Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 15 – 26
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doi:10.3906/mat-1410-35

Extensions of quasipolar rings

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Received: 14.10.2014 • **Accepted/Published Online:** 01.03.2016 • **Final Version:** 16.01.2017

Abstract: An associative ring with identity is called *quasipolar* provided that for each $a \in R$ there exists an idempotent $p \in R$ such that $p \in comm^2(a)$, $a+p \in U(R)$ and $ap \in R^{qnil}$. In this article, we introduce the notion of quasipolar general rings (with or without identity). Some properties of quasipolar general rings are investigated. We prove that a general ring I is quasipolar if and only if every element $a \in I$ can be written in the form $a = s + q$ where s is strongly regular, $s \in comm^2(a)$, q is quasinilpotent, and $sq = qs = 0$. It is shown that every ideal of a quasipolar general ring is quasipolar. Particularly, we show that R is pseudopolar if and only if R is strongly π-rad clean and quasipolar.

Key words: Quasipolar general rings, strongly clean general rings, strongly π-regular general rings, (generalized) Drazin inverse, pseudopolar rings

1. Introduction

Throughout this paper, a ring means an associative ring with identity and a general ring means an associative ring with or without identity. For clarity, R and S will always denote rings, and I and A denote general rings. The notation $U(R)$ denotes the group of units of R, $J(I)$ denotes the Jacobson radical of I, and $Nil(I)$ denotes the set of all nilpotent elements of I. The *commutant* and *double commutant* of an element a in a ring R are defined by $comm_R(a) = \{x \in R \mid xa = ax\}$, $comm^2_R(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm_R(a)\}$, respectively. If there is no ambiguity, we simply use $comm(a)$ and $comm^2(a)$. Let $R^{qnil} = \{a \in R \mid 1 + ax \in U(R) \text{ for every } x \in comm(a)\}$. If $a \in R^{qnil}$, then a is said to be *quasinilpotent* [9]. Set $J^{\#}(R) = \{x \in R \mid \exists\, n \in \mathbb{N} \text{ such that } x^n \in J(R)\}$. Clearly, $J(R) \subseteq J^{\#}(R) \subseteq R^{qnil}$.

An element $a \in R$ is called *quasipolar* provided that there exists an idempotent $p \in comm^2(a)$ such that $a + p \in U(R)$ and $ap \in R^{qnil}$. A ring R is *quasipolar* in case every element in R is quasipolar. This concept ensues from Banach algebra. Indeed, for a Banach algebra R (see [8, page 251]),

$$a \in R^{qnil} \Leftrightarrow \lim_{n\to\infty} \| a^n \|^{\frac{1}{n}} = 0.$$

Quasipolar rings were studied in [6,8–12,21].

Ara [1] defined and investigated the notion of an exchange ring without identity. Chen and Chen [3] introduced the concept of strongly π-regular general rings. In [14], Nicholson and Zhou defined the notion of a clean general ring and they extended some of the basic results about clean rings to general rings. In [17], Wang

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2010 *AMS Mathematics Subject Classification:* 16S70, 16U99, 15A09.

and Chen defined the concept of a strongly clean general ring, and some properties about strongly clean rings were extended. These works motivate us to define quasipolar general rings. In this paper we see that every strongly π-regular general ring is a quasipolar general ring and any quasipolar general ring is a strongly clean general ring. We also see that every (two-sided) ideal of a quasipolar ring is a quasipolar general ring, but there exist quasipolar general rings that are not ideals of quasipolar rings (Example 3.3). In particular, we prove that $a \in R$ is strongly π-regular if and only if there exists a strongly regular element $s \in R$ and $n \in Nil(R)$ such that $a = s + n$ and $sn = ns = 0$ (Theorem 2.14), and $a \in R$ is quasipolar if and only if there exists a strongly regular element $s \in comm^2(a)$ and $q \in R^{qnil}$ such that $a = s + q$ and $sq = qs = 0$ (Corollary 2.17).

An element a of R is *(generalized) Drazin invertible* (see [6, 11, 12]) if there is an element $b \in R$ satisfying $ab^2 = b$, $b \in comm^2(a)$ and ($a^2b - a \in R^{qnil}$) $a^2b - a \in Nil(R)$. Such a b, if it exists, is unique; it is called the *(generalized) Drazin inverse* of a. Koliha [11] showed that an element $a \in R$ is Drazin invertible if and only if a is strongly π-regular [11, Lemma 2.1]. Koliha and Patricio [12] proved that an element $a \in R$ is generalized Drazin invertible if and only if a is quasipolar [12, Theorem 4.2]. With this in mind, we show that, for a general ring I, $a \in I$ is quasipolar if and only if there is an element $b \in I$ satisfying $ab^2 = b$, $b \in comm^2(a)$ and $a^2b - a \in QN(I)$ (Theorem 2.8), and $a \in I$ is strongly π-regular if and only if there is an element $b \in I$ satisfying $ab^2 = b$, $b \in comm^2(a)$ and $a^2b - a \in Nil(I)$ (Theorem 2.10).

Finally, we characterize a pseudopolar element of a ring, and we address the relations among quasipolarity, strong π-rad cleanness, and pseudopolarity. It is shown that R is pseudopolar if and only if R is strongly π-rad clean and quasipolar (Theorem 4.4).

