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On the attached prime ideals of local cohomology modules defined by a pair of ideals

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Abstract: Let I and J be two ideals of a commutative Noetherian ring R and M be an R-module of dimension d. For each $i \in \mathbb{N}_0$ let $\mathrm{H}^i_{I,J}(-)$ denote the i-th right derived functor of $\Gamma_{I,J}(-)$, where $\Gamma_{I,J}(M) := \{x \in M : I^n x \subseteq Jx \text{ for } n \gg 1\}$. If R is a complete local ring and M is finite, then attached prime ideals of $\mathrm{H}^{d-1}_{I,J}(M)$ are computed by means of the concept of co-localization. Moreover, we illustrate the attached prime ideals of $\mathrm{H}^t_{I,J}(M)$ on a nonlocal ring R, for $t = \dim M$ and $t = \mathrm{cd}\,(I,J,M)$, where $\mathrm{cd}\,(I,J,M)$ is the last nonvanishing level of $\mathrm{H}^i_{I,J}(M)$.

Key words: Local cohomology modules with respect to a pair of ideals, attached prime ideals, co-localization

1. Introduction

Throughout this paper, R denotes a commutative Noetherian ring and M an R-module and I and J stand for two ideals of R. For all $i \in \mathbb{N}_0$ the i-th local cohomology functor with respect to (I,J), denoted by $\mathrm{H}^i_{I,J}(-)$, is defined by Takahashi et al. in [11] as the i-th right derived functor of the (I,J)-torsion functor $\Gamma_{I,J}(-)$, where

$$\Gamma_{I,J}(M) := \{x \in M : I^n x \subseteq Jx \text{ for } n \gg 1\}.$$

This notion coincides with the ordinary local cohomology functor $\mathrm{H}^i_I(-)$ when $J = 0$ (see [3]).

The main motivation for this generalization comes from the study of a dual of ordinary local cohomology modules $\mathrm{H}^i_I(M)$ ([10]). Basic facts and more information about local cohomology defined by a pair of ideals can be obtained from [4, 5, 11].

The second section of this paper is devoted to studying the attached prime ideals of local cohomology modules with respect to a pair of ideals by means of co-localization. The concept of co-localization was introduced by Richardson in [9].

Let $(R,\mathfrak{m})$ be local and M be a finite R-module of dimension d. If c is a nonnegative integer such that $\mathrm{H}^i_{I,J}(R) = 0$ for all $i > c$ and $\mathrm{H}^c_{I,J}(R)$ is representable, then we illustrate the attached prime ideals of ${}^{\mathfrak{p}}\mathrm{H}^c_{I,J}(M)$ (see Theorem 2.6). In addition, if R is complete, then we make use of Theorem 2.6 to prove that in a special case

$$\mathrm{Att}\,(\mathrm{H}^{d-1}_{I,J}(M)) \subseteq T \cup \mathrm{Assh}\,(M) \quad \text{and} \quad T \subseteq \mathrm{Att}\,(\mathrm{H}^{d-1}_{I,J}(M)),$$

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where

$$T = \{\mathfrak{p} \in \operatorname{Supp}(M) : \dim M/\mathfrak{p}M = d-1, J \subseteq \mathfrak{p} \text{ and } \sqrt{I+\mathfrak{p}} = \mathfrak{m}\},$$

(see Theorem 2.8).

In [4, Theorem 2.1] the set of attached prime ideals of $\mathrm{H}_{I,J}^{\dim M}(M)$ was computed on a local ring. We generalize this theorem to the nonlocal case (see Proposition 2.13). The authors in [6, 2.4] specified a subset of attached prime ideals of ordinary top local cohomology module $\mathrm{H}_I^{\mathrm{cd}\,(I,M)}(M)$. We improve it for $\mathrm{H}_{I,J}^{\mathrm{cd}\,(I,J,M)}(M)$ over a not necessarily local ring, where $\mathrm{cd}\,(I,J,M) = \sup\{i \in \mathbb{N}_0 : \mathrm{H}_{I,J}^i(M) \neq 0\}$ with the convention that $\mathrm{cd}\,(I,M) = \mathrm{cd}\,(I,0,M)$ (see Theorem 2.15).

