1}{2}}v\right\| \leq \frac{1}{2}\sqrt{\boldsymbol{cond}(G)}\left(\left\|G^{-\frac{1}{2}}u\right\|^2 + \left\|G^{\frac{1}{2}}v\right\|^2\right),$$

where $\boldsymbol{cond}(G) = \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)}$.

The following lemma, which is proved in [14], plays an important rule in our analysis.

Lemma 4.4 *Let $\hat{E}$ and $\hat{F}$ be defined as in (22). Then, for any $P \in \mathcal{C}(X, S)$, one has $\rho\left(\left(\hat{F}\hat{E}\right)^{-1}\right) = \frac{1}{4\lambda_{\min}\left(\hat{X}\hat{S}\right)}$.*

Lemma 4.5 *Let the current iterate* $(X, y, S) \in \mathcal{N}(\tau, \beta)$ *and the predictor and corrector search directions* $\left(\Delta\hat{X}^a, \Delta y^a, \Delta\hat{S}^a\right)$ *and* $\left(\Delta\hat{X}^c, \Delta y^c, \Delta\hat{S}^c\right)$ *be respectively the solutions of systems* (14) *and* (15)*. Then:*

$$\Big[1-\alpha\left(1-\tau\right)\Big]\mu \le \mu(\alpha) \le \Big[1-\alpha\left(1-\beta\tau-\tau\right)\Big]\mu. \tag{23}$$

Proof Using (18) and the facts that $\mathbf{Tr}\left(H\left(M\right)\right) = \mathbf{Tr}\left(M\right)$ for any matrix $M \in \mathbb{R}^{n\times n}$ and $\mathbf{Tr}\left(H\left(\Delta\hat{X}(\alpha)\Delta\hat{S}(\alpha)\right)\right) = 0$, it follows that

$$\begin{aligned}
\mu(\alpha) = \hat{\mu}(\alpha) &= \frac{1}{n}\mathbf{Tr}\left(H\left(\hat{X}(\alpha)\hat{S}(\alpha)\right)\right) \\
&= \frac{1}{n}\Big[\mathbf{Tr}\left(\hat{X}\hat{S}\right) + \alpha\mathbf{Tr}\Big[\hat{R}_c^- + \sqrt{n}\hat{R}_c^+\Big] + \mathbf{Tr}\left(H\left(\Delta\hat{X}(\alpha)\Delta\hat{S}(\alpha)\right)\right)\Big] \\
&= \mu + \frac{\alpha}{n}\mathbf{Tr}\Big[\hat{R}_c + \left(\sqrt{n}-1\right)\hat{R}_c^+\Big] \\
&= \mu + \alpha\Big[\left(\tau-1\right)\mu + \frac{\sqrt{n}-1}{n}\mathbf{Tr}\left(\hat{R}_c^+\right)\Big].
\end{aligned} \tag{24}$$

Using the fact that $\mathbf{Tr}\left(M\right) \le \sqrt{n}\left\|M\right\|_F$ for any matrix $M \in S^n$, we have

$$\begin{aligned}
\mu(\alpha) &\le \mu + \alpha\Big[\left(\tau-1\right)\mu + \frac{\sqrt{n}-1}{\sqrt{n}}\left\|\hat{R}_c^+\right\|_F\Big] \\
&= \mu + \alpha\Big[\left(\tau-1\right)\mu + \frac{\sqrt{n}-1}{\sqrt{n}}\left\|\left(\tau\mu I - X^{\frac{1}{2}}SX^{\frac{1}{2}}\right)^+\right\|_F\Big] \\
&\le \Big[1-\alpha\left(1-\beta\tau-\tau\right)\Big]\mu,
\end{aligned}$$

where the equality is due to the similar property of the matrices $\hat{X}\hat{S}$ and $X^{\frac{1}{2}}SX^{\frac{1}{2}}$. The last inequality holds because of $(X, y, S) \in \mathcal{N}(\tau, \beta)$. This concludes the most right-hand side inequality in (23). Following in the same way as the proof of the right-hand side inequality and using the fact $\mathbf{Tr}\left(\hat{R}_c^+\right) \ge 0$, the most left-hand side inequality in (23) is easily proved. This completes the proof. □

Lemma 4.6 *Let* $(X, y, S) \in \mathcal{N}(\tau, \beta)$, $\hat{R}_c^+, \hat{R}_c^-$, *and* $\hat{E}, \hat{F}$ *be respectively defined as in* (12), (13), *and* (22)*. Then:*

$$\left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\boldsymbol{vec}\left(\hat{R}_c^-\right)\right\|^2 \le n\mu, \tag{25}$$

$$\left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\boldsymbol{vec}\left(\hat{R}_c^+\right)\right\|^2 \le \frac{1}{4}\beta\tau\mu. \tag{26}$$