2. Quasipolar general rings

Let I be a general ring with $p, q \in I$. We write $p * q = p + q - pq$. Let

$$Q(I) = \{q \in I \mid p * q = 0 = q * p \text{ for some } p \in I\}.$$

Note that $J(I) \subseteq Q(I)$. We define a set

$$QN(I) = \{q \in I \mid qx \in Q(I) \text{ for every } x \in comm(q)\}.$$

Clearly, $J(I) \subseteq Q(I)$ and $Nil(I) \subseteq QN(I)$. If R has an identity, then we have $Q(R) = \{q \in R \mid 1 - q \in U(R)\}$ and $QN(R) = R^{qnil}$. Further, if $a \in QN(I)$, then a is also said to be *quasinilpotent*.

Lemma 2.1 *The following conditions are equivalent for a ring R:*

(1) *R is quasipolar.*

(2) *For each $a \in R$, there exists $p^2 = p \in comm^2(a)$ such that $a + p \in Q(R)$ and $a - ap \in QN(R)$.*

Proof (1) ⇒ (2) Let $a \in R$. Since R is quasipolar, there exists an idempotent $1 - p \in R$ such that $1 - p \in comm^2(a)$, $-a + 1 - p = u \in U(R)$, and $a(1 - p) = a - ap \in R^{qnil}$. Then $a + p = q$, $p \in comm^2(a)$ where $q = 1 - u$ and $q * r = 0 = r * q$ with $r = 1 - u^{-1}$. As $R^{qnil} = QN(R)$, $a - ap \in QN(R)$.

(2) ⇒ (1) If $-a+p = q$ where $p^2 = p \in comm^2(a)$, $q \in Q(R)$, and $a-ap \in QN(R)$, then $a+1-p = 1-q$ where $(1 - p)^2 = 1 - p \in comm^2(a)$, $1 - q \in U(R)$ and $a(1 - p) \in R^{qnil}$. □

Definition 2.2 An element a in a general ring I is called a *quasipolar element* if there exists $p^2 = p \in comm^2(a)$ such that $a + p \in Q(I)$ and $a - ap \in QN(I)$, and I is called a *quasipolar general ring* if every element is quasipolar.

Remark 2.3 If I is isomorphic to a general ring K by f, then $a \in I$ is quasipolar if and only if $f(a)$ is quasipolar in K.

Example 2.4 Idempotents, nilpotents, quasinilpotents, and quasiregular elements are all quasipolar.

Recall that an element a in a general ring I is called a *strongly clean element* if it is the sum of an idempotent and an element of $Q(I)$ that commute, and I is called a *strongly clean general ring* if every element is strongly clean [17]. Hence, by Definition 2.2, quasipolar elements (general rings) are strongly clean.

We need the following useful lemma.

Lemma 2.5 *Let a, b, c be elements of a general ring I. If $a \in Q(I) \cap comm(b)$ and $a * c = 0 = c * a$, then $c \in comm(b)$.*

Proof Let $a*c = 0 = c*a$ and $ba = ab$. Then $a+c = ac = ca$. This implies that $ba+bc-bca = 0 = ab+cb-cab$, and so

$$bc - bca = cb - cab. \tag{2.1}$$

Multiplying (2.1) by c from the right yields

$$bcc - bcac = cbc - cabc.$$

This gives $bca = cba = cab$ because $c - ac = -a$. This shows that $bc = cb$ and so $c \in comm(b)$. □

Lemma 2.6 *Let I be a general ring. If $a * b = 0$ and $c * a = 0$, then $b = c$.*

Proof Suppose that $a * b = 0$ and $c * a$ for $a, b, c \in I$. This gives $b = 0 * b = (c * a) * b = c * (a * b) = c * 0 = c$, as desired. □

Lemma 2.7 *Let I be a general ring and assume that $a \in I$ is quasinilpotent. Then $a, -a \in Q(I)$ and $-a \in I$ is quasinilpotent. Further, $QN(I) \subseteq Q(I)$.*

Proof Since $a \in QN(I)$ and $a \in comm(a)$, we get $a^2 \in Q(I)$. That is, there exists $b \in R$ such that $a^2 * b = a^2 + b - a^2b = 0 = b + a^2 - ba^2 = b * a^2$. This implies that $0 = a^2 * b = [a * (-a)] * b = a * [(-a) * b]$ and $0 = b * a^2 = b * [(-a) * a] = [b * (-a)] * a$, and so we have $a \in Q(I)$ by Lemma 2.6. Similarly, it can be shown that $-a \in Q(I)$. On the other hand, we check easily that $-a \in QN(I)$. If $a \in QN(I)$, then $a \in Q(I)$. Hence, $QN(I) \subseteq Q(I)$. The proof is completed. □

The next result was proved in [12, Theorem 4.2] for a in any ring R.

Theorem 2.8 *The following are equivalent for $a \in I$:*

(1) *a is quasipolar in I.*

(2) *There exists* $b \in comm^2(a)$ *such that* $ab^2 = b$ *and* $a^2b - a \in QN(I)$.