2. Attached prime ideals

In this section we study the set of attached prime ideals of local cohomology modules with respect to a pair of ideals.

Remark 2.1 *Following [9], for a multiplicatively closed subset S of the local ring $(R,\mathfrak{m})$, the co-localization of the R-module M relative to S is defined to be the $S^{-1}R$-module $S_{-1}(M) := D_{S^{-1}R}(S^{-1}D_R(M))$, where $D_R(-)$ is the Matlis dual functor* $\mathrm{Hom}\,_R(-,\mathrm{E}_R(R/\mathfrak{m}))$*. If $S = R\setminus\mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec}(R)$ we write ${}^{\mathfrak{p}}M$ for $S_{-1}(M)$.*

Richardson in [9, 2.2] proved that if M is a representable R-module, then so is $S_{-1}(M)$ and $\mathrm{Att}\,(S_{-1}M) = \{S^{-1}\mathfrak{p} : \mathfrak{p} \in \mathrm{Att}\,(M), \mathfrak{p}\cap S = \emptyset\}$*. Therefore, in order to get some results about attached prime ideals of a module, it is convenient to study the attached prime ideals of the co-localization of it.*

Lemma 2.2 *Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ be an ideal of R, and $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\mathfrak{a} \subseteq \mathfrak{p}$. Let $R' = R/\mathfrak{a}$ and $\mathfrak{p}' = \mathfrak{p}/\mathfrak{a}$. Then for any R'-module X and $R'_{\mathfrak{p}'}$-module Y, the following isomorphisms hold:*

(i) $D_R(X) \cong D_{R'}(X)$ *as R-modules.*

(ii) $D_R(X)_{\mathfrak{p}} \cong D_{R'}(X)_{\mathfrak{p}'}$ *as $R_{\mathfrak{p}}$-modules.*

(iii) $D_{R_{\mathfrak{p}}}(Y) \cong D_{R'_{\mathfrak{p}'}}(Y)$ *as $R_{\mathfrak{p}}$-modules.*

Proof (i) By using [3, Lemma 10.1.15], we get

$$\begin{aligned} \mathrm{Hom}\,_R(X,\mathrm{E}_R(R/\mathfrak{m})) &\cong \mathrm{Hom}\,_R(X\otimes_{R'} R', \mathrm{E}_R(R/\mathfrak{m})) \\ &\cong \mathrm{Hom}\,_{R'}(X, \mathrm{Hom}\,_R(R', \mathrm{E}_R(R/\mathfrak{m}))) \\ &\cong \mathrm{Hom}\,_{R'}(X, (0 :_{\mathrm{E}_R(R/\mathfrak{m})} \mathfrak{a})) \\ &\cong \mathrm{Hom}\,_{R'}(X, \mathrm{E}_{R'}(R/\mathfrak{m})). \end{aligned}$$

(ii) It is clear, by (i).

(iii) It is enough to apply (i), by substituting R and X with $R_{\mathfrak{p}}$ and Y, respectively.

□

The following lemmas will be used in the rest of the paper.

Lemma 2.3 *Let T be an R-linear covariant right exact functor on the category of R-modules and R-homomorphisms. Then there is a natural equivalence of functors, $T(-) \cong T(R)\otimes_R -$ on the category of finite R-modules.*

Proof See [3, 6.1.8]. □

Lemma 2.4 *Let* $(R, \mathfrak{m})$ *be a local ring and* M *be a nonzero finite* R*-module of dimension* d*. Then the following statements are equivalent:*

(*i*) $H^d_{I,J}(M) = 0$,

(*ii*) $\dim \hat{R}/(I\hat{R}+\mathfrak{p}) > 0$, *for any prime ideal* $\mathfrak{p}$ *of* $\operatorname{Supp}_{\hat{R}}(\hat{R}\otimes_R M/JM)$ *satisfying* $\dim \hat{R}/\mathfrak{p} = d$, *where* $\hat{R}$ *denotes the* $\mathfrak{m}-$ *adic completion of* R.