Proof To prove inequality (25), we have

$$\begin{aligned}
\left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\mathbf{vec}\left(\hat{R}_c^-\right)\right\|^2 &= \left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\mathbf{vec}\left(\left(\tau\mu I-\hat{X}\hat{S}\right)^-\right)\right\|^2\\
&= \left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\mathbf{vec}\left(\left(\hat{X}\hat{S}-\tau\mu I\right)^+\right)\right\|^2\\
&\leq \left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\left[\mathbf{vec}\left(\left(\hat{X}\hat{S}\right)^+\right)+\mathbf{vec}\left((-\tau\mu I)^+\right)\right]\right\|^2,
\end{aligned}$$

where the inequality is due to lemma 3.1 in [9]. Then:

$$\begin{aligned}
\left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\mathbf{vec}\left(\hat{R}_c^-\right)\right\|^2 &\leq \left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\mathbf{vec}\left(\left(\hat{X}\hat{S}\right)^+\right)\right\|^2\\
&\leq \left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\mathbf{vec}\left(\hat{X}\hat{S}\right)\right\|^2\\
&= \sum_{i=1}^{n}\frac{\lambda_i^2}{\lambda_i}=\sum_{i=1}^{n}\lambda_i=\mathbf{Tr}\left(\hat{X}\hat{S}\right)=n\mu.
\end{aligned}$$

This implies the first inequality in the lemma. To prove inequality (26), using Lemma 4.4, we derive that

$$\begin{aligned}
\left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\mathbf{vec}\left(\hat{R}_c^+\right)\right\|^2 &\leq \left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\right\|^2\left\|\mathbf{vec}\left(\hat{R}_c^+\right)\right\|^2\\
&= \rho\left(\left(\hat{F}\hat{E}\right)^{-1}\right)\left\|\left(\tau\mu I-\hat{X}\hat{S}\right)^+\right\|_F^2\\
&= \frac{1}{4\lambda_{\min}\left(\hat{X}\hat{S}\right)}\left\|\left(\tau\mu I-X^{\frac{1}{2}}SX^{\frac{1}{2}}\right)^+\right\|_F^2\\
&\leq \frac{\beta^2\tau^2\mu^2}{4\left(1-\beta\right)\tau\mu}=\frac{1}{4}\beta\tau\mu,
\end{aligned}$$

where the last inequality follows from $(X,y,S)\in\mathcal{N}(\tau,\beta)$ and the last equality is due to $\beta\in(0,\frac{1}{2}]$. This implies the lemma. □

Lemma 4.7 *Let the current iterate* $(X,y,S)\in\mathcal{N}(\tau,\beta)$ *and let* $\left(\Delta\hat{X}^a,\Delta y^a,\Delta\hat{S}^a\right)$ *be the solution of* (14). *Then:*

$$\left\|\boldsymbol{vec}\left(\Delta\hat{X}^a\right)\right\|\left\|\boldsymbol{vec}\left(\Delta\hat{S}^a\right)\right\|\leq\frac{1}{2}\sqrt{\boldsymbol{cond}(G)}\left(1+\beta\tau\right)n\mu, \tag{27}$$

where $G=\hat{E}^{-1}\hat{F}$.

Proof Multiplying both sides of equation (20) by $\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}$, taking the norm squared on both of its sides, and using Lemma 4.6, we obtain

$$\begin{aligned}
&\left\|G^{-\frac{1}{2}}\mathbf{vec}\left(\Delta\hat{X}^a\right)+G^{\frac{1}{2}}\mathbf{vec}\left(\Delta\hat{S}^a\right)\right\|^2=\left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\left[\mathbf{vec}\left(\hat{R}_c^-\right)+\sqrt{n}\mathbf{vec}\left(\hat{R}_c^+\right)\right]\right\|^2\\
= &\left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\mathbf{vec}\left(\hat{R}_c^-\right)\right\|^2+n\left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\mathbf{vec}\left(\hat{R}_c^+\right)\right\|^2\\
\leq &\ n\mu+n\beta\tau\mu=(1+\beta\tau)n\mu, \qquad (28)
\end{aligned}$$

where the second equality is due to the fact that $(a^-)^T a^+=0$, for any vector $a\in\mathbb{R}^n$. Now, using Lemma 4.3, the fact that $\mathbf{Tr}\left(\Delta\hat{X}^a\Delta\hat{S}^a\right)=0$, and (28), we conclude that