In this case, b *is unique.*

Proof (1) $\Rightarrow$ (2) Write $a+p = q \in Q(I)$ where $p^2 = p \in comm^2(a)$ and $a - ap \in QN(I)$, say $q * r = r * q = 0$ where $r \in I$. Then $r + q = rq = qr$. In view of Lemma 2.5, $rp = pr$ because $q \in Q(I)$ and $q \in comm(p)$. Set $b = rp - p$. It is easy to verify that $p = ab$. Let $ax = xa$ for some $x \in I$. Since $p \in comm^2(a)$, we have $xp = px$ and so $xq = qx$. Moreover, as $r + q = rq = qr$, we see that

$$xr - xrq = rx - rxq. \tag{2.2}$$

Multiplying (2.2) by r from the right yields

$$xrr - xrqr = rxr - rxqr \text{ and so } xrq = rxq = rqx.$$

This shows that $rx = xr$. That is, $r \in comm^2(a)$. Hence, we conclude that $b \in comm^2(a)$. Now we show that $ab^2 = b$ and $a^2b - a \in QN(I)$. We have

$$\begin{aligned}
ab^2 &= (q-p)(rp-p)(rp-p) = (q-p)(r^2p - rp - rp + p)\\
&= qr^2p - qrp - qrp + qp - r^2p + rp + rp - p\\
&= qr^2p - rp - qp - rp - qp + qp - r^2p + rp + rp - p\\
&= qr^2p - r^2p - p - qp\\
&= (qr^2 - r^2 - p - q)p\\
&= (r^2 + rq - r^2 - p - q)p\\
&= (r-p)p\\
&= b.
\end{aligned}$$

Moreover,

$$\begin{aligned}
a^2b - a &= (q-p)(q-p)(rp-p) - (q-p)\\
&= (q^2 - qp - qp + p)(rp - p) - q + p\\
&= q^2rp - q^2p - qrp + qp - qrp + qp + rp - p - q + p\\
&= q^2rp - q^2p - rp - qp + qp - rp - qp + qp + rp - q\\
&= q^2rp - q^2p - rp - q\\
&= qpr + q^2p - q^2p - rp - q\\
&= rp + qp - rp - q\\
&= qp - q\\
&= ap - a \in QN(I).
\end{aligned}$$

Thus (2) holds, as required.

(2) $\Rightarrow$ (1) Set $p = ab$. Then $p \in comm^2(a)$, and $p^2 = abab = a^2b^2 = a(ab^2) = ab = p$. Since $a - ap = a - aab = a - a^2b$ and $a^2b - a \in QN(I)$, we have $a - ap \in QN(I)$. Now we show that $a + p = a + ab \in Q(I)$. We observe that $(a+ab)*(b+ab) = a + ab + b + ab - (a+ab)(b+ab) = a + ab + b + ab - ab - a^2b - b - ab = a - a^2b$. As $a - a^2b \in QN(I)$, $(a - a^2b) * x = x * (a - a^2b) = 0$ for some $x \in I$. This implies that $(a+ab)*(b+ab)*x = 0$ and $x * (b+ab) * (a+ab) = 0$. Further, $(b+ab)*x = 0*(b+ab)*x = \big(x*(b+ab)*(a+ab)\big)*(b+ab)*x = \big(x*(b+ab)\big)*\big((a+ab)*(b+ab)*x\big) = x*(b+ab)*0 = x*(b+ab)$. Then $(b+ab)*x*(a+ab) = x*(b+ab)*(a+ab) = 0$, so we have $a + ab = a + p = q \in Q(I)$. Hence, $a \in I$ is quasipolar. Moreover, as $q \in Q(I)$, there exists $r \in I$ such that $q * r = 0 = r * q$, and so $r + q = rq = qr$. As in the preceding discussion, we see that $r \in comm^2(a)$. Thus, $r * (q * (b + p)) = (r * q) * (b + p) = 0 * (b + p) = b + p = r * (a - ap) = r + a - ap - ra + rap = r + q - p - pq + p - rq + rpq = rpq - pq = rp$. Therefore, $b = rp - p$.

To prove the uniqueness of b, assume that $c \in comm^2(a)$ so that $ac^2 = c$ and $a^2c - a \in QN(I)$. Then $ac - acab = ac - a^2cb = a^2c^2 - a^2cb = (a^2b - a)(b - c)$. Since $a^2b - a \in QN(I)$ and $b - c \in comm(a^2b - a)$, we have $ac - a^2cb \in Q(I)$. This gives that $ac = a^2cb$. Similarly, we show that $ab = a^2cb$, and so $ab = ac$. Thus, $b = rp - p = rab - ab = rac - ac = c$; that is, b is unique. Note that b is unique if and only if p is unique. We complete the proof. □

Corollary 2.9 *Let I be a general ring. If $a \in I$ is quasipolar, then $-a$ is quasipolar.*

Proof It is clear from Theorem 2.8. □

Recall that an element a in a general ring I is called *strongly π-regular* if there exist $n \in \mathbb{N}$ and $x \in I$ such that $a^n = a^{n+1}x$ and $x \in comm(a)$ (see [2, 3, 17]). The next result is known if a is in a ring R (see [11, Lemma 2.1] and [12, Proposition 4.9]).

Theorem 2.10 *The following are equivalent for $a \in I$:*

(1) *a is strongly π-regular in I.*

(2) *There exists $p^2 = p \in comm^2(a)$ such that $a - ap \in Nil(I)$ and $a + p \in Q(I)$.*

(3) *There exists $p^2 = p \in comm(a)$ such that $a - ap \in Nil(I)$ and $a + p \in Q(I)$.*

(4) *There exists $b \in comm^2(a)$ such that $ab^2 = b$ and $a^2b - a \in Nil(I)$.*

(5) *There exists $b \in comm(a)$ such that $ab^2 = b$ and $a^2b - a \in Nil(I)$.*

Proof (1) ⇒ (2) Assume that $a \in I$ is strongly π-regular. Then there exist $n \in \mathbb{N}$ and $x \in I$ such that $a^n = a^{n+1}x$ and $ax = xa$. It is easy to check that $a^nx^n = x^na^n = p = p^2 \in I$. Since $a^n = a^nx^na^n$, we have $(a - ap)^n = 0$, and so $a - ap \in Nil(I)$.

