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A note on dynamics in functional spaces

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Abstract: In this note, we study topologically transitive and hypercyclic composition operators on $C_p(X)$ or $C_k(X)$. We prove that if G is a semigroup of continuous self maps of a countable metric space X with the following properties: (1) every element of G is one-to-one on X, (2) the action of G is strongly run-away on X, then the action of $\hat{G}$ on $C_p(X)$ is topologically transitive and hypercyclic. If G is the set of all one-to-one and continuous self maps of $\mathbb{R}\setminus\mathbb{Z}$, then the action of $\hat{G}$ on $C_k(\mathbb{R}\setminus\mathbb{Z})$ is hypercyclic. We also show that the action of $\hat{G}$ on $C_p(\omega_1)$ is not hypercyclic.

Key words: Topological transitivity, hypercyclicity, composition operators, semigroup actions

1. Introduction

Recall that a map T on a topological space X is called topologically transitive if for every pair of nonempty open subsets U, V of X, there exists some $n \geq 1$ such that $T^n(U) \cap V \neq \emptyset$. A map T on X is called hypercyclic if there exists $x \in X$ such that $\{T^n(x) : n \geq 1\}$ is dense in X. During the last decade hypercyclicity has been thoroughly investigated by several authors. We refer to the recent monograph [6].

If X is a Tychonoff space, then let $C(X)$ denote the set of all continuous functions from X into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers with the natural topology. Any continuous self map $\phi : X \to X$ gives rise to a composition operator C_ϕ, defined by

$$C_\phi(f) = f \circ \phi, \ f \in C(X).$$

The topology under consideration on $C(X)$ is the point-open topology or the compact-open topology. The space $C(X)$ with the point-open topology is denoted by $C_p(X)$. Given a point x of X and an open subset U of $\mathbb{R}$, let $[x, U]$ denote the set of all functions $f \in C(X)$ such that $f(x) \in U$. Then $\{[x, U] : x \in X$ and U is open in $\mathbb{R}\}$ is a subbase for $C_p(X)$. The space $C(X)$ with the compact-open topology is denoted by $C_k(X)$. If K is a compact subset of X and U is an open subset of $\mathbb{R}$, we let $[K, U] = \{f \in C(X) : f(K) \subset U\}$. Then $\{[K, U] : K$ is a compact subset of X and U is open in $\mathbb{R}\}$ is a subbase for $C_k(X)$. We study topologically transitive and hypercyclic composition operators on $C_p(X)$ or $C_k(X)$. Let G be a semigroup acting as continuous functions on a topological space X. Let id_X denote the identity map of X, i.e. $id_X(x) = x$ for all $x \in X$. We say the action of G on X is *topologically transitive*, if for every pair of nonempty open subsets U, V of X, there exists

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$f \in G$ such that $f(U) \cap V \neq \emptyset$. Similarly, the action of G on X is *hypercyclic*, if there exists $x \in X$ such that the G-orbit of x, i.e. $\{f(x) : f \in G\}$ is dense in X [8].

Composition operators on different function spaces have been extensively investigated; see [1, 7, 9]. In [4, 5], composition operators on spaces of real analytic functions were studied. Dynamical properties of topological transitivity and hypercyclicity of a composition operator C_ϕ on certain subspaces of $C(X)$ have also been extensively studied in connection with the topological properties of the underlying map ϕ; see [2, 9].

In [3], Bonet and Domański characterized when composition operators $C_\phi : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ are topologically transitive, where $\mathcal{A}(\Omega)$ is the space of real analytic functions defined on an open subset Ω of $\mathbb{R}^d$. They proved that topological transitivity, hypercyclicity, and sequential hypercyclicity of $C_\phi : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ are equivalent when ϕ is a self map on a simply connected complex neighborhood U of $\mathbb{R}$, $U \neq \mathbb{R}$.

