

Some fixed point theorems for single valued strongly contractive mappings in partially ordered ultrametric and non-Archimedean normed spaces

Hamid MAMGHADERI, Hashem PARVANEH MASIHA*, Meraj HOSSEINI
Faculty of Mathematics, K. N. Toosi University of Technology, Tehran, Iran

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Abstract: Let (X, d, \preceq) be a partially ordered ultrametric space and $f : X \rightarrow X$ a single valued mapping. We obtain sufficient conditions for the existence of a fixed point for the strongly contractive mapping f . We also investigate the existence of a fixed point for strongly contractive mappings defined on partially ordered non-Archimedean normed spaces under the same conditions. Finally, we give some examples to discuss the assumptions of the theorems.

Key words: Fixed point, spherically complete ultrametric space, non-Archimedean normed space, strongly contractive mapping, partially ordered set

1. Introduction and preliminaries

Banach's contraction principle [3] is a fundamental and useful tool in mathematics. A number of authors have defined strongly contractive type mappings [8] on a complete metric space X that are generalizations of the Banach's contraction principle. Because of its simplicity it has been used in solving existence problems in many branches of mathematics [12]. Ran and Reurings [7] initiated the trend of weakening the contraction condition by considering single valued mappings on a partially ordered metric space.

In this paper, motivated by the work of Ran and Reurings[7], we introduce two new strongly contractive conditions for mappings on spherically complete ultrametric spaces and non-Archimedean normed spaces and, using these strongly contractive conditions, obtain some fixed point theorems. We first recall some basic notions in ultrametric spaces and non-Archimedean normed spaces. For more details the reader is referred to [11].

Van Rooij [11] introduced the concept of an ultrametric space as follows:

Let (X, d) be a metric space. Then (X, d) is called an ultrametric space if the metric d satisfies the strong triangle inequality, i.e. for all $x, y, z \in X$:

$$d(x, y) \leq \max\{d(x, z), d(y, z)\}.$$

In this case, d is called ultrametric. An ultrametric space (X, d) is said to be spherically complete if every shrinking collection of balls in X has a nonempty intersection. A non-Archimedean valued field is a field \mathbb{K} equipped with a function (valuation) $|\cdot|$ from \mathbb{K} into $[0, \infty)$ such that $|x| = 0$ if and only if $x = 0$, $|x + y| \leq \max\{|x|, |y|\}$ and $|xy| = |x||y|$ for all $x, y \in \mathbb{K}$. Clearly, $|1| = |-1| = 1$ and $|n \cdot 1_{\mathbb{K}}| \leq 1$ for all $n \in \mathbb{N}$.

*Correspondence: masiha@kntu.ac.ir

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An example of a non-Archimedean valuation is the mapping $|\cdot|$ taking each point of an arbitrary field but 0 into 1 and $|0| = 0$. This valuation is called trivial. The set $\{|x| : x \in \mathbb{K}, x \neq 0\}$ is a subgroup of the multiplicative group $(0, +\infty)$ and it is called the value group of the valuation. The valuation is called trivial, discrete, or dense accordingly as its value group is $\{1\}$, a discrete subset of $(0, +\infty)$, or a dense subset of $(0, \infty)$, respectively.

Definition 1 ([11]) *Let \mathbb{K} be a non-Archimedean valued field. A norm on a vector space X over \mathbb{K} is a map $\|\cdot\|$ from X into $[0, \infty)$ with the following properties:*

- 1) $\|x\| \neq 0$ if $x \in E \setminus \{0\}$;
- 2) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ ($x, y \in X$);
- 3) $\|\alpha x\| = |\alpha| \|x\|$ ($\alpha \in \mathbb{K}, x \in X$).

Since every ultrametric space is a metric space, all of the proved theorems in metric fixed point theory are established. If $f : X \rightarrow X$, then f is said to be strongly contractive if whenever x and y are distinct points in X ,

$$d(fx, fy) < d(x, y).$$

It is known that a strongly contractive mapping in a complete metric space need not have a fixed point. For more details the reader is referred to [1, 6]. Generally to prove fixed point theorems in metric spaces, for maps satisfying strongly contractive conditions, one has to assume the continuity of maps and compactness of spaces. However, in spherically complete ultrametric spaces, the continuity of maps are not necessary to obtain fixed points. In 2001, Gajic [2] proved the following fixed point theorem for a class of generalized strongly contractive mapping on ultrametric spaces.