Proof See [4, Theorem 2.4]. □

Lemma 2.5 *Let* M *and* N *be two* R*-modules such that* M *has a secondary representation. Then* $M \otimes_R N$ *has a secondary representation and* $\operatorname{Att}_R(M \otimes_R N) \subseteq \operatorname{Att}_R(M) \cap \operatorname{Supp}_R(N)$.

Proof See [12, 3.1]. □

In [8, 2.1 and 2.2] the following theorems have been proved for the attached prime ideals of $\mathrm{H}^d_I(R)$ and $\mathrm{H}^{d-1}_I(R)$, where $d = \dim R$. Here we generalize these theorems for the local cohomology modules of M with respect to a pair of ideals when M is a finite R-module with $\dim M = d$.

Theorem 2.6 *Let* $(R, \mathfrak{m})$ *be a local ring,* M *be a finite* R*-module, and* $\mathfrak{p} \in \operatorname{Spec}(R)$*. Assume that* $c = \operatorname{cd}(I, J, R)$ *and* $H^c_{I,J}(R)$ *is representable. Then*

1. $\operatorname{Att}_{R_\mathfrak{p}}({}^{\mathfrak{p}}H^c_{I,J}(M)) \subseteq \{\mathfrak{q}R_\mathfrak{p} : \dim M/\mathfrak{q}M \geq c,\ \mathfrak{q} \subseteq \mathfrak{p},$ *and* $\mathfrak{q} \in \operatorname{Spec}(R)\}$.
2. *If* R *is complete, then*

$$\operatorname{Att}_{R_\mathfrak{p}}({}^{\mathfrak{p}}H^{\dim M}_{I,J}(M)) = \{ \mathfrak{q}R_\mathfrak{p} : \mathfrak{q} \in \operatorname{Supp}(M), \dim M/\mathfrak{q}M = \dim M, J \subseteq \mathfrak{q} \subseteq \mathfrak{p}, \text{ and } \sqrt{I+\mathfrak{q}} = \mathfrak{m}\}.$$

Proof

(1) Let $\mathfrak{q}R_\mathfrak{p} \in \operatorname{Att}_{R_\mathfrak{p}}({}^{\mathfrak{p}}\mathrm{H}^c_{I,J}(M))$. By Lemma 2.5 and Remark 2.1, we have $\mathrm{H}^c_{I,J}(M)$ is representable and $\operatorname{Att}_{R_\mathfrak{p}}({}^{\mathfrak{p}}\mathrm{H}^c_{I,J}(M)) = \{\mathfrak{q}R_\mathfrak{p} : \mathfrak{q} \in \operatorname{Att}(\mathrm{H}^c_{I,J}(M))$ *and* $\mathfrak{q} \subseteq \mathfrak{p}\}$. Moreover, using Lemma 2.3 and [1, 2.11]

$$\operatorname{Att}(\mathrm{H}^c_{I,J}(M/\mathfrak{q}M)) = \operatorname{Att}(\mathrm{H}^c_{I,J}(M)) \cap \operatorname{Supp}(R/\mathfrak{q}).$$

This implies that $\mathrm{H}^c_{I,J}(M/\mathfrak{q}M) \neq 0$ and consequently $\dim M/\mathfrak{q}M \geq c$.

(2) Let $\mathfrak{p} \in \operatorname{Supp}(M)$. Put $d := \dim M$, $\overline{R} = R/\operatorname{Ann}_R M$, and

$$T := \{\mathfrak{q}R_\mathfrak{p} : \mathfrak{q} \in \operatorname{Supp}(M), \dim M/\mathfrak{q}M = d, J \subseteq \mathfrak{q} \subseteq \mathfrak{p} \text{ and } \sqrt{I+\mathfrak{q}} = \mathfrak{m}\}.$$