$$\begin{aligned}
&\left\|\mathbf{vec}\left(\Delta\hat{X}^a\right)\right\|\left\|\mathbf{vec}\left(\Delta\hat{S}^a\right)\right\|\\
\leq &\ \frac{1}{2}\sqrt{\mathbf{cond}(G)}\left[\left\|G^{-\frac{1}{2}}\mathbf{vec}\left(\Delta\hat{X}^a\right)\right\|^2+\left\|G^{\frac{1}{2}}\mathbf{vec}\left(\Delta\hat{S}^a\right)\right\|^2\right]\\
= &\ \frac{1}{2}\sqrt{\mathbf{cond}(G)}\left\|G^{-\frac{1}{2}}\mathbf{vec}\left(\Delta\hat{X}^a\right)+G^{\frac{1}{2}}\mathbf{vec}\left(\Delta\hat{S}^a\right)\right\|^2\\
\leq &\ \frac{1}{2}\sqrt{\mathbf{cond}(G)}(1+\beta\tau)n\mu.
\end{aligned}$$

This proves the lemma. □

The following corollary is a direct result of the above lemma.

Corollary 4.8 *Let* $\left(\Delta\hat{X}^a,\Delta y^a,\Delta\hat{S}^a\right)$ *be the solution of system* (14). *Then*

$$\left\|G^{-\frac{1}{2}}\boldsymbol{vec}\left(\Delta\hat{X}^a\right)\right\| \leq \sqrt{1+\beta\tau}\sqrt{n\mu}, \qquad (29)$$

$$\left\|G^{\frac{1}{2}}\boldsymbol{vec}\left(\Delta\hat{S}^a\right)\right\| \leq \sqrt{1+\beta\tau}\sqrt{n\mu}. \qquad (30)$$

Corollary 4.9 *Let* $\left(\Delta\hat{X}^a,\Delta y^a,\Delta\hat{S}^a\right)$ *be the solution of system* (14). *Then*

$$\left\|H\left(\Delta\hat{X}^a\Delta\hat{S}^a\right)\right\|_F\leq\frac{1}{2}\sqrt{\boldsymbol{cond}(G)}(1+\beta\tau)n\mu. \qquad (31)$$

Proof Due to the definition of the operator $H(\cdot)$, we have

$$\begin{aligned}
\left\|H\left(\Delta\hat{X}^a\Delta\hat{S}^a\right)\right\|_F &= \left\|\frac{\Delta\hat{X}^a\Delta\hat{S}^a+\left(\Delta\hat{X}^a\Delta\hat{S}^a\right)^T}{2}\right\|_F\leq\left\|\Delta\hat{X}^a\right\|_F\left\|\Delta\hat{S}^a\right\|_F\\
&= \left\|\mathbf{vec}\left(\Delta\hat{X}^a\right)\right\|\left\|\mathbf{vec}\left(\Delta\hat{S}^a\right)\right\|\leq\frac{1}{2}\sqrt{\mathbf{cond}(G)}(1+\beta\tau)n\mu,
\end{aligned}$$

where the last inequality follows from Lemma 4.7. □

Lemma 4.10 *Let the current iterate* $(X, y, S) \in \mathcal{N}(\tau, \beta)$ *and* $\left(\Delta\hat{X}^c, \Delta y^c, \Delta\hat{S}^c\right)$ *be the solution of* (15). *Then*

$$\left\|\boldsymbol{vec}\left(\Delta\hat{X}^c\right)\right\|\left\|\boldsymbol{vec}\left(\Delta\hat{S}^c\right)\right\| \leq \frac{1}{8\tau}\left(\boldsymbol{cond}(G)\right)^{\frac{3}{2}}(1+\beta\tau)^2 n^2\mu. \tag{32}$$

Proof Multiplying both sides of equation (21) by $\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}$, taking the norm squared on both of its sides, and using Lemma 4.4 and Corollary 4.9, we derive that

$$\begin{aligned}
\left\|G^{-\frac{1}{2}}\mathbf{vec}\left(\Delta\hat{X}^c\right)+G^{\frac{1}{2}}\mathbf{vec}\left(\Delta\hat{S}^c\right)\right\|^2 &= \left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\mathbf{vec}\left(H\left(\Delta\hat{X}^a\Delta\hat{S}^a\right)\right)\right\|^2\\
&\leq \left\|\left(\hat{F}\hat{E}\right)^{-\frac{1}{2}}\right\|^2\left\|\mathbf{vec}\left(H\left(\Delta\hat{X}^a\Delta\hat{S}^a\right)\right)\right\|^2\\
&= \rho\left(\left(\hat{F}\hat{E}\right)^{-1}\right)\left\|\mathbf{vec}\left(H\left(\Delta\hat{X}^a\Delta\hat{S}^a\right)\right)\right\|^2\\
&= \frac{1}{4\lambda_{\min}\left(\hat{X}\hat{S}\right)}\left\|H\left(\Delta\hat{X}^a\Delta\hat{S}^a\right)\right\|_F^2\\
&\leq \frac{1}{16(1-\beta)\tau\mu}\mathbf{cond}(G)(1+\beta\tau)^2 n^2\mu^2\\
&= \frac{1}{8\tau}\mathbf{cond}(G)(1+\beta\tau)^2 n^2\mu.
\end{aligned}\tag{33}$$