Recently, Javaheri in [8] proved that if X is a separable locally compact metric space, G is a semigroup of continuous self maps of X with the following properties: (i) every element of G is one-to-one on X, (ii) the action of G is run-away on X (G is called run-away on X if for every compact subset $K \subset X$, there exists some $\phi \in G$ such that $\phi(K) \cap K = \emptyset$), then the action of $\hat{G} = \{C_\phi : \phi \in G\}$ on $C_k(X)$ is topologically transitive and hypercyclic. We discuss characterizations of the action of $\hat{G}$ on $C_p(X)$ or $C_k(X)$ under which X is not a locally compact space. We prove that if G is a semigroup of continuous self maps of a countable metric space X with the following properties: (1) every element of G is one-to-one on X, (2) the action of G is strongly run-away on X, then the action of $\hat{G}$ on $C_p(X)$ is topologically transitive and hypercyclic. If G is the set of all one-to-one and continuous self maps of $\mathbb{R} \setminus \mathbb{Z}$, then the action of $\hat{G}$ on $C_k(\mathbb{R} \setminus \mathbb{Z})$ is hypercyclic. We also show that the action of $\hat{G}$ on $C_p(\omega_1)$ is not hypercyclic.

The sets of real, rational, and integer numbers will be denoted by $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$, respectively. $\mathbb{N}$ denotes the set of positive integers. The set $\mathbb{N} \cup \{0\}$ is denoted by ω. In this note, we let $(\mathbb{Q}, d)$ be a subspace of $\mathbb{R}$ with the nature topology.

2. Main results

Inspired by the definition of run-away, we define a notion of strongly run-away.

Definition 1 *Let (X, d) be a metric space. Suppose that G is a semigroup of continuous self maps of X, G is called strongly run-away if there exists some $\delta > 0$ such that for every compact subset K of X, there exists some $\phi \in G$ such that $d(\phi(K), K) \geq \delta$.*

Remark 2 *Let $(\mathbb{Q}, d)$ be a topological space with the nature topology. If G is the semigroup of all one-to-one and continuous self maps of $\mathbb{Q}$, then the action of G on $\mathbb{Q}$ is strongly run-away.*

Proof Let $\delta > 0$. Then there is some $c \in \mathbb{Q}$ such that $c > \delta$. If K is a compact subset of $\mathbb{Q}$, then there exist $a, b \in \mathbb{Q}$ such that $K \subset (a, b)$. If $\phi : \mathbb{Q} \to \mathbb{Q}$ is a map such that $\phi(x) = x + (b - a) + c$ for each $x \in \mathbb{Q}$, then the map ϕ is one-to-one and continuous and $d(\phi(K), K) \geq c > \delta$. □

The following theorem is a main result of this note.

Theorem 3 *Let (X, d) be a countable metric space. Suppose that G is a semigroup of continuous self maps of X with the following properties:*

(1) Every element of G is one-to-one on X.

(2) The action of G is strongly run-away on X.

Let $\hat{G}$ be the semigroup of composition operators induced by the elements of G i.e. $\hat{G} = \{C_\phi : \phi \in G\}$, then the action of $\hat{G}$ on $C_p(X)$ is topologically transitive and hypercyclic.

Proof Let V and W be open subsets of $C_p(X)$. Let $f \in V$. Then there exists a finite set $F_1 = \{x_1, x_2, \ldots, x_n\} \subset X$ and open subsets $U_1, U_2, \ldots, U_n$ of $\mathbb{R}$ such that $f \in [x_1, U_1] \cap \ldots \cap [x_n, U_n] \subset V$. Let $g \in W$. There exists a finite set $F_2 = \{y_1, y_2, \ldots, y_m\} \subset X$ and open subsets $V_1, V_2, \ldots, V_m$ of $\mathbb{R}$ such that $g \in [y_1, V_1] \cap \ldots \cap [y_m, V_m] \subset W$.

Since the action of G on X is strongly run-away, there exists some $\delta > 0$ such that for every compact subset K of X, there exists some $\phi \in G$ such that $d(\phi(K), K) \geq \delta$. Let $F = F_1 \cup F_2$. Then there exists some $\phi \in G$ such that $d(\phi(F), F) \geq \delta$.

Define $\hat{f} : F_1 \cup \phi(F_2) \to \mathbb{R}$ such that if $x \in F_1$ then $\hat{f}(x) = f(x)$. If $x \in \phi(F_2)$, $\hat{f}(x) = g(\phi^{-1}(x))$. Then $\hat{f}$ is continuous on $F_1 \cup \phi(F_2)$. Since X is normal, there exists a continuous mapping $\hat{f}^*$ such that $\hat{f}^*|_{F_1 \cup \phi(F_2)} = \hat{f}$. This implies $\hat{f}^* \in V$ and $\hat{f}^* \circ \phi \in W$. Thus $C_\phi(V) \cap W \neq \emptyset$. Thus the action of $\hat{G}$ on $C_p(X)$ is topologically transitive.