Since $\dim_{\overline{R}} M = \dim_R M$, [11, 2.7] and Lemma 2.2 imply that ${}^{\overline{\mathfrak{p}}}\mathrm{H}^d_{I\overline{R},J\overline{R}}(M) \cong {}^{\mathfrak{p}}\mathrm{H}^d_{I,J}(M)$, as $R_\mathfrak{p}$-modules. Therefore, by [3, 8.2.5], $\mathfrak{q} \in \operatorname{Att}_{\overline{R}_{\overline{\mathfrak{p}}}}({}^{\overline{\mathfrak{p}}}\mathrm{H}^d_{I\overline{R},J\overline{R}}(M))$ if and only if

$$\mathfrak{q} \cap R_\mathfrak{p} \in \operatorname{Att}_{R_\mathfrak{p}}({}^{\overline{\mathfrak{p}}}\mathrm{H}^d_{I\overline{R},J\overline{R}}(M)) = \operatorname{Att}_{R_\mathfrak{p}}({}^{\mathfrak{p}}\mathrm{H}^d_{I,J}(M)).$$

Now, without loss of generality, we may assume that M is faithful and $\dim R = d$. If $\mathrm{H}^d_{I,J}(M) = 0$, then $\mathrm{Att}_{R_\mathfrak{p}}({}^\mathfrak{p}\mathrm{H}^d_{I,J}(M)) = \emptyset$. Assume that $T \neq \emptyset$ and $\mathfrak{q}R_\mathfrak{p} \in T$. Since $\dim M/\mathfrak{q}M = \dim R$, we have $\dim R/\mathfrak{q} = d$. On the other hand, $\mathfrak{q} \in \mathrm{Supp}\,(M/JM)$. Thus, by Lemma 2.4, $\dim R/(I+\mathfrak{q}) > 0$, which contradicts $\sqrt{I+\mathfrak{q}} = \mathfrak{m}$. Hence $T = \emptyset$.

Now we assume that $\mathrm{H}^d_{I,J}(M) \neq 0$.

$\supseteq$: Let $\mathfrak{q}R_\mathfrak{p} \in T$. Since $\mathrm{H}^d_{I,J}(M)$ is an Artinian R-module (see [5, 2.1]), by Remark 2.1, it is enough to show that $\mathfrak{q} \in \mathrm{Att}\,(\mathrm{H}^d_{I,J}(M))$. As $M/\mathfrak{q}M$ is J-torsion with dimension d and $\sqrt{I+\mathfrak{q}} = \mathfrak{m}$, by [3, 4.2.1 and 6.1.4],

$$\mathrm{H}^d_{I,J}(M/\mathfrak{q}M) \cong \mathrm{H}^d_{I}(M/\mathfrak{q}M) \cong \mathrm{H}^d_{I(R/\mathfrak{q})}(M/\mathfrak{q}M) \cong \mathrm{H}^d_{\mathfrak{m}/\mathfrak{q}}(M/\mathfrak{q}M) \neq 0.$$

Hence Lemma 2.3 and [1, 2.11] imply that $\emptyset \neq \mathrm{Att}\,(H^d_{I,J}(M/\mathfrak{q}M)) = \mathrm{Att}\,(\mathrm{H}^d_{I,J}(M)) \cap \mathrm{Supp}\,(R/\mathfrak{q})$. Let $\mathfrak{q}_0 \in \mathrm{Att}\,(\mathrm{H}^d_{I,J}(M))$ be such that $\mathfrak{q} \subset \mathfrak{q}_0$. Thus $\dim M/\mathfrak{q}_0 M < d$. On the other hand, by Remark 2.1, $\mathfrak{q}_0 R_{\mathfrak{q}_0} \in \mathrm{Att}_{R_{\mathfrak{q}_0}}({}^{\mathfrak{q}_0}\mathrm{H}^d_{I,J}(M))$ and this implies that $\dim M/\mathfrak{q}_0 M \geq d$, which is a contradiction. Therefore, $\mathfrak{q} = \mathfrak{q}_0$.

