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On the equivalence of Alexandrov curvature and Busemann curvature

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Abstract: It is shown that the curvature bounded above (resp. below) in the sense of Alexandrov is equivalent to the curvature bounded above (resp. below) in the sense of Busemann if and only if the sum of adjacent average angles is at least (resp. at most) π.

Key words: Alexandrov curvature, Busemann curvature, average angles

1. Introduction

It is well known that the curvature bounded above (resp. below) in the sense of Alexandrov is stronger than the curvature bounded above (resp. below) in the sense of Busemann (see, e.g., [7, p. 107] or [9, p. 57]). The classical example that shows that the converse statement does not hold is the finite dimensional normed vector space $\mathbb{R}^2$ equipped with one of the l^p-norms defined for $p > 2$ or $p < 2$. In the case where $p = 2$ these two kinds of curvatures coincide. Note also a theorem of Bridson and Haefliger [6, p. 173] saying that if the length space X is a sufficiently smooth Riemannian manifold, then Alexandrov curvature is equivalent to Busemann curvature. Little has been done, however, to detect the agreement of these two different notions of curvatures in more general spaces, for instance, a geodesic length space. Spaces bounded by Alexandrov curvature have been deeply studied by Burago, Gromov, Perel'man (see, e.g., [7, 8]), Berestovskii [2–5], etc. Many famous conjectures and open problems have been solved on Alexandrov spaces. The paramount result is Perel'man's stability theorem, which plays a role in his work on the geometrization conjecture. In contrast, the study of Busemann curvature, especially nonnegative curvature in the sense of Busemann, is surprisingly poised.

Here we strive to demonstrate a criterion to judge when those two curvatures are actually equivalent by introducing an angle called an average angle. This term is explained in the next section. The following is the main result.

Theorem (Main theorem)**.** *A geodesic length space X with curvature bounded above (resp. below) in the sense of Alexandrov is equivalent to X with curvature bounded above (resp. below) in the sense of Busemann if and only if the sum of adjacent average angles $\geq \pi$ (resp. $\leq \pi$).*

This result might provide an initial step to generalize some theorems in terms of Alexandrov curvature to ones in terms of Busemann curvature.

The strategy of the proof is greatly inspired by Theorem 4.3.5 [7, p. 116]. However, their strategy only applies for spaces with curvature defined in the sense of Alexandrov. Therefore, for spaces with curvature

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defined in the sense of Busemann, we shall extend the idea of the comparison angle in Section 2 to produce the triangle condition, the monotonicity property, and the angle property.

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2. Basic concepts

Let (X, d) be a metric space. The symbol $B(p, \delta_p)$ is used to denote the metric ball of small radius $\delta_p > 0$ centered at a point $p \in X$. We will denote by $|\cdot,\cdot|$ or $d(\cdot,\cdot)$ the distance function.

The metric space (X, d) is called an *intrinsic metric space* if for any $x, y \in X$, $\delta > 0$, there exists a finite sequence of points $z_0 = x, z_1, \ldots, z_k = y$ such that $|z_i z_{i+1}| < \delta$ $(0 \le i \le k-1)$ and $\sum_{i=0}^{k-1} |z_i z_{i+1}| < |xy| + \delta$. A *geodesic* is a curve whose length is equal to the distance between its ends. A collection of three points $p, q, r \in X$ and three geodesic pq, pr, qr is called a *triangle* in X and is denoted by Δpqr.

Fix a real number k. A *k-plane* M_k^2 is a 2-dimensional complete simply-connected Riemannian manifold of curvature k. Place a triad of points p, q, r in a space X with intrinsic metric. We associate a triangle $\tilde{\Delta} pqr$ on M_k^2 with vertices $\tilde{p}, \tilde{q}, \tilde{r}$ and sides of lengths $|\tilde{p}\tilde{q}| = |pq|$, $|\tilde{p}\tilde{r}| = |pr|$ and $|\tilde{q}\tilde{r}| = |qr|$. It is well known that for $k \le 0$, the triangle $\tilde{\Delta} pqr$ always exists and is unique up to a rigid motion; for $k > 0$, it exists by assuming that the perimeter of Δpqr is less than $2\pi/\sqrt{k}$. Let $\tilde{\measuredangle} pqr$ denote that angle (i.e. comparison angle) at the vertex $\tilde{q}$ of the triangle $\tilde{\Delta} pqr$.