Claim 1. $p \in comm^2(a)$.

Proof. Let $ay = ya$. This implies that $py - pyp = a^nx^ny - a^nx^nyp = a^nx^ny - x^nya^np = a^nx^ny - x^nya^n = a^nx^ny - a^nx^ny = 0$ because $ax = xa$ and $a^nx^n = x^na^n$, so $py = pyp$. Similarly, we see that $yp = pyp$. Then $py = yp$ and so $p \in comm^2(a)$.

The remaining proof is to show that $q = a + p$ is a quasiregular element of I. Set $t = a + a^2 + a^3 + \cdots + a^{n-1}$ and $r = tp - t + a^{n-1}x^np + p$. Hence,

$$\begin{aligned} q * r \quad &= a + p + tp - t + a^{n-1}x^np + p - \\ & atp - at - p - ap - a^{n-1}x^np - p \\ &= a + p + ap - a - a^np + a^np - p - ap \\ &= 0. \end{aligned}$$

Analogously, we have $r * q = 0$. Thus (2) holds.

(2) ⇒ (3) Clear by $comm^2(a) \subseteq comm(a)$.

(3) ⇒ (4) Assume that $a + p = q \in Q(I)$ where $p^2 = p \in comm(a)$ and $a - ap \in Nil(I)$, say $q * r = r * q = 0$ and $(a - ap)^k = a^k - a^kp = 0$ where $r \in I$ and $k \in \mathbb{N}$. By Lemma 2.5, $rp = pr$ because $q \in Q(I)$ and $q \in comm(p)$. Set $b = rp - p$ and let $ax = xa$ for some $x \in I$. Then we have $ab = p = ba$, and so $xp - pxp = xa^kb^k - pxa^kb^k = a^kxb^k - pa^kxb^k = (a^k - pa^k)xb^k = 0$. That is, $xp = pxp$. Analogously,

we see that $px = pxp$. This gives $xp = px$, so $p \in comm^2(a)$. Therefore, an argument similar to the proof of Theorem 2.8 shows that $b \in comm^2(a)$, $ab^2 = b$, and $a^2b - a = ap - a \in Nil(I)$.

(4) $\Rightarrow$ (5) It is obvious.

(5) $\Rightarrow$ (1) Let $ab = p$. Since $ab^2 = b$, we have $p = p^2$. As $a^2b - a \in Nil(I)$, there exists $k \in \mathbb{N}$ such that $(a^2b - a)^k = 0$. This implies that $(a^2b - a)^k = a^kp - a^k = 0$. Then $a^k = a^kp = a^kab = a^{k+1}b$ and $b \in comm(a)$. Hence, $a \in I$ is strongly π-regular, and so (1) holds. □

Remark 2.11 If an element a of a general ring I is strongly π-regular, then b and p in Theorem 2.10 are unique (indeed, as in the proof of Theorem 2.8, we see that b and p are unique).

By Theorem 2.10, the following result is immediate.

Corollary 2.12 *Any strongly π-regular element in a general ring is strongly clean.*

Recall that an element a of a general ring I is *strongly regular* if $a = aba$ and $b \in comm(a)$ for some $b \in I$. I is *strongly regular* if every element in I is strongly regular.

Lemma 2.13 *Let I be a general ring and $a \in I$. Then the following are equivalent:*

(1) *a is strongly regular in I.*

(2) *There exists $b \in comm^2(a)$ such that $a = a^2b$.*

Proof It is similar to the proof of [2, Lemma 1]. □

Theorem 2.14 was proved for a in any ring R in [15].

Theorem 2.14 *For an element a in a general ring I, the following are equivalent:*

(1) *a is strongly π-regular in I.*

(2) *$a \in I$ can be written in the form $a = s + n$ where s is strongly regular, n is nilpotent, and $sn = ns = 0$.*

Proof (1) $\Rightarrow$ (2) Suppose that $a \in I$ is strongly π-regular. It is well known that a is strongly π-regular if and only if a is pseudoinvertible; that is, there exist $c \in I$ and $m \in \mathbb{N}$ such that $ac = ca$, $a^m = a^{m+1}c$, and $c = c^2a$ (see [6, Theorem 4]). Set $s = aca$ and $n = a - aca$. Then $sn = ns = aca(a - aca) = 0$ because $ac = ca$ and ac is idempotent in I. It is easy to check that $s = s^2c$ and so s is strongly regular in I. Write $ca = ac = e = e^2 \in I$. Hence, $(a - aca)^m = (a - ae)^m = a^m - a^me = a^m - a^mac = a^m - a^{m+1}c = 0$. Thus, $n \in I$ is nilpotent and so (2) holds.

(2) $\Rightarrow$ (1) Assume that $a = s + n$ where s is strongly regular, n is nilpotent, and $sn = ns = 0$. Since n is nilpotent, there exists $k \in \mathbb{N}$ such that $n^k = 0$. As s is strongly regular, there exists $x \in I$ such that $s = s^2x$ and $x \in comm^2(s)$ by Lemma 2.13. Then it is easy to see that $a^k = (s + n)^k = s^k$ and $a^{k+1} = (s + n)^k = s^{k+1}$ because $sn = ns = 0$. This gives that $a^k = s^k = s^{k-1}s = s^{k-1}s^2x = s^{k+1}x = a^{k+1}x$. Further, as $as = sa$ and $x \in comm^2(s)$, we have $ax = xa$. Hence, a is strongly π-regular in I. □

The following result is well known for a ring (see [2]).