$\subseteq$: Let $\mathfrak{q}R_\mathfrak{p} \in \mathrm{Att}_{R_\mathfrak{p}}({}^\mathfrak{p}\mathrm{H}^d_{I,J}(M))$. As we have seen in the proof of part (1), $\dim M/\mathfrak{q}M = d$ and $\mathfrak{q} \subseteq \mathfrak{p}$. Thus, by [11, 2.7],

$$\mathrm{H}^d_{IR/\mathfrak{q},JR/\mathfrak{q}}(M/\mathfrak{q}M) \cong \mathrm{H}^d_{I,J}(M/\mathfrak{q}M) \neq 0.$$

Now, by Lemma 2.4, there exists $\mathfrak{r}/\mathfrak{q} \in \mathrm{Supp}\,(R/\mathfrak{q} \otimes_{R/\mathfrak{q}} \frac{M/\mathfrak{q}M}{(JR/\mathfrak{q})(M/\mathfrak{q}M)})$ such that $\dim \frac{R/\mathfrak{q}}{\mathfrak{r}/\mathfrak{q}} = d$ and $\dim \frac{R/\mathfrak{q}}{IR/\mathfrak{q}+\mathfrak{r}/\mathfrak{q}} = 0$. Since $\mathfrak{q}R_\mathfrak{p} \in \mathrm{Att}_{R_\mathfrak{p}}({}^\mathfrak{p}\mathrm{H}^d_{I,J}(M))$, we have $\mathfrak{q} \in \mathrm{Att}\,(\mathrm{H}^d_{I,J}(M))$ and so $\mathfrak{q} \in \mathrm{Supp}\,(M) \cap V(J)$. Hence $\mathfrak{q}/\mathfrak{q} \in \mathrm{Supp}_{R/\mathfrak{q}}(M/\mathfrak{q}M)$ and then

$$\dim R/\mathfrak{q} = \dim M/\mathfrak{q}M = d = \dim \frac{R/\mathfrak{q}}{\mathfrak{r}/\mathfrak{q}} = \dim R/\mathfrak{r}.$$

Therefore, $\dim R/\mathfrak{q} = \dim R/\mathfrak{r}$, which shows that $\mathfrak{q} = \mathfrak{r}$. Thus $\sqrt{I+\mathfrak{q}} = \mathfrak{m}$. □

Remark 2.7 *The inclusion in Theorem 2.6 (1) is not an equality in general. Let the assumption be as in Theorem 2.6. Assume that $H^d_{I,J}(M) = 0$, $\mathfrak{p} \in Min\,(M)$, and $\dim M/\mathfrak{p}M = d$. Then $\mathrm{Att}_{R_\mathfrak{p}}({}^\mathfrak{p}H^d_{I,J}(M)) = \emptyset$. However,*

$$\{\mathfrak{q}R_\mathfrak{p} : \dim M/\mathfrak{q}M = d, \mathfrak{q} \subseteq \mathfrak{p} \text{ and } \mathfrak{q} \in \mathrm{Supp}\,(M)\} = \{\mathfrak{p}R_\mathfrak{p}\}.$$

Theorem 2.8 *Let $(R,\mathfrak{m})$ be a complete local ring and M be a finite R-module with dimension d. Assume that $H^i_{I,J}(R) = 0$ for all $i > d-1$ and $H^{d-1}_{I,J}(R)$ is representable. Then*

1.

$$\mathrm{Att}_R(H^{d-1}_{I,J}(M)) \subseteq \{\mathfrak{p} \in \mathrm{Supp}\,(M) : \dim M/\mathfrak{p}M = d-1, J \subseteq \mathfrak{p} \text{ and } \sqrt{I+\mathfrak{p}} = \mathfrak{m}\} \cup \mathrm{Assh}\,(M).$$

2.
$$\{\mathfrak{p} \in \operatorname{Supp}(M) : \dim M/\mathfrak{p}M = d-1, J \subseteq \mathfrak{p} \text{ and } \sqrt{I+\mathfrak{p}} = \mathfrak{m}\} \subseteq \operatorname{Att}(H^{d-1}_{I,J}(M)).$$

Proof (1) In the case where $\mathrm{H}^{d-1}_{I,J}(M) = 0$ there is nothing to say. Therefore, we assume that $\mathrm{H}^{d-1}_{I,J}(M) \neq 0$. Note that, by [11, 4.8] and Lemma 2.5, $\mathrm{H}^{d-1}_{I,J}(M)$ is representable and $\operatorname{Att}(\mathrm{H}^{d-1}_{I,J}(M)) \subseteq \operatorname{Supp}(M)$. Now let $\mathfrak{p} \in \operatorname{Att}(\mathrm{H}^{d-1}_{I,J}(M))$. Since $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}\mathrm{H}^{d-1}_{I,J}(M))$, by Theorem 2.6 (1), $\dim M/\mathfrak{p}M \geq d-1$.