Since X is countable, let $X = \{q_n : n \in \mathbb{N}\}$. If $F_m = \{q_1, q_2, \ldots, q_m\}$, then $|F_m| < \omega$ and $F_m \subset F_{m+1}$ for each $m \in \mathbb{N}$. Let $\mathcal{F} = \{h : h : F \to \mathbb{Q}$ is a map, $F \subset X, 1 \leq |F| < \omega\}$. Since $\mathcal{F}$ is countable, denote $\mathcal{F} = \{h_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ there exists some $m_n \in \mathbb{N}$ such that $dom(h_n) \subset F_{m_n}$, and we assume that $m_n > m_l$ if $n > l$.

For $n = 1$, the set $dom(h_1) \subset F_{m_1}$. By (2), there exists a continuous and one-to-one map $\phi_1 \in G$ such that $d(\phi_1(F_{m_1}), F_{m_1}) \geq \delta$. Let $k_1 = m_1$. For $n = 2$, there exists some $k_2 > \max\{k_1, m_2\}$ such that $(dom(h_2) \cup \phi_1(F_{k_1})) \subset F_{k_2}$. By (2), there exists some continuous and one-to-one map $\phi_2 \in G$ such that $d(\phi_2(F_{k_2}), F_{k_2}) \geq \delta$.

By induction, there exists a sequence $\{\phi_n : n \in \mathbb{N}\} \subset G$ of continuous and one-to-one maps on X and an increasing sequence $\{k_i : i \in \mathbb{N}\}$ with $k_i > \max\{m_i, k_{i-1}\}$ if $i > 1$ such that $(dom(h_i) \cup \phi_{i-1}(F_{k_{i-1}})) \subset F_{k_i}$ and $d(\phi_i(F_{k_i}), F_{k_i}) \geq \delta$ for each $i \in \mathbb{N}$. Thus $d(\phi_i(F_{k_i}), \phi_j(F_{k_j})) \geq \delta$ if $i \neq j$. Thus $dom(h_n) \subset F_{m_n} \subset F_{k_n}$ and $\phi_n(dom(h_n)) \cap F_{k_n} = \emptyset$ for each n.

Since $\{\phi_n(dom(h_n)) : n \in \mathbb{N}\}$ is a family of finite sets of X and $d(\phi_n(dom\ (h_n)), \phi_m(dom(h_m))) \geq \delta$ if $n \neq m$, the set $D = \bigcup\{\phi_n(dom(h_n)) : n \in \mathbb{N}\}$ is closed and discrete in X.

Define $f : D \to \mathbb{R}$ such that if $x \in \phi_n(dom(h_n))$ for some $n \in \mathbb{N}$ then $f(x) = h_n(\phi_n^{-1}(x))$. Thus $f(\phi_n(x)) = h_n(x)$ if $x \in dom(h_n)$ for some n. Hence f is a continuous map on D. Since X is normal and D is closed discrete in X, there exists a continuous mapping $f^* : X \to \mathbb{R}$ such that $f^*|_D = f$.

Let g be an arbitrary element of $C_p(X)$, and let W be an arbitrary open neighborhood of g in $C_p(X)$. Then there exists a finite set $F' = \{z_1, z_2, \ldots, z_l\} \subset X$ and a number $k \in \mathbb{N}$ such that $\{g' : g' \in C_p(X), |g'(z_i) - g(z_i)| < \frac{1}{k}, 1 \leq i \leq l\} \subset W$.

For every $1 \leq i \leq l$, there exists some $b_i \in (g(z_i) - \frac{1}{k}, g(z_i) + \frac{1}{k}) \cap \mathbb{Q}$. Let $h' : F' \to \mathbb{Q}$ be a map such that $h'(z_i) = b_i$ for every $1 \leq i \leq l$. Then there exists some $n \in \mathbb{N}$ such that $h' = h_n$. Thus $dom(h_n) = F' \subset F_{m_n} \subset F_{k_n}$. For every $x \in dom(h_n)$, we have $f^*(\phi_n(x)) = h_n(x)$. Then for every $x \in F'$, $|C_{\phi_n}(f^*)(x) - g(x)| = |(f^* \circ \phi_n)(x) - g(x)| = |(f^*(\phi_n(x)) - g(x)| = |h_n(x) - g(x)| = |h'(x) - g(x)| < \frac{1}{k}$. Hence

$C_{\phi_n}(f^*) \in W$. Thus $\{C_{\phi_n}(f^*) : n \in \mathbb{N}\}$ is dense in $C_p(X)$. □