Theorem 1 ([2]) *Let (X, d) be a spherically complete ultrametric space. If $f : X \rightarrow X$ is a mapping such that*

$$d(fx, fy) < \max\{d(x, y), d(x, fx), d(y, fy)\} \quad (x, y \in X, x \neq y),$$

then f has a unique fixed point in X .

Theorem 1 was proved in [2] using Zorn's Lemma. Kirk and Shahzad [4] give a constructive proof that seems to be more illuminating. Specifically, the conclusion holds in every ball of the form $B(x, d(x, fx))$.

Theorem 2 ([4]) *Suppose (X, d) is a spherically complete ultrametric space and suppose $f : X \rightarrow X$ is strongly contractive. Then every ball of the form $B(x, d(x, fx))$ contains a fixed point of f .*

In 1993, Petalas and Vidales [5] proved that every strongly contractive mapping on a spherically complete non-Archimedean normed space has a unique fixed point.

A very different and unexpected application of ultrametric dynamics is found in the determination of solutions of the famous Fermat equation in square matrices with entries in a p-adic field [9]. Moreover, methods of ultrametric dynamics find applications in the study of differential equations over rings of power series, as in the work of van der Hoeven, for example see his lecture notes [10]. Thus it is quite natural to consider various results of fixed point in order to address the needs of these applied sciences.

2. Main results

In this section, first we begin with the following theorem that gives the existence of a fixed point (not necessarily unique) in partially ordered ultrametric spaces for the single valued strongly contractive mappings, where a partial order on a nonempty set X is a binary relation \preceq over X satisfying the following conditions:

- 1) $x \preceq x$ for all $x \in X$ (reflexivity);
- 2) $x \preceq y$ and $y \preceq x$ imply $x = y$ for all $x, y \in X$ (antisymmetry);
- 3) $x \preceq y$ and $y \preceq z$ imply $x \preceq z$ for all $x, y, z \in X$ (transitivity).

The set X with a partial order \preceq is called a partially ordered set and it is denoted by the pair (X, \preceq) . If (X, \preceq) is a partially ordered set and $x, y \in X$, then x and y are said to be comparable elements of X if either $x \preceq y$ or $y \preceq x$.

Definition 2 Suppose (X, d, \preceq) is a partially ordered ultrametric space and $f : X \rightarrow X$ a mapping. We would say that the $B(x, r)$ is partially f -invariant if for any $u \in B(x, r)$ that u is comparable with x (with respect to \preceq),

$$fu \in B(x, r).$$

Theorem 3 Let (X, d, \preceq) be a partially ordered ultrametric space, and $f : X \rightarrow X$ satisfying the following conditions:

- (H1) If $x, y \in X$ and $x \preceq y$, then $fx \preceq fy$;
- (H2) $d(fx, fy) < d(x, y)$, for all $x, y \in X, x \preceq y, x \neq y$;
- (H3) The quadruple (X, f, d, \preceq) has the following property:

If $\{x_n\}$ is a nondecreasing sequence in X and $\{B(x_n, r_n)\}$ a nonincreasing sequence of closed balls in X , then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an upper bound $z \in X$ of the sequence $\{x_{n_k}\}$ in $\bigcap_{k=1}^{\infty} B(x_{n_k}, r_{n_k})$ such that $z \preceq fz$.

Then for any $x \in X$ with $x \preceq fx$, the closed ball $B(x, d(x, fx))$ contains a fixed point of f .

Proof We first assert that for all $z \in X$, the ball $B(z, d(z, fz))$ is partially f -invariant. To see this, let $z \in X$, $r = d(z, fz)$ and let $u \in B(z, r)$ such that u and z are comparable with respect to \preceq ; then

$$\begin{aligned} d(fu, z) &\leq \max\{d(fu, fz), d(z, fz)\} \\ &\leq \max\{d(u, z), d(z, fz)\} \\ &= r. \end{aligned}$$

Now let $x_0 \in X$ and $x_0 \preceq fx_0$. We shall show that $B(x_0, d(x_0, fx_0))$ contains a fixed point of f . Assume that f has no fixed point in $B(x_0, d(x_0, fx_0))$. Let $x_1 = x_0$, $r_1 = d(x_1, fx_1)$, and

$$\lambda_1 = \inf\{d(x, fx) \mid x \in B(x_1, r_1) \text{ and } x_1 \preceq x \preceq fx\}.$$