Now, using Lemma 4.3, (33), and the fact $\mathbf{Tr}\left(\Delta\hat{X}^c\Delta\hat{S}^c\right)=0$, we have

$$\begin{aligned}
\left\|\mathbf{vec}\left(\Delta\hat{X}^c\right)\right\|\left\|\mathbf{vec}\left(\Delta\hat{S}^c\right)\right\| &\leq \frac{1}{2}\sqrt{\mathbf{cond}(G)}\Big[\left\|G^{-\frac{1}{2}}\mathbf{vec}\left(\Delta\hat{X}^c\right)\right\|^2\\
&\quad+\left\|G^{\frac{1}{2}}\mathbf{vec}\left(\Delta\hat{S}^c\right)\right\|^2\Big]\\
&= \frac{1}{2}\sqrt{\mathbf{cond}(G)}\left\|G^{-\frac{1}{2}}\mathbf{vec}\left(\Delta\hat{X}^c\right)+G^{\frac{1}{2}}\mathbf{vec}\left(\Delta\hat{S}^c\right)\right\|^2\\
&\leq \frac{1}{16\tau}\mathbf{cond}(G)^{\frac{3}{2}}(1+\beta\tau)^2 n^2\mu,
\end{aligned}$$

which concludes the result. □

The following corollary is the main result of Lemma 4.10.

Corollary 4.11 *Let* $\left(\Delta\hat{X}^c, \Delta y^c, \Delta\hat{S}^c\right)$ *be the solution of system* (15). *Then*

$$\left\|G^{-\frac{1}{2}}\boldsymbol{vec}\left(\Delta\hat{X}^c\right)\right\| \leq \frac{1}{\sqrt{8\tau}}\sqrt{\boldsymbol{cond}(G)}(1+\beta\tau)n\sqrt{\mu}, \tag{34}$$

$$\left\|G^{\frac{1}{2}}\boldsymbol{vec}\left(\Delta\hat{S}^c\right)\right\| \leq \frac{1}{\sqrt{8\tau}}\sqrt{\boldsymbol{cond}(G)}(1+\beta\tau)n\sqrt{\mu}. \tag{35}$$

Corollary 4.12 *Let $\left(\Delta\hat{X}^c, \Delta y^c, \Delta\hat{S}^c\right)$ be the solution of system (15). Then*

$$\left\|H\left(\Delta\hat{X}^c\Delta\hat{S}^c\right)\right\|_F \leq \frac{1}{16\tau}\left(\boldsymbol{cond}(G)\right)^{\frac{3}{2}}(1+\beta\tau)^2 n^2\mu. \tag{36}$$

Proof Using Lemma 4.10, in the same way as in the proof of Corollary 4.9, the result is proved. □

Corollary 4.13 *Let $\left(\Delta\hat{X}^a, \Delta y^a, \Delta\hat{S}^a\right)$ and $\left(\Delta\hat{X}^c, \Delta y^c, \Delta\hat{S}^c\right)$ respectively be the solution of systems (14) and (15). Then*

$$\left\|\Delta\hat{X}^a\right\|_F \left\|\Delta\hat{S}^c\right\|_F \leq \frac{1}{\sqrt{8}\tau}\boldsymbol{cond}(G)\,(1+\beta\tau)^{\frac{3}{2}}\, n^{\frac{3}{2}}\mu, \tag{37}$$

$$\left\|\Delta\hat{S}^a\right\|_F \left\|\Delta\hat{X}^c\right\|_F \leq \frac{1}{\sqrt{8}\tau}\boldsymbol{cond}(G)\,(1+\beta\tau)^{\frac{3}{2}}\, n^{\frac{3}{2}}\mu. \tag{38}$$

4.2. Step size selection

In this subsection, we investigate how to choose the step size α so that the convergence of the proposed MPC interior-point algorithm in the previous section is reached. More precisely, our choice of the step size α should be based on some considerations that the convergence of the algorithm is obtained. To this end, we obtain a lower bound for the largest step size $\bar{\alpha}$ such that it not only guarantees the sufficient reduction of the duality gap $\mu(\alpha)$ but also it ensures that the new iterate $\left(\hat{X}(\alpha), y(\alpha), \hat{S}(\alpha)\right)$ belongs to $\mathcal{N}(\tau,\beta)$ for $\alpha \in [0,\bar{\alpha}]$.