Let $n \in \mathbb{N}$ and C be a compact subset of $\mathbb{Q}^n$, where $\mathbb{Q}^n$ is the nth power of $(\mathbb{Q}, d)$. Let d' be the natural metric on $\mathbb{Q}^n$. For each $1 \leq i \leq n$, the set $\pi_i(C)$ is compact in $\mathbb{Q}$, where $\pi_i : \mathbb{Q}^n \to \mathbb{Q}$ is the projection into the ith coordinate. By Remark 2, there exists some $\delta > 0$ and a map $\phi_i \in G$ such that $d(\phi_i(\pi_i(C)), \pi_i(C)) \geq \delta$ for each $i \leq n$, where G is the semigroup of one-to-one and continuous self maps of $\mathbb{Q}$. If $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ such that $\phi(x) = \langle \phi_1(x_1), \phi_2(x_2), \ldots, \phi_n(x_n) \rangle$ for each $x = \langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{Q}^n$, then ϕ is a one-to-one and continuous self map of $\mathbb{Q}^n$. Since $C \subset \prod_{1 \leq i \leq n} \pi_i(C)$ and $d(\phi_i(\pi_i(C)), \pi_i(C)) \geq \delta$ for each $i \leq n$, then $d'(\phi(C), C) \geq \sqrt{n}\delta$. Thus the action of G^* is strongly run-away on $\mathbb{Q}^n$ if G^* is the set of one-to-one and continuous self maps of $\mathbb{Q}^n$.

Corollary 4 *Let $n \in \mathbb{N}$ and G be a semigroup of continuous self maps of $\mathbb{Q}^n$ with the following properties:*

(1) Every element of G is one-to-one on $\mathbb{Q}^n$.

(2) The action of G is strongly run-away on $\mathbb{Q}^n$.

Then the action of $\hat{G}$ on $C_p(\mathbb{Q}^n)$ is topologically transitive and hypercyclic.

Remark 5 *In Theorem 3, every element of G is one-to-one and continuous and the action of G on X is strongly run-away. By Remark 2, for every $\delta > 0$ and every compact subset F of $\mathbb{Q}$, there exists a one-to-one and continuous self map ϕ of $\mathbb{Q}$ such that $d(\phi(F), F) \geq \delta$. Thus if G is the set of one-to-one and continuous self maps of $\mathbb{Q}$, then the action of $\hat{G}$ on $C_p(\mathbb{Q})$ is topologically transitive and hypercyclic.*

Corollary 6 *Let $n \in \mathbb{N}$. If G is the semigroup of one-to-one and continuous self maps of $\mathbb{Q}^n$, then the action of $\hat{G}$ on $C_p(\mathbb{Q}^n)$ is topologically transitive and hypercyclic.*

Corollary 7 *Let X be a countable discrete space. If G is the set of one-to-one and continuous self maps of X, then $C_p(X) = C_k(X)$ and the action of $\hat{G}$ on $C_p(X)$ is topologically transitive and hypercyclic.*

There exists a countable topological space X such that the action of $\hat{G}$ on $C_p(X)$ is not hypercyclic.

Proposition 8 *Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Suppose that G is the set of one-to-one and continuous self maps of X; then the action of $\hat{G}$ on $C_p(X)$ is not hypercyclic.*

Proof Let $\phi \in G$. Since ϕ is continuous and one-to-one, $\phi(0) = 0$. If $f \in C_p(X)$ and $f(0) = b$, then $C_\phi(f)(0) = (f \circ \phi)(0) = b$ for each $\phi \in G$. Then $\{f \circ \phi : \phi \in G\}$ is not dense in $C_p(X)$. Thus the action of $\hat{G}$ on $C_p(X)$ is not hypercyclic. □

If X is a finite topological space, then the set G of one-to-one and continuous self maps is finite. Thus for each $f \in C_p(X)$, the set $\{C_\phi(f) : \phi \in G\}$ is a finite subset of $C_p(X)$. Thus $\{C_\phi(f) : \phi \in G\}$ cannot be dense in $C_p(X)$. Therefore, we have

Proposition 9 *Let X be a finite topological space. If G is the set of one-to-one and continuous self maps on X, then the action of $\hat{G}$ on $C_p(X)$ is not hypercyclic.*

Since $(\mathbb{R}, d)$ is connected, every continuous function $f : \mathbb{R} \to \{0, 1\}$ is a constant map, where $\{0, 1\}$ is a discrete space. Thus we have

