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# More characterizations of Dedekind domains and V-rings

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**Abstract:** In this paper, cyclic $c-$injective modules are introduced and investigated. It is shown that a commutative Noetherian domain is Dedekind if and only if every simple module is cyclic $c-$injective. Finally, it is shown that injectivity, principal injectivity, mininjectivity, and simple injectivity are all equal to characterize right V-rings, right GV-rings, right pV-rings, and WV-rings.

**Key words:** $c-$Injective modules, Dedekind domains, right V-rings, right GV-rings, WV-rings

## 1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary (unless otherwise stated).

Recall that an integral domain $R$ that is not a field is called a Dedekind domain if every nonzero proper ideal factors into primes. This is equivalent to $R$ being an integrally closed, Noetherian domain with Krull dimension one (i.e. every nonzero prime ideal is maximal). Mermut et al. [10] showed that a commutative Noetherian domain is Dedekind if and only if every simple module is $c-$injective. According to Baccella [1], a ring $R$ is called a *right V-ring* (resp. *right GV-ring*) if every simple (resp. simple singular) right $R-$module is $R_R-$injective, i.e. injective. According to López-Permouth et al. [9], a ring $R$ is called a *right pV-ring* provided every simple right $R-$module is $p-$injective (there, $p-$injectivity is exactly principal injectivity). It is shown that being a right or left pV-ring is a Morita invariant. Following Holston [7], a ring $R$ is called a *WV-ring* if every simple right $R-$module is injective relative to proper cyclics, i.e $M$ is $R/I-$injective for every right ideal $I$ of $R$ satisfying $R/I \ncong R$. We ask here:

*Can we characterize Dedekind domains, right V-rings, right GV-rings, right pV-rings, and WV-rings by weaker forms of injectivity instead of the defined ones?*

The answer is definitely yes as we shall see in the last section of this text. In fact, commutative Noetherian domains are characterized by cyclic $c-$injective modules. Right V-rings, right GV-rings, right pV-rings, and WV-rings are characterized by mininjective modules and simple injective modules.

Let $M$ be a right $R-$module and $S = \text{End}(M_R)$ its endomorphism ring. $M$ is called a *simple module* if $M$ and 0 are the only submodules of $M$. $M$ is called a *cyclic module* if $M = mR$ for some $m \in M$, i.e. $M$

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is generated by a single element. A submodule $X$ of $M$ (denoted by $X \hookrightarrow M$) is called a *cyclic submodule* (resp. *simple or minimal submodule*) if $X$ is a cyclic module (resp. simple module). A submodule $N$ of $M$ is said to be *essential* in $M$ (denoted by $N \overset{*}{\hookrightarrow} M$) if $N$ has nonzero intersection with every nonzero submodule of $M$, i.e. $N \cap A \neq 0$ for every $0 \neq A \hookrightarrow M$. A submodule $X \hookrightarrow M$ is called a *closed* submodule if it has no proper essential extension in $M$, i.e. for every submodule $B$ of $M$, $X \overset{*}{\hookrightarrow} B$ implies $X = B$. Note that if $X$ is a closed submodule of $Y$ and $Y$ is a closed submodule of $M$, then $X$ is closed in $M$ (see [12, section 1.29]). A submodule $K$ of $M$ is called an *$M$-cyclic submodule* if $K = s(M)$ for some $s \in S$. A fact that cyclic submodules and $M-$cyclic submodules are distinguished as we will see in Example 1. However, in the case of rings, they meet each other.

We denote $\mathbf{r}_X(Y), \mathbf{l}_X(Y)$ for the right annihilator, the left annihilator of $Y$ in $X$, respectively. We simply write $\mathbf{r}(Y), \mathbf{l}(Y)$ if $X$ is a ring $R$ and $Y \subset R$. According to [6, 12], a module $M_R$ is called a *CS module* (or *extending module*) if every submodule of $M$ is essential in a direct summand, or, equivalently, every closed submodule of $M$ is a direct summand.