Definition 2.1 (Alexandrov triangle condition)**.** *A geodesic length space (X, d) is said to be a space with curvature bounded above in the sense of Alexandrov if in some neighborhood of each point the following holds:*

For every Δpqr and every point $s \in qr$, one has $|ps| \le |\tilde{p}\tilde{s}|$ where $\tilde{s}$ is the point on the side $\tilde{q}\tilde{r}$ of a comparison triangle $\tilde{\Delta} pqr$ such that $|qs| = |\tilde{q}\tilde{s}|$.

To define the curvature bounded below in the sense of Alexandrov, just reverse the inequality above.

Definition 2.2. *A geodesic length space (X, d) is said to be a space with nonnegative curvature in the sense of Busemann if for every $p \in X$ there exists $\delta_p > 0$ such that for all $x, y, z \in B(p, \delta_p)$ and for midpoints m and n on the sides xy and xz, we have the inequality*

$$d(m, n) \ge \frac{1}{2} d(y, z).$$

In other words, for any two geodesics $\alpha : [0, a] \to X$ and $\beta : [0, b] \to X$ with $\alpha(0) = \beta(0) = x \in B(p, \delta_p)$ and with endpoints $\alpha(a), \beta(b) \in B(p, \delta_p)$, we have

$$d\left(\alpha\left(\frac{a}{2}\right), \beta\left(\frac{b}{2}\right)\right) \ge \frac{1}{2} d(\alpha(a), \beta(b)).$$

The space with nonpositive curvature in the sense of Busemann can be defined by just reversing the inequalities.

Now we give an interpretation of Busemann curvature in view of the comparison triangle.

Definition 2.3 (Busemann triangle condition)**.** *Let Δ be a geodesic triangle in a geodesic length space X that consists of three points $p, q, r \in X$. Let k be a real number.** *Let $\tilde{\Delta} \subset M_k^2$ be a comparison triangle for Δ.*

*The perimeter of Δ is less than $2\pi/\sqrt{k}$ when $k > 0$.

Then Δ is said to have Busemann curvature $\leq k$ if for any two midpoints m,n on two sides of Δ and the corresponding points $\tilde{m},\tilde{n}$ on two sides of $\tilde{\Delta}$, the inequality

$$d(m,n) \leq d(\tilde{m},\tilde{n})$$

holds.

For the definition of Busemann curvature $\geq k$, just reverse the above inequality.

The angle is defined in terms of the law of cosines.

Definition 2.4. *Given $\epsilon > 0$, let $\alpha : [0,\epsilon) \to X$ and $\beta : [0,\epsilon) \to X$ be two paths in a length space X emanating from the same point $p = \alpha(0) = \beta(0)$. We define the angle $\angle(\alpha,p,\beta)$ between α and β as*

$$\angle(\alpha,p,\beta) = \lim_{s,t\to 0} \tilde{\angle}(\alpha(s),p,\beta(t)),$$

if the limit exists, where

$$\tilde{\angle}(\alpha(s),p,\beta(t)) =: \arccos \frac{s^2+t^2-d(\alpha(s),\beta(t))^2}{2st}.$$

It is well known that the following condition is equivalent to Definition 2.1.

Definition 2.5 (Alexandrov monotonicity condition)**.** *A geodesic length space X is said to be a space with curvature bounded above (resp. below) in the sense of Alexandrov if it can be covered by neighborhoods such that, for two any shortest segments α and β contained in the neighborhood (and starting from the same point p), the corresponding function is nondecreasing (resp. nonincreasing) in each variable s and t.*

Since the existence of angles between geodesics as defined in Definition 2.5 may not be valid with respect to Busemann curvature, we introduce the following new angles based on the Busemann triangle condition.