Proposition 10 *Let $(\mathbb{R}, d)$ be the real line with the nature topology. Suppose that G is the set of one-to-one and continuous self maps of $\mathbb{R}$; then the action of $\hat{G}$ on $C_p(\mathbb{R}, 2)$ is not hypercyclic, where $C_p(\mathbb{R}, 2) = \{f : f : \mathbb{R} \to \{0, 1\}$ is continuous$\}$.*

Theorem 11 *Let X and Y be homeomorphic topological spaces. If G_1 and G_2 are the sets of one-to-one and continuous self maps of X and Y, respectively, then the action of $\hat{G}_1$ on $C_k(X)$ is topologically transitive and hypercyclic if and only if the action of $\hat{G}_2$ on $C_k(Y)$ is topologically transitive and hypercyclic.*

Proof Let $\theta : X \to Y$ be a homeomorphism. Define $\hat{\theta} : C_k(Y) \to C_k(X)$ by

$$\hat{\theta}(f) = f \circ \theta, \ f \in C_k(Y).$$

Now we prove that $\hat{\theta}$ is a homeomorphism from $C_k(Y)$ to $C_k(X)$. Obviously, $\hat{\theta}$ is a bijection.

For any set $[K, U] \subset C_k(X)$, where $K \subset X$ is compact and U is an open subset of $\mathbb{R}$. Since θ is a homeomorphism, $\theta(K)$ is a compact set in Y. Hence $\hat{\theta}^{-1}([K, U]) = [\theta(K), U]$ and $[\theta(K), U]$ is open in $C_k(Y)$. Thus the map $\hat{\theta}$ is continuous. Let $W = [K_1, U_1] \cap [K_2, U_2] \cap \ldots \cap [K_n, U_n]$ be an arbitrary element of the base of $C_k(Y)$, since θ is a homeomorphism, $\theta^{-1}(K_i)$ is compact in X for $1 \leq i \leq n$. Therefore, $\hat{\theta}(W) = [\theta^{-1}(K_1), U_1] \cap [\theta^{-1}(K_2), U_2] \cap \ldots \cap [\theta^{-1}(K_n), U_n]$ is open in $C_k(X)$. The map $\hat{\theta}$ is an open map. Thus $\hat{\theta}$ is a homeomorphism from $C_k(Y)$ to $C_k(X)$. Then the action of $\hat{G}_1$ on $C_k(X)$ is topologically transitive and hypercyclic if and only if the action of $\hat{G}_2$ on $C_k(Y)$ is topologically transitive and hypercyclic. □

Remark 12 *Let G_1 be the set of one-to-one and continuous self maps of $\mathbb{R}$. By Theorem 1 in [8], the action of $\hat{G}_1$ on $C_k(\mathbb{R})$ is hypercyclic. Let G_2 be the set of all one-to-one and continuous self maps of $(0, 1)$. Since $(0, 1)$ and $\mathbb{R}$ are homeomorphic, the action of $\hat{G}_2$ on $C_k((0, 1))$ is hypercyclic.*

Finally, we discuss some properties of hypercyclicity in functional spaces on sums of spaces.

Theorem 13 *Let $X = \oplus_{i \in \mathbb{N}} X_i$. For each $i \in \mathbb{N}$, suppose that G_i is a semigroup of continuous self maps of X_i and $\hat{G}_i$ is the semigroup of composition operators induced by elements of G_i. If the action of $\hat{G}_i$ on $C_k(X_i)$ is hypercyclic, then there exists a semigroup G of continuous self maps of X such that the action of $\hat{G}$ on $C_k(X)$ is hypercyclic.*

Proof For each $i \in \mathbb{N}$, since $\hat{G}_i$ on $C_k(X_i)$ is hypercyclic, there exists a continuous function $f_i \in C_k(X_i)$ and a countable set $A_i = \{\phi_{i,1}, \phi_{i,2}, \ldots\}$ of continuous self maps of X_i such that $\{C_{\phi_{i,j}}(f_i) : j \in \mathbb{N}\}$ is dense in $C_k(X_i)$.

We first construct a continuous function $f \in C_k(X)$. Let $f(x) = f_i(x)$ if $x \in X_i$ for each i. Then f is well-defined and continuous on X.