Let $M$ and $N$ be two right $R-$modules. The following notions of injectivity have been established and investigated extensively.

1. $N$ is called an *$M-$injective module* if every homomorphism from any submodule of $M$ to $N$ extends to an element of $\mathrm{Hom}_R(M, N)$. $N$ is called a *quasi-injective module* if it is $N-$injective. $N$ is called an *injective module* if it is $M-$injective for every right $R-$module $M$.

2. $N$ is called an *$M-c-$injective module* if every homomorphism from any closed submodule of $M$ to $N$ extends to an element of $\mathrm{Hom}_R(M, N)$. $N$ is called *quasi$-c-$injective* if it is $N-c-$injective. $N$ is called a *$c-$injective module* if it is $M-c-$injective for every right $R-$module $M$ (see [5, 10]).

3. $N$ is called an *$M-cp-$injective module* if every homomorphism from any closed and $M-$cyclic submodule of $M$ to $N$ extends to an element of $\mathrm{Hom}_R(M, N)$. $N$ is called a *cp*$-$injective module if it is $M-$ *cp*$-$injective for every right $R-$module $M$ (see [3]).

4. $N$ is called a *principally $M-$injective module* if every homomorphism from any cyclic submodule of $M$ to $N$ extends to an element of $\mathrm{Hom}_R(M, N)$. According to [11], $N$ is called a *principally quasi-injective module* if $N$ is principally $N-$injective.

5. $N$ is called an *$M-$mininjective module* if every homomorphism from any minimal submodule of $M$ to $N$ extends to an element of $\mathrm{Hom}_R(M, N)$. $N$ is called a *quasi-mininjective module* if it is $N-$mininjective (see [12]). In [14], quasi-mininjective modules are referred to by the term *minimal quasi-injective modules.*

6. $N$ is called a *simple $M-$injective module* if for every $A \hookrightarrow M$, any $f \in \mathrm{Hom}_R(A, N)$ with simple image extends to an element of $\mathrm{Hom}_R(M, N)$. $N$ is called a *simple quasi-injective module* if it is simple $N-$injective (see [12]).

Motivated by the notions above, we investigate some weak forms of principal injectivity relative to the classes of cyclic submodules, closed submodules, and minimal submodules. Cyclic $M-c-$injective modules and cyclic quasi$-c-$injective modules are defined and studied in sections 2 and 3. Our main results that answer our question are shown in the last section as applications of cyclic $c-$injectivity, simple injectivity, and mininjectivity. Surprisingly, in spite of weak injectivity, characterizations of right V-rings, right GV-rings, right pV-rings, WV-rings, and commutative Noetherian domains that are Dedekind are obtained.

## 2. On cyclic $M-c-$injective modules

**Definition 1** *Let $M$ and $N$ be two right $R-$modules. $N$ is called a cyclic closed $M-$injective (or cyclic $M-c-$injective) module if every homomorphism from any closed and cyclic submodule of $M$ to $N$ extends to an element of* $\mathrm{Hom}_R(M,N)$.

*$M$ is called a cyclic quasi$-c-$injective module if it is cyclic $M-c-$injective. $M$ is said to be cyclic $c-$injective if it is cyclic $N-c-$injective for every right $R-$module $N$. In particular, a ring $R$ is called a right cyclic $c-$injective ring if $R_R$ is a cyclic quasi$-c-$injective module.*

Note that, in general, $cp-$injective modules (see [3]) and cyclic $c-$injective modules are different, since cyclic submodules and $M-$cyclic submodules are distinguished. However, for a ring, being right $cp-$injective (by sense of [3]) and being right cyclic $c-$injective are the same. Therefore, in order to unify terminologies in the case of rings, we say right cyclic $c-$injective rings.

Let $M$ and $N$ be right $R-$modules. Then, we have

(1) If $N$ is $M-$injective, then $N$ is $M-c-$injective.

(2) If $N$ is $M-c-$injective, then $N$ is cyclic $M-c-$injective and $M-cp-$injective.