$f^n x_1$ belongs to $B(x_1, r_1)$ and $f^n x_1 \preceq f^{n+1} x_1$ for all $n \in \mathbb{N}$, and so $d(f^n x_1, f^{n+1} x_1) < r_1$. Thus $\lambda_1 < r_1$. Now let ε_n be a sequence of positive numbers such that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. We can choose $x_2 \in B(x_1, r_1)$ such that

$$x_1 \preceq x_2 \preceq f x_2, \quad r_2 := d(x_2, f x_2) < \min\{r_1, \lambda_1 + \varepsilon_1\}.$$

Let

$$\lambda_2 = \inf\{d(x, f x) \mid x \in B(x_2, r_2) \text{ and } x_2 \preceq x \preceq f x\}.$$

As seen above, we have $\lambda_2 < r_2$, and select $x_3 \in B(x_2, r_2)$ with

$$x_2 \preceq x_3 \preceq f x_3, \quad r_3 := d(x_3, f x_3) < \min\{r_2, \lambda_2 + \varepsilon_2\}.$$

Having defined $x_n \in X$, let

$$\lambda_n = \inf\{d(x, f x) \mid x \in B(x_n, r_n) \text{ and } x_n \preceq x \preceq f x\};$$

since we assumed that f has no fixed point in $B(x, d(x, f x))$, we have $\lambda_n < r_n$, and select $x_{n+1} \in B(x_n, r_n)$ with

$$x_n \preceq x_{n+1} \preceq f x_{n+1}, \quad r_{n+1} := d(x_{n+1}, f x_{n+1}) < \min\{r_n, \lambda_n + \varepsilon_n\}.$$

The sequence $\{x_n\}$ is nondecreasing and $\{B(x_n, r_n)\}$ is a descending sequence of balls. Thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in \bigcap_{k=1}^{\infty} B(x_{n_k}, r_{n_k})$ such that $z \preceq Tz$ and $x_{n_k} \preceq z$ for all $k \in \mathbb{N}$. Now since $\{r_n\}$ is decreasing, $r := \lim_{n \rightarrow \infty} r_n$ exists. Furthermore, λ_n is nondecreasing and bounded above, and so $\lambda := \lim_{n \rightarrow \infty} \lambda_n$ also exists. Then for each n ,

$$d(z, f z) \leq \max\{d(z, x_{n_k}), d(x_{n_k}, f z)\} \leq r_{n_k}.$$

Moreover,

$$\lambda_{n_k} \leq d(z, f z) \leq r \leq r_{n_{k+1}} \leq \lambda_{n_k} + \varepsilon_{n_k}, \quad (k \in \mathbb{N}).$$

Letting $k \rightarrow \infty$, we see that $d(z, f z) = \lambda = r$. Set

$$a = \inf\{d(x, f x) \mid x \in B(z, d(z, f z)) : z \preceq x \preceq f x\}.$$

Since $z \in B(x_{n_k}, r_{n_k})$ and $x_{n_k} \preceq z$ for all $k \in \mathbb{N}$, we conclude that if $x \in B(z, d(z, f z))$ and $z \preceq x \preceq f x$, then

$$d(x, f x) \leq d(z, f z) \leq r_{n_k};$$

hence, $a \leq r_{n_k}$. Moreover, $\lambda_{n_k} \leq a$ since every closed ball in X is partially f -invariant. Thus

$$a = \inf\{d(x, f x) \mid x \in B(z, d(z, f z)) : z \preceq x \preceq f x\} = r = d(z, f z).$$

Now $f^n z$ belongs to $B(z, d(z, f z))$ and $f^n z \preceq f^{n+1} z$ for all $n \in \mathbb{N}$. Thus

$$d(f^n z, f^{n+1} z) < d(z, f z), \quad \forall n \in \mathbb{N}$$

this is contradiction. Hence, f has a fixed point in $B(z, d(z, f z))$. □

Theorem 4 Suppose $(X, \|\cdot\|)$ is a non-Archimedean vector space over a non-Archimedean valued field \mathbb{K} and $f : X \rightarrow X$ is a mapping such that the conditions (H1, H2, H3) in Theorem 3 hold. Then for every $x \in X$ with $x \preceq fx$, the ball $B(x, \|x - fx\|)$ contains a fixed point of f .

Proof The proof of this theorem is similar to the proof of Theorem 3. □

We will give an example to show this fact where the property (H3) in Theorem 3 and 4 is necessary.