For each $n \in \mathbb{N}$, let $B_n = \{\phi : \phi : X \to X$ such that for each $i \leq n$ there exists some $j \in \mathbb{N}$ such that $\phi|_{X_i} = \phi_{i,j}$ and $\phi(x) = x$ for each $x \in X_i$ if $i > n\}$. Then $|B_n| \leq \omega$. If $B = \bigcup\{B_n : n \in \mathbb{N}\}$, then $|B| \leq \omega$ and B is a subfamily of the semigroup of continuous self maps of X.

Let $g \in C_k(X)$ be an arbitrary element. Let $K \subset X$ be an arbitrary compact set and ϵ be an arbitrary positive number. We prove that there exists $\phi \in G$ such that $|C_\phi(f)(x) - g(x)|_K < \epsilon$. Since $X = \oplus_{i \in \mathbb{N}} X_i$, there exists a finite set $\{n_1, n_2, \ldots, n_s\} \subset \mathbb{N}$ such that $K \subset \bigcup\{X_{n_l} : 1 \leq l \leq s\}$ and $X_{n_l} \cap K \neq \emptyset$. Let $X_{n_l} \cap K = K_l$. For every $l \in \{1, 2, \ldots, s\}$, let $g_{n_l} = g|_{X_{n_l}}$; then g_{n_l} is continuous on X_{n_l}. Since $f|_{X_{n_l}} = f_{n_l}$ and $\{C_{\phi_{n_l,j}}(f_{n_l}) : j \in \mathbb{N}\}$ is dense in $C_k(X_{n_l})$, there exists $j_l \in \mathbb{N}$ such that $|C_{\phi_{n_l,j_l}}(f_{n_l})(x) - g_{n_l}(x)| < \epsilon$ for all $x \in K_l$. By the construction of B, there exists a $\phi \in B$ such that $\phi|_{X_{n_l}} = \phi_{n_l,j_l}$ for every $l \in \{1, 2, \ldots, s\}$. Thus we have $|C_\phi(f)(x) - g(x)| < \epsilon$ for all $x \in K$. □

By Theorem 13 and Remark 12, we have the following corollary.

Corollary 14 *Suppose that G is the set of one-to-one and continuous self maps of $\mathbb{R} \setminus \mathbb{Z}$; then the action of $\hat{G}$ on $C_k(\mathbb{R} \setminus \mathbb{Z})$ is hypercyclic.*

It is well known that if $f : \omega_1 \to \mathbb{R}$ is continuous, then there exists some $x_0 \in \omega_1$ such that $f(x) = f(x_0)$ for all $x \geq x_0$. In what follows, we prove that the action of $\hat{G}$ on $C_p(\omega_1)$ is not hypercyclic. We need the following lemma.

Lemma 15 *If $\phi : \omega_1 \to \omega_1$ is a one-to-one and continuous self map of ω_1, then for every $\alpha \in \omega_1$ there exists an $a_\alpha \in \omega_1$ such that $\phi(\beta) > \alpha$ for every $\beta > a_\alpha$.*

Proof Suppose that there exists some $\alpha \in \omega_1$ such that for every $a \in \omega_1$ there exists $\beta_a > a$ with $\phi(\beta_a) \leq \alpha$.

If $A = \{\beta_a : a \in \omega_1\}$, then A is unbounded in ω_1. Since $\phi(A) \subset [0, \alpha]$ and $|A| = \omega_1$, there are $p, q \in A$ such that $p \neq q$ and $\phi(p) = \phi(q)$. This contradicts with ϕ being one-to-one. □

Theorem 16 *Let $X = \omega_1$. If G is the set of one-to-one and continuous self maps of X, then the action of $\hat{G}$ on $C_p(X)$ is not hypercyclic.*

Proof Let $f \in C_p(X)$. Then there exists an $\alpha_f \in \omega_1$ such that $f(\alpha) = f(\alpha_f)$ for every $\alpha > \alpha_f$. Let $\{\phi_n : n \in \mathbb{N}\} \subset G$ be any sequence. For each $n \in \mathbb{N}$ there is an $a_n \in \omega_1$ such that $\phi_n(\beta) > \alpha_f$ for every $\beta > a_n$ by Lemma 15.

If $\gamma = \sup\{a_n : n \in \mathbb{N}\}$, then $\gamma \in \omega_1$. For each $n \in \mathbb{N}$ and for every $\beta > \gamma$, $C_{\phi_n}(f)(\beta) = f(\phi_n(\beta)) = f(\alpha_f)$. Then the action of $\hat{G}$ on $C_p(X)$ is not hypercyclic. □

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