Let $\alpha_g := \arg\min\{\mu(\alpha) : \alpha \in [0,1]\}$. The following lemma establishes $\alpha_g = 1$.

Lemma 4.14 *Let $(X,y,S) \in \mathcal{N}(\tau,\beta)$, $\beta \leq \frac{1}{2}$ and $\tau \leq \frac{1}{4}$. Then $\mu(\alpha)$ is strictly monotonically decreasing in $\alpha \in [0,1]$.*

Proof From (24) and the fact that $\mathbf{Tr}\left(\hat{R}_c^+\right) \leq \sqrt{n}\left\|\hat{R}_c^+\right\|_F$, we have

$$\begin{aligned} \mu'(\alpha) = \left[(\tau-1)\,\mu + \frac{\sqrt{n}-1}{n}\mathbf{Tr}\left(\hat{R}_c^+\right)\right] &\leq \left[(\tau-1)\,\mu + \frac{\sqrt{n}-1}{\sqrt{n}}\left\|\hat{R}_c^+\right\|_F\right] \\ &\leq (-1+\tau+\beta\tau)\,\mu < 0, \end{aligned}$$

where the second inequality is due to $(X,y,S)\in\mathcal{N}(\tau,\beta)$ and the last one follows from $\beta \leq \frac{1}{2}$ and $\tau \leq \frac{1}{4}$. This concludes that the duality gap $\mu(\alpha)$ is decreasing in $\alpha \in [0,1]$ and the lemma follows. □

Due to the above lemma, the largest step size $\bar{\alpha}$ will be computed as follows:

$$\bar{\alpha} = \max\left\{\alpha : \left(\hat{X}(\alpha), y(\alpha), \hat{S}(\alpha)\right) \in \mathcal{N}(\tau,\beta),\ \forall\alpha\in[0,1]\right\}. \tag{39}$$

Lemma 4.15 *Let $(X,y,S)\in\mathcal{N}(\tau,\beta)$. Then, if $\alpha \geq \frac{1}{\sqrt{n}}$, then*

$$\left\|\left(\tau\mu(\alpha)I - \hat{X}\hat{S} - \alpha\left(\hat{R}_c^- + \sqrt{n}\hat{R}_c^+\right)\right)^+\right\|_F = 0, \tag{40}$$

and if $\alpha < \frac{1}{\sqrt{n}}$*, then*

$$\left\| \left(\tau\mu(\alpha)I - \hat{X}\hat{S} - \alpha\left(\hat{R}_c^- + \sqrt{n}\hat{R}_c^+\right)\right)^+ \right\|_F \leq \left(1 - \alpha\sqrt{n}\right)\beta\tau\mu. \tag{41}$$

Proof Since $\mu(\alpha) \leq \mu$, Lemma 4.14 implies that

$$\begin{aligned} \left\| \left(\tau\mu(\alpha)I - \hat{X}\hat{S} - \alpha\left(\hat{R}_c^- + \sqrt{n}\hat{R}_c^+\right)\right)^+ \right\|_F &\leq \left\| \left(\tau\mu I - \hat{X}\hat{S} - \alpha\left(\hat{R}_c^- + \sqrt{n}\hat{R}_c^+\right)\right)^+ \right\|_F \\ &= \left\| \left[(1-\alpha)\hat{R}_c^- + \left(1 - \alpha\sqrt{n}\right)\hat{R}_c^+ \right]^+ \right\|_F \\ &\leq \left(1 - \alpha\sqrt{n}\right)^+ \left\| \hat{R}_c^+ \right\|_F \\ &\leq \left(1 - \alpha\sqrt{n}\right)^+ \beta\tau\mu, \end{aligned}$$

where the second inequality follows from Lemma 3.1 in [9]. If $\alpha \geq \frac{1}{\sqrt{n}}$, then $1 - \alpha\sqrt{n} \leq 0$. Therefore, the first claim is designed. On the other hand, if $\alpha < \frac{1}{\sqrt{n}}$, then $1 - \alpha\sqrt{n} > 0$ and therefore, using the definition of $\mathcal{N}(\tau, \beta)$, we derive the second claim. □

To proceed, we show that $\bar{\alpha}^0 = \frac{\beta\tau}{\sqrt{\mathbf{cond}(G)}\sqrt{n}}$ is a lower bound on the largest step size $\bar{\alpha}$, which is equivalent to proving that for every $\alpha \in [0, \bar{\alpha}^0]$, $\left(\hat{X}(\alpha), y(\alpha), \hat{S}(\alpha)\right)$ belongs to $\mathcal{N}(\tau, \beta)$. The following lemma will be used in the proof of Lemma 4.17.