(3) If $M$ is principally quasi-injective or quasi$-c-$injective, then $M$ is cyclic quasi$-c-$injective.

For a ring $R$, right (self) injectivity implies right principal injectivity and right $c-$injectivity, and right principal injectivity or right $c-$injectivity implies right cyclic $c-$injectivity.

**Proposition 1** *Let $M_1$ and $M_2$ be right $R-$modules. If $M=M_1\oplus M_2$ is a cyclic quasi$-c-$injective module, then $M_i$ is cyclic $M_j-c-$injective, for $i,j=1,2$. In particular, if $M^n$ is a cyclic quasi$-c-$injective for any integer $n\geq 2$, then $M$ is cyclic quasi$-c-$injective.*

An element $m$ of a module $M=M_R$ is called a *closed element* if $mR$ is a closed submodule of $M$. In particular, an element $x$ of a ring $R$ is said to be *right closed* (resp. *left closed*) if $xR$ (resp. $Rx$) is a closed right ideal (resp. closed left ideal) of $R$.

According to [8], $M$ is called a *P-extending module* provided every cyclic submodule is essential in a direct summand. By [13], *ECS modules* are those whose every closed submodule containing essentially a cyclic submodule is a direct summand. Following [4], for *CMS modules* all closed and $M-$cyclic submodules are direct summands. For *right C-regular ring* every closed principal right ideal is a direct summand. Next, we study a weak CS property.

**Definition 2** *A right $R-$module $M$ is called a CCS module if for every closed element $m\in M, mR$ is a direct summand. In particular, a ring $R$ is called a right CCS ring if every closed, principal right ideal is a direct summand.*

**Example 1** *We consider $Z-$modules, $M=Z\oplus Z$, and $A=2Z\oplus 3Z$. Then it is easy to check that $A$ is an $M-$cyclic submodule of $M$ but is not cyclic. On the other hand, $(2,3)Z$ is not $M-$cyclic.*

*Considering $Z-$module $M=Q\oplus Z/Zp$, where $Q$ is the set of all rational numbers, $Z$ is the set of all integers, and $p$ is a prime number. Note that $Z/Zp$ is a simple $Z-$module, $Q$ is a uniform $Z-$module, and $Q$ has no closed proper submodule. Therefore, the only closed and cyclic submodules of $M$ are 0, $Q$, and $Z/Zp$. Thus $M$ is CCS and so is cyclic quasi$-c-$injective (by Proposition 2 in the following) but not principally quasi-injective.*

Left CCS rings are defined analogously. We have the relation of extending properties on a module: being CS $\Rightarrow$ being ECS $\Rightarrow$ being P-extending $\Rightarrow$ being CCS. Every CCS module is cyclic quasi$-c-$injective (by the following result). By Example 1, we see that CCS modules and CMS modules are distinguished. However, for a ring $R$, $R_R-$cyclic submodules and cyclic submodules of $R_R$ coincide (in fact, both are principal (or cyclic) right ideal of $R$). Thus, saying that $R$ is right CCS, or $R$ is right $C-$regular [4], or $R_R$ is CMS [4] is the same. Therefore, we say right CCS rings in unity.

**Proposition 2** *The following conditions are equivalent for a right $R-$module $M$:*

*(1) $M$ is a CCS module;*

*(2) Every right $R-$module is cyclic $M-c-$injective;*

*(3) Every cyclic right $R-$module is cyclic $M-c-$injective;*

*(4) Every closed and cyclic submodule of $M$ is cyclic $M-c-$injective.*

**Proof** The proof is routine. □

We say that a right $R-$module $M$ has the *CC property* if every closed submodule of $M$ is cyclic.