Corollary 2.15 *If an element a in a general ring I is strongly π-regular, then a^k is strongly regular for some $k \in \mathbb{N}$.*

A new characterization of a quasipolar element in a general ring is given as follows.

Theorem 2.16 *For an element a in a general ring I, the following are equivalent:*

(1) *a is quasipolar in I.*

(2) *$a \in I$ can be written in the form $a = s + q$ where s is strongly regular, $s \in comm^2(a)$, $q \in QN(I)$, and $sq = qs = 0$.*

Proof (1) $\Rightarrow$ (2) Assume that $a \in I$ is quasipolar. By Theorem 2.8, there exists $b \in comm^2(a)$ such that $ab^2 = b$ and $a^2b - a \in QN(I)$. Set $s = a^2b$ and $q = a - a^2b$. Further, we have $s \in comm^2(a)$ and $sq = qs = a^2b(a - a^2b) = 0$ because $ab = ba$ and ab is idempotent in I. It is easy to see that $s = s^2b$ and so $s \in I$ is strongly regular.

(2) $\Rightarrow$ (1) Suppose that $a = s + q$ where s is strongly regular, $s \in comm^2(a)$, $q \in QN(I)$, and $sq = qs = 0$. Since s is strongly regular, there exists $y \in comm^2(s)$ such that $s = s^2y$ by Lemma 2.13. Then we have that $sy = ys$ is an idempotent and $yq = qy$. Hence, $a + sy = s + sy + q = (s + sy) * q = q * (s + sy)$ and $(s + sy) * (y^2s + sy) = (y^2s + sy) * (s + sy) = 0$. This implies that $(a + sy) * (y^2s + sy) = (y^2s + sy) * (a + sy) = (s + sy) * q * (y^2s + sy) = (y^2s + sy) * (s + sy) * q = q$. As $q \in Q(I)$, it can be checked that $a + sy \in Q(I)$. Further, $a - asy = s + q - s^2y - qsy = q \in QN(I)$ and $sy \in comm^2(a)$. Thus, $a \in I$ is quasipolar, and so (1) holds. □

The following result is a direct consequence of Theorem 2.16.

Corollary 2.17 *Let R be a ring and let $a \in R$. Then the following are equivalent:*

(1) *a is quasipolar.*

(2) *$a = s + q$ where s is strongly regular, $s \in comm^2(a)$, $q \in R^{qnil}$, and $sq = qs = 0$.*

Proposition 2.18 *A general ring I is strongly regular if and only if I is quasipolar and $QN(I) = 0$.*

Proof Assume that I is strongly regular. Then I is strongly π-regular and so I is quasipolar by Theorem 2.10. Let $a \in QN(I)$. By hypothesis, $a = aba$ and $b \in comm(a)$ for some $b \in I$. Since $ab = ba$, we have $ab \in Q(I)$. This implies that $ab = 0$ and so $a = 0$. Hence, $QN(I) = 0$. Conversely, let $a \in I$. Since $QN(I) = 0$, a is strongly regular by Theorem 2.16. □

The following result follows from Proposition 2.18.

Corollary 2.19 [4, Theorem 2.4] Let R be a ring. Then R is strongly regular if and only if R is quasipolar and $R^{qnil} = 0$.

Remark 2.20 (1) In Proposition 2.18, it was proved that if $a \in QN(I)$ and a is strongly regular, then $a = 0$.

(2) If a is strongly regular, then a^k is strongly regular for any $k \in \mathbb{N}$.

(3) If $a \in QN(I)$ and a^k is strongly regular for some $k \in \mathbb{N}$, then $a \in Nil(I)$.

Proposition 2.21 *A general ring I is strongly π-regular if and only if I is quasipolar and $QN(I) \subseteq Nil(I)$.*

Proof Assume that I is strongly π-regular. Then, by Theorem 2.10, I is quasipolar because $Nil(I) \subseteq QN(I)$. Let $a \in QN(I)$. As I is strongly π-regular, by Theorem 2.14, $a = s + n$ where s is strongly regular, n is nilpotent, and $sn = ns = 0$. Since n is nilpotent, there exists $k \in \mathbb{N}$ such that $n^k = 0$. Hence, we have $a^k = s^k$. As s^k is strongly regular and $a \in QN(I)$, by Remark 2.20, we see that $a \in Nil(I)$. Thus, $QN(I) \subseteq Nil(I)$. Conversely, suppose that I is quasipolar and $QN(I) \subseteq Nil(I)$. In view of Theorem 2.16 and Theorem 2.14, I is strongly π-regular. □

The following result is a direct consequence of Proposition 2.21.

Corollary 2.22 [4, Theorem 2.6] Let R be a ring. Then R is strongly π-regular if and only if R is quasipolar and $R^{qnil} \subseteq Nil(R)$.

An element a of a ring R is called *semiregular* if there exists $b \in R$ with $bab = b$ and $a - aba \in J(R)$. A ring is a *semiregular ring* if each of its elements is semiregular ([13, Proposition 2.2]).

We give a different proof of [19, Theorem 3.2].