Lemma 4.16 *Let* $\bar{\alpha}^0 = \frac{\beta\tau}{\sqrt{\mathbf{cond}(G)}\sqrt{n}}$*. Then, for all* $\alpha \in [0, \bar{\alpha}^0]$*, we have*

$$\left\| H\left(\Delta\hat{X}(\alpha)\Delta\hat{S}(\alpha)\right)\right\|_F \leq \alpha\sqrt{n}\beta\tau\mu\left[\frac{9\beta^2\tau^2}{16n} + \frac{1}{\sqrt{2}}\left(\frac{9}{8}\right)^{\frac{3}{2}}\beta + \frac{81}{256}\beta^2\tau\right].$$

Proof Using (19), the triangle inequality, and the fact that $g(\alpha) \leq \alpha^2$, we have

$$\begin{aligned} \left\| H\left(\Delta\hat{X}(\alpha)\Delta\hat{S}(\alpha)\right)\right\|_F \leq\ & \alpha^4 \left\| H\left(\Delta\hat{X}^a\Delta\hat{S}^a\right)\right\|_F + 2\alpha^3 \left\| H\left(\Delta\hat{X}^a\Delta\hat{S}^c + \Delta\hat{X}^c\Delta\hat{S}^a\right)\right\|_F \\ & +4\alpha^4 \left\| H\left(\Delta\hat{X}^c\Delta\hat{S}^c\right)\right\|_F, \end{aligned}$$

from which, from Corollaries 4.9, 4.12, and 4.13, it follows that

$$\begin{aligned}
&\left\|H\left(\Delta\hat{X}(\alpha)\Delta\hat{S}(\alpha)\right)\right\|_F \\
\leq\;& \frac{1}{2}\alpha^4\sqrt{\mathbf{cond}(G)}\left(1+\beta\tau\right)n\mu+\sqrt{\frac{2}{\tau}}\alpha^3\mathbf{cond}(G)\left(1+\beta\tau\right)^{\frac{3}{2}}n^{\frac{3}{2}}\mu \\
&+\frac{1}{4\tau}\alpha^4\left(\mathbf{cond}(G)\right)^{\frac{3}{2}}\left(1+\beta\tau\right)^2n^2\mu \\
\leq\;& \alpha\sqrt{n}\mu\Big[\frac{1}{2}\alpha^3\sqrt{\mathbf{cond}(G)}\left(1+\beta\tau\right)\sqrt{n} \\
&+\sqrt{\frac{2}{\tau}}\alpha^2\mathbf{cond}(G)\left(1+\beta\tau\right)^{\frac{3}{2}}n+\frac{1}{4\tau}\alpha^3\left(\mathbf{cond}(G)\right)^{\frac{3}{2}}\left(1+\beta\tau\right)^2n^{\frac{3}{2}}\Big].
\end{aligned}$$

Therefore, for $\alpha\in[0,\bar{\alpha}^0]$ with $\bar{\alpha}^0=\frac{\beta\tau}{\sqrt{\mathbf{cond}(G)}\sqrt{n}}$, we may write

$$\begin{aligned}
\left\|H\left(\Delta\hat{X}(\alpha)\Delta\hat{S}(\alpha)\right)\right\|_F \leq\;& \alpha\sqrt{n}\beta\tau\mu\Big[\frac{1}{2}\frac{\beta^2\tau^2}{n\mathbf{cond}(G)}\left(1+\beta\tau\right)+\sqrt{\frac{2}{\tau}}\beta\tau\left(1+\beta\tau\right)^{\frac{3}{2}} \\
&+\frac{1}{4\tau}\beta^2\tau^2\left(1+\beta\tau\right)^2\Big] \\
\leq\;& \alpha\sqrt{n}\beta\tau\mu\Big[\frac{9\beta^2\tau^2}{16n}+\frac{1}{\sqrt{2}}\left(\frac{9}{8}\right)^{\frac{3}{2}}\beta+\frac{81}{256}\beta^2\tau\Big].
\end{aligned}$$

This completes the proof. □

Lemma 4.17 *Let $\bar{\alpha}$ be defined as (39). Then $\bar{\alpha}\geq\frac{\beta\tau}{\sqrt{cond(G)}\sqrt{n}}$.*