Example 1 Let \mathbb{N} be the set of all positive integers, and d be an ultrametric on \mathbb{N} defined by

$$d(m, n) := \begin{cases} 0, & m = n, \\ \max\{1 + \frac{1}{m}, 1 + \frac{1}{n}\}, & m \neq n, \end{cases}$$

and consider an order relation \preceq defined by

$$x \preceq y \Leftrightarrow ((x \text{ and } y \text{ are odd and } x \leq y) \vee (x \text{ is even and } y = x)).$$

Let k be an even positive integer and define $f : \mathbb{N} \rightarrow \mathbb{N}$ by

$$fn := \begin{cases} n + k, & n \text{ is odd,} \\ 1, & n \text{ is even.} \end{cases}$$

Then f is a nondecreasing strongly contractive mapping with respect to the defined order on \mathbb{N} and it is not difficult to show that f has the property (H1, H2) and does not satisfy the property (H3). Moreover, f has no fixed points in \mathbb{N} .

Definition 3 Let X be the space ℓ^∞ over a non-Archimedean valued field \mathbb{K} and $x, y \in X$. We say that y is a sub-member of x if $y = (0, 0, \dots, 0, x_n, 0, \dots, 0, x_m, \dots)$, where $x = (x_1, x_2, \dots)$. If y is a submember of x , then we write $y \subset x$.

Example 2 Let X be the space c_0 over a non-Archimedean valued field \mathbb{K} with the valuation of \mathbb{K} discrete and $\pi \in \mathbb{K}$ with $|\pi| > 1$. Let $z \in B(0, 1)$ be such that for each $n \in \mathbb{N}$, $z_{\pi^n} \subset z_\pi$, where $z_{\pi^n} = (\frac{z_1}{\pi^{n+1}}, \frac{z_2}{\pi^{n+2}}, \frac{z_3}{\pi^{n+3}}, \dots)$. Define

$$x \preceq y \Leftrightarrow \{(x, y \in B(0, 1), x_\pi \subset y_\pi \subset z_\pi\}$$

and for each $n \in \mathbb{N}$,

$$\{(x_{\pi^n} \subset x_\pi, y_{\pi^n} \subset y_\pi) \vee (x = y)\}.$$

Suppose $f : c_0 \rightarrow c_0$ is the mapping defined by

$$f(x) = \begin{cases} (\frac{x_1}{\pi}, \frac{x_2}{\pi^2}, \frac{x_3}{\pi^3}, \dots), & x \in B(0, 1), \\ 2x, & \text{otherwise.} \end{cases}$$

f has fixed point 0. We want to show that properties H_1, H_2 , and H_3 hold. Obviously H_1 and H_2 hold; we show that H_3 holds also. Let $\{x^n\}$ be a sequence of nondecreasing points in X and $\{B(x^n, r_n)\}$ be a nonincreasing sequence of closed balls. If for each $n \in \mathbb{N}$, $x^n = x^{n+1}$, then we are finished. Without loss of generality, we may assume that for each $n \in \mathbb{N}$, $x^n \neq x^{n+1}$. Let $i \in \mathbb{N}$ be arbitrary. If there exist $n \in \mathbb{N}$, such that $x_i^n \neq 0$, set $v_i = x_i^n$, otherwise set $v_i = 0$, and put $v = (v_1, v_2, v_3, \dots)$. For each $n \leq m \in \mathbb{N}$, $\|x^n - x^m\| \leq r_n$, because $\{B(x^n, r_n)\}$ is a family of descending balls. We claim that $\|x^n - v\| \leq r_n$. Suppose

there exists $n \in \mathbb{N}$ such that $\|x^n - v\| > r_n$. Therefore, there exists $i \in \mathbb{N}$ such that $|x_i^n - v_i| > r_n$. By definition of v there exists $m > n$ such that $v_i = x_i^m$. Thus $|x_i^n - x_i^m| > r_n$; this is a contradiction, because $\|x^n - x^m\| \leq r_n$. Therefore, $v \in \cap B(x^n, r_n)$. Because for each $n \in \mathbb{N}$, $x^n \subset x_\pi^n \subset z_\pi$ and therefore $v \subset v_\pi \subset z_\pi$ and since $x_{\pi^m}^n \subset x^n$ for each $m, n \in \mathbb{N}$, and so $v \subset v_{\pi^m}$, for each $m \in \mathbb{N}$, and so $v \preceq fv$ and H_3 holds.

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