If $\dim M/\mathfrak{p}M = d$, then $\dim R/\mathfrak{p} = d$ and so $\mathfrak{p} \in \operatorname{Assh}(M)$.

Now assume that $\dim M/\mathfrak{p}M = d-1$. Since $\mathfrak{p} \in \operatorname{Att}(\mathrm{H}^{d-1}_{I,J}(M))$, $\mathrm{H}^{d-1}_{IR/\mathfrak{p},JR/\mathfrak{p}}(M/\mathfrak{p}M) \cong \mathrm{H}^{d-1}_{I,J}(M/\mathfrak{p}M) \neq 0$. Thus, by Lemma 2.4, there exists $\mathfrak{r}/\mathfrak{p} \in \operatorname{Supp}(\frac{M/\mathfrak{p}M}{(JR/\mathfrak{p})(M/\mathfrak{p}M)})$ such that $\dim \frac{R}{\mathfrak{r}} = d-1$ and $\dim \frac{R}{I+\mathfrak{r}} = 0$. Hence $\mathfrak{r} = \mathfrak{p}, J \subseteq \mathfrak{p}$, and $\sqrt{I+\mathfrak{p}} = \mathfrak{m}$.

(2) Let $\mathfrak{p} \in \operatorname{Supp}(M)$, $J \subseteq \mathfrak{p}$, $\dim M/\mathfrak{p}M = d-1$, and $\sqrt{I+\mathfrak{p}} = \mathfrak{m}$. Then, by Lemma 2.5 and Theorem 2.6 (2), $\mathrm{H}^{d-1}_{I,J}(M)$ is representable, $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}\mathrm{H}^{d-1}_{I,J}(M/\mathfrak{p}M))$, and so $\mathfrak{p} \in \operatorname{Att}(\mathrm{H}^{d-1}_{I,J}(M/\mathfrak{p}M))$. Now the proof is completed by considering the epimorphism
$\mathrm{H}^{d-1}_{I,J}(M) \to \mathrm{H}^{d-1}_{I,J}(M/\mathfrak{p}M)$.

□

In the rest of the paper, following [11], we use the notation
$$W(I,J) := \{\mathfrak{p} \in Spec(R) : I^n \subseteq \mathfrak{p} + J \text{ for an integer } n \geq 1\}$$
and
$$\widetilde{W}(I,J) := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of } R; I^n \subseteq \mathfrak{a} + J \text{ for an integer } n \geq 1\}.$$
The following lemma can be proved using [11, 3.2].

Lemma 2.9 *For any nonnegative integer i and R-module M,*

(*i*) $\operatorname{Supp}(H^i_{I,J}(M)) \subseteq \bigcup_{\mathfrak{a} \in \widetilde{W}(I,J)} \operatorname{Supp}(H^i_{\mathfrak{a}}(M))$.

(*ii*) $\operatorname{Supp}(H^i_{I,J}(M)) \subseteq \operatorname{Supp}(M) \cap W(I,J)$.

Proof

(*ii*) By (*i*), $\operatorname{Supp}(\mathrm{H}^i_{I,J}(M)) \subseteq \operatorname{Supp}(M) \cap V(I) \subseteq \operatorname{Supp}(M) \cap W(I,J)$. □

Corollary 2.10 *Let M be an R-module and $c = \operatorname{cd}(I,J,R)$. Assume that M is representable or $H^c_{I,J}(R)$ is finite. Then*
$$\operatorname{Att}(H^c_{I,J}(M)) \subseteq \operatorname{Att}(M) \cap W(I,J).$$

Proof

By [11, 4.8], [1, 2.11], Lemma 2.5, and Lemma 2.9 (*ii*), we have
$$\begin{aligned}\operatorname{Att}(\mathrm{H}^c_{I,J}(M)) &= \operatorname{Att}(M \otimes \mathrm{H}^c_{I,J}(R)) \subseteq \operatorname{Att}(M) \cap \operatorname{Supp}(H^c_{I,J}(R)) \\ &\subseteq \operatorname{Att}(M) \cap W(I,J).\end{aligned}$$

□

Applying the set of attached prime ideals of the top local cohomology module in [4, Theorem 2.1], we obtain another presentation for it.