Theorem 2.23 *Let R be a ring. If R is quasipolar and $R^{qnil} \subseteq J(R)$, then R is semiregular. The converse holds if R is abelian.*

Proof Assume that R is a quasipolar ring and $R^{qnil} \subseteq J(R)$. Then we have $J(R) = R^{qnil}$. In view of Corollary 2.17, $R/J(R)$ is strongly regular. As R is quasipolar, R is strongly clean and so idempotents lift modulo $J(R)$. Then R is semiregular by [13, Theorem 2.9]. Conversely, let $a \in R$. Then there exists $b \in R$ with $bab = b$ and $a - aba \in J(R)$. Write $a = aba + (a - aba)$, say $s = aba$ and $q = a - aba$. Since $a - aba \in J(R) \subseteq R^{qnil}$ and R is abelian, we see that $s \in comm^2(a)$, $q \in R^{qnil}$, $s = aba = (aba)^2 b = s^2 b$, and $sq = qs = aba(a - aba) = a^2ba - a^2ba = 0$. By Corollary 2.17, a is quasipolar, and so R is quasipolar. Take $x \in R^{qnil}$. By assumption, there exists $y \in R$ with $yxy = y$ and $x - xyx \in J(R)$. Note that $x \cdot 0 = 0$ and $x^2 \cdot 0 - x = -x \in R^{qnil}$. By Theorem 2.8, we get $y = 0$. This gives that $x \in J(R)$. □

3. Extensions of quasipolar general rings

Let S be a ring and I an (S, S)-bimodule, which is a general ring in which $(vw)s = v(ws)$, $(vs)w = v(sw)$, and $(sv)w = s(vw)$ hold for all $v, w \in I$ and $s \in S$. Then the *ideal-extension* (it is also called the Dorroh extension) $I(S; I)$ of S by I is defined to be the additive abelian group $E(S; I) = S \oplus I$ with multiplication $(s, v)(r, w) = (sr, sw + vr + vw)$. In this case, $I \lhd E(S; I)$, and $E(S; I)/I \cong S$. In particular, $E(\mathbb{Z}; I)$ is the standard unitization of the general ring I.

Clean general ideal-extensions were considered in [14, Proposition 7]. Now we deal with quasipolar general ideal-extensions.

Proposition 3.1 *The following are equivalent for a general ring I:*

(1) *I is quasipolar.*

(2) *$(0, a)$ is quasipolar in $E(\mathbb{Z}; I)$ for all $a \in I$.*

(3) *There exists a ring* S *such that* $I =_S I_S$ *and* $(0,a)$ *is quasipolar in* $E(S;I)$ *for all* $a \in I$.

Proof (1) $\Rightarrow$ (2) Let $a \in I$ and $R = E(\mathbb{Z};I)$. By Theorem 2.16, we have $-a = s+q$ where $s \in I$ is strongly regular, $q \in QN(I)$, and $sq = qs = 0$. Write $(0,a) = (0,-s)+(0,-q)$. Since s is strongly regular, there exists $y \in comm^2(s)$ such that $s = s^2y$ by Lemma 2.13. This implies that $(0,-s) = (0,-s)^2(0,-y)$ and $(0,-y) \in comm^2\big((0,-s)\big)$, and so, by Lemma 2.13, $(0,-s)$ is strongly regular in R. Assume that $(x,y) \in comm\big((0,q)\big)$. Then we have $x+y \in comm(q)$ and so $(x+y)q \in Q(I)$ because $q \in QN(I)$. This gives $(1,0)+(x,y)(0,-q) = \big(1,-(x+y)q\big) \in U(R)$ (the inverse is $(1,-t)$ where $(x+y)q * t = 0 = t*(x+y)q$). Hence, $(0,-q) \in R^{qnil}$. As $sq = qs = 0$, we see that $(0,-s)(0,-q) = (0,-q)(0,-s) = (0,0)$, and so $(0,a) \in R$ is quasipolar by Corollary 2.17.

(2) $\Rightarrow$ (3) It is clear with $S = \mathbb{Z}$.

(3) $\Rightarrow$ (1) Let $a \in I$ and $R = E(S;I)$. By (3), $(0,-a)+(e,p) = (e,p-a)$ where $(e,p)^2 = (e,p) \in comm^2\big((0,-a)\big)$, $(e,p-a) \in U(R)$, and $(0,-a)(e,p) = \big(0,-a(e+p)\big) \in R^{qnil}$. Since $(e,p)^2 = (e,p)$, we have $e^2 = e$ and $p = ep+pe+p^2$. This gives that $e = 1_S$ because $(e,p-a) \in U(R)$, so $-p$ is an idempotent in I. As $(-1,a-p) \in U(R)$, there exists $q \in I$ such that $q*(a-p) = 0 = (a-p)*q$. This implies that $a+(-p) \in Q(I)$. If $ax = xa$, then we have $(0,x) \in comm\big((0,-a)\big)$ and so $xp = px$ because $(1,p) \in comm^2\big((0,-a)\big)$. Hence, $-p \in comm^2(a)$. Now we show that $a+ap \in QN(I)$. Let $x(a+ap) = (a+ap)x$. As $\big(0,-a(1_S+p)\big) \in R^{qnil}$, it follows that $x(a+ap) \in Q(I)$, so $a \in I$ is quasipolar. The proof is completed. □

Theorem 3.2 *Let* I *be a quasipolar general ring and* $A \lhd I$. *Then* A *is quasipolar.*