Proof We have $\bar{\alpha}\geq\frac{1}{\sqrt{n}}$ or $\bar{\alpha}<\frac{1}{\sqrt{n}}$. If $\bar{\alpha}\geq\frac{1}{\sqrt{n}}$, we immediately obtain the lower bound on $\bar{\alpha}$. Thus, we only consider $\bar{\alpha}<\frac{1}{\sqrt{n}}$. Letting $\alpha=\frac{\beta\tau}{\sqrt{\mathbf{cond}(G)}\sqrt{n}}$, we conclude that $(X(\alpha),y(\alpha),S(\alpha))\in\mathcal{N}(\tau,\beta)$ if $X(\alpha)$ and $S(\alpha)$ are positive definite matrices and

$$\left\|\left(\tau\mu(\alpha)I-X^{\frac{1}{2}}(\alpha)S(\alpha)X^{\frac{1}{2}}(\alpha)\right)^+\right\|_F\leq\beta\tau\mu(\alpha).$$

Using Lemmas 3.1 and 3.3 in [9], we have

$$\begin{aligned}
\left\|\left(\tau\mu(\alpha)I-X^{\frac{1}{2}}(\alpha)S(\alpha)X^{\frac{1}{2}}(\alpha)\right)^+\right\|_F \leq\;& \left\|\left[H_P\left(\tau\mu(\alpha)I-X(\alpha)S(\alpha)\right)\right]^+\right\|_F \\
=\;& \left\|\left(\tau\mu(\alpha)I-H_P\left(X(\alpha)S(\alpha)\right)\right)^+\right\|_F \\
=\;& \left\|\left(\tau\mu(\alpha)I-H\left(\hat{X}(\alpha)\hat{S}(\alpha)\right)\right)^+\right\|_F.
\end{aligned}$$

Using (18), we obtain

$$
\begin{aligned}
&\left\|\left(\tau\mu(\alpha)I - X^{\frac{1}{2}}(\alpha)S(\alpha)X^{\frac{1}{2}}(\alpha)\right)^{+}\right\|_F \leq \left\|\left(\tau\mu(\alpha)I - H\left(\hat{X}(\alpha)\hat{S}(\alpha)\right)\right)^{+}\right\|_F \\
&= \left\|\left(\tau\mu(\alpha)I - \hat{X}\hat{S} - \alpha\left(\hat{R}_c^{-} + \sqrt{n}\hat{R}_c^{+}\right) - H\left(\Delta\hat{X}(\alpha)\Delta\hat{S}(\alpha)\right)\right)^{+}\right\|_F \\
&\leq \left\|\left(\tau\mu(\alpha)I - \hat{X}\hat{S} - \alpha\left(\hat{R}_c^{-} + \sqrt{n}\hat{R}_c^{+}\right)\right)^{+}\right\| + \left\|-H\left(\Delta\hat{X}(\alpha)\Delta\hat{S}(\alpha)\right)^{+}\right\|_F \\
&\leq \left\|\left(\tau\mu(\alpha)I - \hat{X}\hat{S} - \alpha\left(\hat{R}_c^{-} + \sqrt{n}\hat{R}_c^{+}\right)\right)^{+}\right\| + \left\|-H\left(\Delta\hat{X}(\alpha)\Delta\hat{S}(\alpha)\right)\right\|_F.
\end{aligned}
$$

Therefore, from Lemmas 4.5, 4.15, and 4.16, we further conclude that

$$
\begin{aligned}
&\left\|\left(\tau\mu(\alpha)I - \hat{X}\hat{S} - \alpha\left(\hat{R}_c^{-} + \sqrt{n}\hat{R}_c^{+}\right)\right)^{+}\right\| + \left\|-H\left(\Delta\hat{X}(\alpha)\Delta\hat{S}(\alpha)\right)\right\|_F - \beta\tau\mu(\alpha) \\
&\leq \left(1-\alpha\sqrt{n}\right)\beta\tau\mu + \alpha\sqrt{n}\beta\tau\mu\left[\frac{9\beta^2\tau^2}{16n} + \frac{1}{\sqrt{2}}\left(\frac{9}{8}\right)^{\frac{3}{2}}\beta + \frac{81}{256}\beta^2\tau\right] \\
&-\beta\tau\left[\mu + \alpha\left(\tau-1\right)\mu\right] \\
&= \alpha\sqrt{n}\beta\tau\mu\left[\frac{9\beta^2\tau^2}{16n} + \frac{1}{\sqrt{2}}\left(\frac{9}{8}\right)^{\frac{3}{2}}\beta + \frac{81}{256}\beta^2\tau + \frac{1-\tau}{\sqrt{n}} - 1\right] \leq 0,
\end{aligned}
$$