Definition 2.6 (Average angle)**.** *Suppose X is a length space. Let $\alpha : [0,a] \to X$ and $\beta : [0,b] \to X$ be two geodesic segments with $p = \alpha(0) = \beta(0)$. The average angle between α and β at p is defined to be $\measuredangle\alpha p\beta = \lim_{n\to\infty} A_{\alpha,\beta}\left(\frac{a}{2^n},\frac{b}{2^n}\right)$ if the limit of the sequence exists, where the comparison angle*

$$\tilde{\measuredangle}\alpha p\beta = A_{\alpha,\beta}(a,b) := \arccos \frac{a^2+b^2-d(\alpha(a),\beta(b))^2}{2ab}.$$

Let q be an inner point of a shortest path pr and qs be a shortest path. *The sum of adjacent average angles is at least π* means $\measuredangle pqs + \measuredangle sqr \geq \pi$.

3. Proof of the main theorem

The proof of the main theorem depends on two properties of average angles: the Busemann monotonicity property and the Busemann angle property, which can be deduced from the Busemann triangle condition.

Lemma 3.1 (Busemann monotonicity property)**.** *The Busemann triangle condition implies the monotonicity of the sequence of average angles, that is, if X is a geodesic length space with curvature bounded above (resp. below) in the sense of Busemann, the corresponding sequence $\{A_{\alpha,\beta}\left(\frac{a}{2^n},\frac{b}{2^n}\right)\}_{n\in\mathbb{N}}$ as defined in Definition 2.6 is nonincreasing (resp. nondecreasing) with a and b remaining fixed.*

Proof Without loss of generality, it suffices to prove the case with curvature bounded above. Assume the Busemann triangle condition holds. Consider a hinge of two geodesics $\alpha : [0, a] \to X$ and $\beta : [0, b] \to X$ starting from the same point $p \in X$. Let m and n be the midpoints of α and β such that $|\tilde{p}\tilde{m}| = a/2$ and $|\tilde{p}\tilde{n}| = b/2$. Then the triangle condition implies $|mn| \leq |\tilde{m}\tilde{n}|$. This shows $\tilde{\measuredangle} mpn \leq \tilde{\measuredangle} \alpha p \beta$. Arguing inductively, it follows that the nonincreasing condition of the angles holds.

Corollary 3.1. *Let X be a geodesic length space with the Busemann curvature bounded below (resp. above). If a sequence of pairs of geodesics $\{p_iq_i, p_ir_i\}$ converge uniformly to the geodesics pq and pr, then*

$$\measuredangle qpr \leq \lim_{i\to\infty} \inf \measuredangle q_ip_ir_i \ (resp. \measuredangle qpr \geq \lim_{i\to\infty} \sup \measuredangle q_ip_ir_i).$$

Proof Let A and A_i be the angles $\measuredangle qpr$ and $\measuredangle q_ip_ir_i$. Let α, β be the corresponding geodesics such that $p = \alpha(0) = \beta(0)$ and $q = \alpha(a), r = \beta(b)$. Pick a big natural number N, and let $m \in pq, n \in pr$ and $m' \in p_iq_i, n' \in p_ir_i$ be the points at the distance $a/2^N$ and $b/2^N$ from p and p_i respectively. Denote by $\theta(N)$ and $\theta_i(N)$ the comparison angles $\tilde{\measuredangle} mpn$ and $\tilde{\measuredangle} m'p_in'$. By the uniform convergence of $\{p_iq_i, p_ir_i\}$, $\theta(N) = \lim_{i\to\infty} \theta_i(N)$ for fixed N. The definition of average angles implies $A = \lim_{N\to\infty} \theta(N)$ and $A_i = \lim_{N\to\infty} \theta_i(N)$. Apply Lemma 3.1; the sequences $\theta(n)$ and $\theta_i(n)$ are nondecreasing. Therefore, $\theta_i(N) \leq A_i$ for all N. It follows that $\theta(N) = \lim_{i\to\infty} \theta_i(N) \leq \lim_{i\to\infty} \inf A_i$. Hence, $\measuredangle qpr = A = \lim_{N\to\infty} \theta(N) \leq \lim_{i\to\infty} \inf A_i = \lim_{i\to\infty} \inf q_ip_ir_i$.