Proposition 2.11 *Let* $(R,\mathfrak{m})$ *be a local ring and* $\hat{R}$ *denotes the* $\mathfrak{m}-$ *adic completion of* R. *Suppose that* M *is a finite* R*-module of dimension* d. *Then*

$$\begin{aligned}\mathrm{Att}_{R}(H^d_{I,J}(M)) = \{ & \mathfrak{q}\cap R : \mathfrak{q}\in \mathrm{Supp}_{\hat{R}}(\hat{R}\otimes_R M/JM), \dim(\hat{R}/\mathfrak{q}) = d, \\ & \textit{and } \dim \hat{R}/(I\hat{R}+\mathfrak{q}) = 0\}.\end{aligned}$$

Proof Denote the set on the right-hand side of the desired equality by T. It is clear that by Lemma 2.4, $\mathrm{H}^d_{I,J}(M) = 0$ if and only if $T = \emptyset$. Assume that $\mathrm{H}^d_{I,J}(M) \neq 0$ and $\mathfrak{p}\in \mathrm{Supp}(M/JM)$ with the property that $\mathrm{cd}(I, R/\mathfrak{p}) = d$. Let $\mathfrak{q}\in \mathrm{Ass}(M/JM)$ be such that $\mathfrak{q}\subseteq \mathfrak{p}$. Then

$$d = \mathrm{cd}(I, R/\mathfrak{p}) \leq \mathrm{cd}(I, R/\mathfrak{q}) \leq \dim R/\mathfrak{q} \leq \dim M/JM \leq \dim M = d$$

implies that $\mathfrak{p} = \mathfrak{q} \in \mathrm{Ass}(M/JM)$ and $\dim M/JM = d$. Now the claim follows from [12, 3.10] and [4, Theorem 2.1].

□

The following lemma, which can be proved by using an argument similar to the proof of [11, 4.3], will be applied in the rest of the paper.

Lemma 2.12 *Let* M *be a finite* R*-module. Suppose that* $J\subseteq J(R)$, *where* $J(R)$ *denotes the Jacobson radical of* R, *and let* $d = \dim M/JM$. *Then* $H^i_{I,J}(M) = 0$ *for all* $i > d$.

Using Lemma 2.12, we can compute $\mathrm{Att}(H^{\dim M}_{I,J}(M))$ in the nonlocal case as a generalization of [7, 2.5].

Proposition 2.13 *Let* M *be a finite* R*-module of dimension* d *and* $J\subseteq J(R)$. *Then* $H^d_{I,J}(M)$ *is an Artinian* R*-module and*

$$\begin{aligned}\mathrm{Att}(H^d_{I,J}(M)) &= \mathrm{Att}(H^d_I(M/JM)) \\ &= \{\mathfrak{p}\in \mathrm{Ass}(M)\cap V(J) : \mathrm{cd}(I, R/\mathfrak{p}) = d\}.\end{aligned}$$

Proof In view of Lemma 2.12, [11, 4.3] holds for the nonlocal case. Now the assertion follows by applying the same method of the proofs of [5, 2.1] and [4, Theorem 2.1 and Proposition 2.1]. □

Corollary 2.14 *Suppose that* $J\subseteq J(R)$ *and* M *is a finite* R*-module such that* $\dim M = d$. *Then*

$$\mathrm{Att}\left(\frac{H^d_{I,J}(M)}{JH^d_{I,J}(M)}\right) = \{\mathfrak{p}\in \mathrm{Supp}(M)\cap V(J) : \mathrm{cd}(I, R/\mathfrak{p}) = d\}.$$