which implies that $\left\|\left(\tau\mu(\alpha)I - X^{\frac{1}{2}}(\alpha)S(\alpha)X^{\frac{1}{2}}(\alpha)\right)^{+}\right\|_F \leq \beta\tau\mu(\alpha)$. On the other hand, due to the similarity of matrices $X(\alpha)S(\alpha)$ and $X(\alpha)^{\frac{1}{2}}S(\alpha)X(\alpha)^{\frac{1}{2}}$, we have

$$
\begin{aligned}
\lambda_i\left(X(\alpha)S(\alpha)\right) &= \lambda_i\left(X(\alpha)^{\frac{1}{2}}S(\alpha)X(\alpha)^{\frac{1}{2}}\right) \geq (1-\beta)\,\tau\mu(\alpha) \\
&\geq (1-\beta)\,\tau[1-\alpha(1-\tau)]\mu > 0,
\end{aligned}
$$

which reveals that $X(\alpha)S(\alpha)$ is a nonsingular matrix and further implies that $X(\alpha)$ and $S(\alpha)$ are nonsingular as well. Using continuity of the eigenvalues of a symmetric matrix, it follows that $X(\alpha)$ and $S(\alpha)$ are positive definite matrices for all $\alpha \in [0,1]$, since X, S are positive definite matrices. This implies that $(X(\alpha), y(\alpha), S(\alpha)) \in \mathcal{N}(\tau,\beta)$. Thus, from the definition of $\bar{\alpha}$ in (39), we have $\bar{\alpha} \geq \frac{\beta\tau}{\sqrt{\mathbf{cond}(G)}\sqrt{n}}$. This completes the proof. □

4.3. Polynomial complexity

In this subsection, we present the main result of the paper. The following lemma gives an upper bound for the number of iterations in which the algorithm terminates with an ε-approximate solution.

Lemma 4.18 *Let $\sqrt{\boldsymbol{cond}(G)} \leq \kappa$ for all iterations. Then the proposed MPC interior-point algorithm will terminate with $\left(X^k, y^k, S^k\right)$ such that $\boldsymbol{Tr}\left(X^kS^k\right) \leq \varepsilon\,\boldsymbol{Tr}\left(X^0S^0\right)$ in $O\left(\kappa\sqrt{n}\log\varepsilon^{-1}\right)$ iterations.*

Proof Let $\bar{\alpha}^0 = \frac{\beta\tau}{\sqrt{\mathbf{cond}(G)}\sqrt{n}}$. Therefore, using Lemma 4.5 and the definition of $\bar{\alpha}$ in (39), we obtain

$$\mu(\bar{\alpha}) \leq \mu(\bar{\alpha}^0) = (1 - \varrho\bar{\alpha}^0)\mu,$$

where $\varrho = (1 - \tau - \beta\tau)$. Since the algorithm terminates when $\mu(\bar{\alpha}) \leq \varepsilon\mu^0$, it suffices to have

$$\left(1 - \frac{\beta\tau\varrho}{\sqrt{\mathbf{cond}(G)}\sqrt{n}}\right)^k \mu^0 \leq \left(1 - \frac{\beta\tau\varrho}{\kappa\sqrt{n}}\right)^k \mu^0 \leq \varepsilon\mu^0,$$

which implies that the algorithm stops after at most $k \geq \frac{1}{\beta\tau}\left(\kappa\sqrt{n}\log\varepsilon^{-1}\right)$ iterations. $\square$

In order to obtain an exact upper bound for the number of iterations, we need to recall the following key lemma in [14].

Lemma 4.19 (Lemma 3.1 in [14]) *For the NT search direction* $\boldsymbol{cond}(G) = 1$, *while if the HKM search directions are used, then* $\boldsymbol{cond}(G) \leq \frac{n}{1-\beta}$.

Now we are in a position to present the complexity bound of the proposed algorithm.

Corollary 4.20 *If the NT search direction is used, the iteration complexity of the algorithm is* $O\left(\sqrt{n}\log\varepsilon^{-1}\right)$. *If the HKM search directions are used, then the algorithm stops in at most* $O\left(n\log\varepsilon^{-1}\right)$ *iterations.*

5. Concluding remarks

In this paper, we proposed a second-order MPC feasible interior-point algorithm for SDO problems. The algorithm computes the Newton search directions based on a new form of combination of the predictor and corrector directions. Using the NT direction as the Newton search direction, we proved that the iteration-complexity bound of the algorithm is $O\left(\sqrt{n}\log\varepsilon^{-1}\right)$, while for HKM search directions the proposed algorithm terminates in at most $O\left(n\log\varepsilon^{-1}\right)$ iterations.

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