If X is of the Busemann curvature bounded above, we can prove it in a similar manner by applying the nonincreasing angular property.

Lemma 3.2 (Busemann angle property)**.** *If a geodesic length space X has curvature bounded above in the sense of Busemann, then the following average angle property holds:*

If every point for X has a neighborhood such that for every triangle Δpqr contained in this neighborhood, then average angles $\measuredangle qpr, \measuredangle rqp$, and $\measuredangle pqr$ are well defined and satisfy the inequalities

$$\measuredangle qpr \leq \tilde{\measuredangle} qpr, \measuredangle rqp \leq \tilde{\measuredangle} rqp, \measuredangle pqr \leq \tilde{\measuredangle} pqr,$$

where $\tilde{\measuredangle} qpr, \tilde{\measuredangle} rqp$, and $\tilde{\measuredangle} pqr$ are comparison angles.

Likewise, if X has curvature bounded below in the sense of Busemann, then the above inequalities will be reversed.

Proof Without loss of generality, we only prove the case with curvature bounded above. Let us consider a triangle Δpqr. Let the side pq, pr be the geodesics α, β with $p = \alpha(0) = \beta(0)$ and $\alpha(a) = q, \beta(b) = r$. By the monontonicity of average angles proved in Lemma 3.1,

$$\measuredangle qpr := \measuredangle \alpha p \beta = \lim_{n\to\infty} A_{\alpha,\beta}\left(\frac{a}{2^n}, \frac{b}{2^n}\right) \leq \tilde{\measuredangle} \alpha p \beta.$$

Proof [Proof of the main theorem] Apply the Alexandrov monotonicity condition and Theorem 2.3.2 [9, p. 57]; the necessity is clear.

Let us show the sufficiency. Assume X is geodesic length space with curvature bounded above in the sense of Busemann. Consider a triangle Δpqr and a point s in its side pr. Note that the sum of adjacent average angles is not less than π, i.e. $\measuredangle qsp + \measuredangle qsr \geq \pi$. Place the comparison triangles $\tilde{\measuredangle} psq$ and $\tilde{\measuredangle} rsq$ along the side $\tilde{q}\tilde{s}$. By Lemma 3.2, we have $\tilde{\measuredangle} psq + \tilde{\measuredangle} rsq \geq \pi$. Consider a comparison triangle $\tilde{\Delta} p_1q_1r_1$ for Δpqr

such that $|ps| = |\tilde{p_1}\tilde{s_1}|$ and $|rs| = |\tilde{r_1}\tilde{s_1}|$. Then apply Alexandrov's lemma [1], and it follows that $|\tilde{q_1}\tilde{s_1}| \geq |qs|$. Hence, the Busemann triangle condition implies the triangle condition defined in the sense of Alexandrov.

To complete the proof for spaces with Busemann curvature bounded below, it suffices to reverse the inequalities above and apply the second part of Lemma 3.2.

Corollary 3.2. *Let X be a geodesic length space with curvature bounded below in the sense of Busemann. If the sum of adjacent angles is at most π, and pa, pb, pc are geodesics, then $\measuredangle apb + \measuredangle bpc + \measuredangle cpa \leq 2\pi$.*

Proof Let $d \in ap$. By the main theorem, $\measuredangle adb + \measuredangle adc + \measuredangle bdc \leq (\measuredangle adb + \measuredangle bdp) + (\measuredangle adc + \measuredangle cdp) \leq 2\pi$. Then apply Corollary 3.1 when $d \to p$ if the geodesics pb, pc are unique. Otherwise, we replace the points b and c with points in pb and pc and repeat the argument.

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