Proof Let $R = E(\mathbb{Z};I)$ and $a \in A$. By Theorem 2.16, we have $-a = s+q$ where $s \in I$ is strongly regular, $s \in comm^2(a)$, $q \in QN(I)$, and $sq = qs = 0$. Write $(0,a) = (0,-s)+(0,-q)$. Since s is strongly regular, there exists $y \in comm^2(s)$ such that $s = s^2y$ by Lemma 2.13. This implies that $(0,-s) = (0,-s)^2(0,-y)$ and $(0,-y) \in comm^2\big((0,-s)\big)$, and so, by Lemma 2.13, $(0,-s)$ is strongly regular in R. Assume that $(m,n) \in comm\big((0,q)\big)$. Then we have $x+y \in comm(q)$ and so $(m+n)q \in Q(I)$ because $q \in QN(I)$. This gives $(1,0)+(m,n)(0,-q) = \big(1,-(m+n)q\big) \in U(R)$ (the inverse is $(1,-t)$ where $(m+n)q*t = 0 = t*(m+n)q$). Hence, $(0,-q) \in R^{qnil}$. As $sq = qs = 0$, we see that $(0,-s)(0,-q) = (0,-q)(0,-s) = (0,0)$. Let $(u,v)(0,a) = (0,a)(u,v)$. Then $(u+v) \in comm(a)$ and so $(u+v) \in comm(s)$ since $s \in comm^2(a)$. This proves $(u,v)(0,-s) = (0,-s)(u,v)$. That is, $(0,-s) \in comm^2\big((0,a)\big)$, so $(0,a) \in R$ is quasipolar by Corollary 2.17. As $A \cong (0,A) \lhd R$, A is quasipolar by Proposition 3.1 and Remark 2.3. □

This result shows that any ideal of a quasipolar general ring is a quasipolar general ring. However, the converse need not be true in general, as the following example shows.

Given a ring R, the set $I = \{(a,b) \mid a,b \in R\}$ becomes a general ring (without identity) with addition defined componentwise and multiplication defined by $(a,b)(c,d) = (ac,ad)$. Then $I \cong \begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix} = J$ where J is a right ideal of $M_2(R)$.

Example 3.3 Consider the local ring $R = \mathbb{Z}_{(2)} = \{\frac{m}{n} \in \mathbb{Q} \mid 2 \nmid n\}$ and $(a,b) \in I$. If $a \in J(R)$, then it is easy to verify that $(a,b) \in J(I)$ and so (a,b) is quasipolar in I. If $a \notin J(R)$, then $a \in 1+J(R)$, so $(a,b)+(1,a^{-1}b) =$

$(a+1, b+a^{-1}b)$ where $(1, a^{-1}b)^2 = (1, a^{-1}b) \in comm^2\big((a,b)\big)$ and $(a+1, b+a^{-1}b) \in J(I) \subseteq Q(I)$. Further, since $(a,b)-(a,b)(1,a^{-1}b) = (0,0) \in QN(I)$, (a,b) is quasipolar in I. Hence, I is a quasipolar general ring. On the other hand, $M_2(R)$ is not a quasipolar ring because $M_2(R)$ is not a strongly clean ring (see [16]).

Lemma 3.4 *Let $e^2 = e \in I$. Then $QN(eIe) = eIe \cap QN(I)$.*

Proof Let $a \in QN(eIe)$ and $ab = ba$ for some $b \in I$. Then $a \cdot ebe = abe = bae = ba$ and $ebe \cdot a = eba = eab = ab$, so $ebe \in comm(a)$. Since $a \in QN(eIe)$, we have $ab * x = 0 = x * ab$ for some $x \in eIe$. Hence, $a \in eIe \cap QN(I)$. This gives that $QN(eIe) \subseteq eIe \cap QN(I)$. Conversely, let $a \in eIe \cap QN(I)$ and $aere = erea$ for some $ere \in eIe$. This implies that $ae = ea = a$. Since $a \in QN(I)$, $are + y - arey = 0 = are + y - yare$ for some $y \in I$. Then $are + eye - areye = 0 = are + eye - eyare$ and so $are \in Q(eIe)$. Therefore, $eIe \cap QN(I) \subseteq QN(eIe)$. We complete the proof. □

Theorem 3.5 *Let I be a quasipolar general ring with $e^2 = e \in I$. Then eIe is quasipolar.*

Proof Let $a \in eIe$. Then there exists $p^2 = p \in comm^2(a)$ such that $a + p = q \in Q(I)$ and $a - ap \in QN(I)$. Since $ae = ea$, we have $ep = pe$. This implies that $a + epe = eqe$ where $epe^2 = epe$ and $eqe \in Q(I) \cap eIe = Q(eIe)$. It is easy to see that $epe \in comm^2(a)$ because $p^2 = p \in comm^2(a)$. As $a - ap \in QN(I)$, we have $a - ap = a - aep = a - aepe = a - ape = e(a-ap)e \in QN(I) \cap eIe = QN(eIe)$ by Lemma 3.4. Hence, eIe is quasipolar. □

Corollary 3.6 [19, Proposition 3.6] Let R be a ring with $e^2 = e \in R$. If R is quasipolar, then so is eRe.

4. Pseudopolar elements

An element a of R is *pseudo-Drazin invertible* if there exist $b \in R$ and $k \in \mathbb{N}$ satisfying $ab^2 = b$, $b \in comm^2(a)$, and $(a - a^2b)^k \in J(R)$. Such a b, if it exists, is unique; it is called a *pseudo-Drazin inverse* of a. Wang and Chen [18] showed that an element $a \in R$ is pseudo-Drazin invertible if and only if a is pseudopolar; that is, there exist $p \in R$ and $k \in \mathbb{N}$ such that $p^2 = p \in comm^2(a)$, $a + p \in U(R)$, and $a^k p \in J(R)$.