Proof Let $\overline{R} = R/\mathrm{Ann}_R M$. Using [11, 2.7], $\mathrm{H}^d_{I,J}(M)\cong \mathrm{H}^d_{I\overline{R},J\overline{R}}(M)$ and also for a prime $\mathfrak{p}\in \mathrm{Supp}(M)\cap V(J)$, $\mathrm{cd}(I\overline{R}, \overline{R}/\mathfrak{p}) = \mathrm{cd}(I, R/\mathfrak{p})$. Thus we may assume that M is faithful and so $\dim R = d$. By virtue of Lemma 2.3, $\mathrm{H}^d_I(M/JM)\cong \mathrm{H}^d_{I,J}(M/JM)\cong \frac{\mathrm{H}^d_{I,J}(M)}{J\mathrm{H}^d_{I,J}(M)}$. Now the assertion follows by Proposition 2.13. □

The final result of this section is a generalization of [6, 2.4] in the nonlocal case for local cohomology modules with respect to a pair of ideals. Since this reference is unpublished, we provide the proof of [6, 2.4] here (with the needed changes) for the reader's convenience.

Theorem 2.15 *Let* $J \subseteq J(R)$ *and* M *be a finite* R*-module. Then*

$$\{\mathfrak{p} \in \mathrm{Ass}(M) \cap V(J) : \mathrm{cd}\,(I, R/\mathfrak{p}) = \dim\, R/\mathfrak{p} = \mathrm{cd}\,(I, J, M)\} \subseteq \mathrm{Att}\,(H_{I,J}^{\mathrm{cd}\,(I,J,M)}(M)).$$

Equality holds if $\mathrm{cd}\,(I, J, M) = \dim\, M$*.*

Proof Let T be the set on the left-hand side of the desired inclusion. Suppose that $S := \{\mathfrak{p} \in \mathrm{Ass}(M) \cap V(J) : \dim\,(R/\mathfrak{p}) = \mathrm{cd}\,(I, J, M)\}$ is nonempty. By [2, page 263, Proposition 4], there exists a submodule N of M with $\mathrm{Ass}(N) = \mathrm{Ass}(M) \backslash S$ and $\mathrm{Ass}(M/N) = S$. Since $\mathrm{Supp}\,(N) \subseteq \mathrm{Supp}\,(M)$, by virtue of [5, 3.2], we have $\mathrm{cd}\,(I, J, N) \leq \mathrm{cd}\,(I, J, M)$, and so there is an exact sequence

$$\mathrm{H}_{I,J}^{\mathrm{cd}\,(I,J,M)}(N) \to \mathrm{H}_{I,J}^{\mathrm{cd}\,(I,J,M)}(M) \to \mathrm{H}_{I,J}^{\mathrm{cd}\,(I,J,M)}(M/N) \to 0.$$

It follows that $\mathrm{Att}\,(\mathrm{H}_{I,J}^{\mathrm{cd}\,(I,J,M)}(M/N)) \subseteq \mathrm{Att}\,(\mathrm{H}_{I,J}^{\mathrm{cd}\,(I,J,M)}(M))$. Therefore, it is sufficient to show that $T = \mathrm{Att}\,(\mathrm{H}_{I,J}^{\mathrm{cd}\,(I,J,M)}(M/N))$. Note that $\dim\, R/\mathfrak{p} = \mathrm{cd}\,(I, J, M)$ for all $\mathfrak{p} \in \mathrm{Ass}(M/N)$. Thus, $\mathrm{cd}\,(I, J, M) = \dim\, M/N$, and hence $\mathrm{H}_{I,J}^{\mathrm{cd}\,(I,J,M)}(M/N) = \mathrm{H}_{I,J}^{\dim\, M/N}(M/N)$ is an Artinian R-module and $\mathrm{Att}\,(\mathrm{H}_{I,J}^{\mathrm{cd}\,(I,J,M)}(M/N)) = T$, by Proposition 2.13.

For the special case, $\mathrm{cd}\,(I, J, M) = \dim\, M$, again see Proposition 2.13. □

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