A characterization of pseudopolar elements can be given as follows.

Theorem 4.1 *Let R be a ring and let $a \in R$. Then the following are equivalent:*

(1) *a is pseudopolar.*

(2) *$a = s + q$ where s is strongly regular, $s \in comm^2(a)$, $q \in J^\#(R)$, and $sq = qs = 0$.*

Proof (1) ⇒ (2) Assume that $a \in R$ is pseudopolar. Then there exist $b \in comm^2(a)$ and $k \in \mathbb{N}$ such that $ab^2 = b$ and $(a - a^2b)^k \in J(R)$. Set $s = a^2b$ and $q = a - a^2b$. This gives $s \in comm^2(a)$, $q \in J^\#(R)$ and $sq = qs = a^2b(a - a^2b) = 0$. It is easy to see that $s = s^2b$ and so $s \in R$ is strongly regular.

(2) ⇒ (1) Suppose that $a = s + q$ where s is strongly regular, $s \in comm^2(a)$, $q \in J^\#(R)$, and $sq = qs = 0$. Since s is strongly regular, there exists $y \in comm^2(s)$ such that $s = s^2y$ by Lemma 2.13. Then we have that $1 - p = sy = ys$ is an idempotent, $p \in comm^2(a)$, and $yq = qy$. As $q \in J^\#(R)$, we see that

$q^n \in J(R)$ and so $1+q \in U(R)$ for some $n \in \mathbb{N}$. Hence, $(a+p)(y^2s+p) = 1+q \in U(R)$ and so $a+p \in U(R)$. Moreover, $a^np = (s^n+q^n)(1-sy) = q^n \in J(R)$ because $s^n = s^{n+1}y$, so (1) holds. □

Note that if R is pseudopolar, then R is quasipolar by Theorem 4.1 and Corollary 2.17. Further, if $-a$ is pseudopolar, then so is a by Theorem 4.1.

Combining Theorem 2.10 with Theorem 4.1, we obtain the following result.

Corollary 4.2 [18, Theorem 2.1] Let R be a ring. Then R is strongly π-regular if and only if R is pseudopolar and $J(R)$ is nil.

We give a different proof of the [18, Theorem 2.4].

Theorem 4.3 *Let R be a ring. If R is pseudopolar and $J^{\#}(R) = J(R)$, then R is semiregular. The converse holds if R is abelian.*

Proof Assume that R is pseudopolar and $J^{\#}(R) = J(R)$. According to Theorem 4.1, $R/J(R)$ is strongly regular. Hence, R is semiregular by [13, Theorem 2.9]. Conversely, let $a \in R$. Then there exists $b \in R$ with $bab = b$ and $a - aba \in J(R)$. Write $a = aba + (a - aba)$, say $s = aba$ and $q = a - aba$. Since $a - aba \in J(R) \subseteq J^{\#}(R)$ and R is abelian, we see that $s \in comm^2(a)$, $q \in J^{\#}(R)$, $s = aba = (aba)^2b = s^2b$, and $sq = qs = aba(a-aba) = a^2ba - a^2ba = 0$. By Theorem 4.1, a is pseudopolar. In view of Theorem 2.23, we see that $J^{\#}(R) = J(R)$. □

Recall that an element $a \in R$ is *strongly π-rad clean* provided that there exists an idempotent $e \in R$ such that $ae = ea$ and $a - e \in U(R)$ and $a^ne \in J(R)$ for some $n \in \mathbb{N}$. A ring R is *strongly π-rad clean* if every element in R is strongly π-rad clean (see [5]). We now give the relations among quasipolarity, strong π-rad cleanness, and pseudopolarity.

Theorem 4.4 *Let R be a ring. Then R is pseudopolar if and only if R is strongly π-rad clean and quasipolar.*

Proof The "only if" part is easy to see and so we only have to prove the "if" part. Let $a \in R$. Then there exists $p^2 = p \in comm^2(a)$ such that $a+p \in U(R)$ and $ap \in R^{qnil}$ since R is quasipolar. Further, there exists $q \in comm(a)$ such that $-a-q \in U(R)$ and $a^nq \in J(R)$ for some $n \in \mathbb{N}$ because R is strongly π-rad clean. Since $a^nq \in J(R)$, we have $aq \in R^{qnil}$. By [12, Proposition 2.3], we see that $p = q$. Hence, a is pseudopolar, as desired. □

Corollary 4.5 [18, Corollary 2.12] Let R be a ring with $e^2 = e \in R$. If R is pseudopolar, then so is eRe.

Proof Assume that R is pseudopolar. Then R is strongly π-rad clean and quasipolar by Theorem 4.4. In view of [5, Corollary 4.2.2] and Corollary 3.6, eRe is strongly π-rad clean and quasipolar. Hence, eRe is pseudopolar again by Theorem 4.4. □

Remark 4.6 Let S be a commutative ring and $R = M_2(S)$. By [18, Example 4.3], we have $J^{\#}(R) = R^{qnil}$. Hence, by Theorem 4.1 and Corollary 2.17, R is quasipolar if and only if R is pseudopolar. Further, if S is commutative local, then R is pseudopolar if and only if R is quasipolar if and only if R is strongly clean (by [7, Corollary 2.13]) if and only if R is strongly π-rad clean (by [5, Corollary 4.3.7]).

Acknowledgment The author would like to thank the referee(s) for careful reading of the manuscript and valuable suggestions. The author also thanks the Scientific and Technological Research Council of Turkey (TÜBİTAK) for the grant.

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