**Proposition 3** *The following statements are equivalent for a right $R-$module $M$ with the CC property:*

*(1) $M$ is a CS module;*

*(2) Every right $R-$module is $M-c-$injective;*

*(3) Every right $R-$module is cyclic $M-c-$injective;*

*(4) $M$ is a CCS module.*

**Proof** The implications (*1*) $\Rightarrow$ (*2*) $\Rightarrow$ (*3*) are routine.

(*3*) $\Rightarrow$ (*4*) Let $A$ be an arbitrary closed and cyclic submodule of $M$. Then $A$ is cyclic $M-c-$injective and so is a direct summand of $M$. This implies that $M$ is a CCS module.

(*4*) $\Rightarrow$ (*1*) Let $A$ be an arbitrary closed submodule of $M$. Since $M$ has the CC property, $A$ is cyclic. Thus $A$ is a direct summand of $M$, since $M$ is CCS. This shows that $M$ is a CS module, completing the proof. □

**Example 2** *Let $Z$ be the set of all integers. Consider $Z-$module $M = (Z/Z2) \oplus (Z/Z8)$. We observe that all closed submodules of $M$ are cyclic. However, there are cyclic submodules that are not closed. It is easy to check that $M$ has the CC property but it is not a CCS module and so it is not extending by Proposition 3 (in fact, $((1+Z2) \oplus (2+Z8))Z$ is a cyclic closed submodule of $M$ but not a direct summand). By this example, we also see that a direct sum of two CCS (even extending) modules may not be a CCS module.*

**Theorem 1** *Let $M$ be a projective right $R-$module. Then the following conditions are equivalent:*

*(1) Every homomorphic image of a cyclic $M-c-$injective module is cyclic $M-c-$injective;*

*(2) Every homomorphic image of an $M-c-$injective module is cyclic $M-c-$injective;*

*(3) Every homomorphic image of an $M-$injective module is cyclic $M-c-$injective;*

*(4) Every homomorphic image of an injective module is cyclic $M-c-$injective;*

*(5) Every cyclic and closed submodule of $M$ is projective.*

*(6) For every closed element $m \in M$, $\mathbf{r}_R(m)$ is a direct summand of $R$.*

**Proof** The implications *(1)* $\Rightarrow$ *(2)* $\Rightarrow$ *(3)*$\Rightarrow$ *(4)* are obvious.

*(4)* $\Rightarrow$ *(5)* Let $X$ be a closed and cyclic submodule of $M$. We assume that $N$ and $K$ are two right $R-$modules, $N$ is injective, and $\phi : N \to K$ is an epimorphism. By [2, Proposition 5.1], in order to prove that $X$ is projective, it is sufficient to show that every homomorphism $\psi : X \to K$ lifts to a homomorphism from $X$ to $N$.

By *(4)*, $K$ is cyclic $M - c-$injective; thus $\psi$ extends to a homomorphism $\alpha : M \to K$. Since $M$ is a projective module, there exists $\beta : M \to N$ such that $\phi\beta = \alpha$. Then we see that $\beta$ lifts $\psi$ to $N$, implying that $X$ is projective.

$$\begin{array}{ccc} 0 \to & X \hookrightarrow M \\ & \psi \searrow \downarrow \alpha \\ & N \xrightarrow{\phi} K \to 0 \end{array}$$

*(5)* $\Rightarrow$ *(1)* Assuming that $N$ is cyclic $M - c-$injective and $X$ is projective for any closed and cyclic submodule $X$ of $M$. Then any homomorphism $\psi : X \to K$ lifts to a homomorphism $\gamma : X \to N$ and $\gamma$ extends to a homomorphism $\lambda : M \to N$. Thus, $\phi\lambda : M \to K$ is an extension of $\psi$, showing that $K$ is cyclic $M - c-$injective.

*(5)* $\Leftrightarrow$ *(6)* It is clear, finishing the proof. □

Let $M$ and $N$ be right $R-$modules. It is obvious to say that if every quotient of $N$ is cyclic $M - c-$injective and then so is $N$. However, the converse need not be true. Theorem 1 gives us a necessary and sufficient condition for a factor module of a cyclic $M - c-$injective module to be cyclic $M - c-$injective.

**Corollary 1** *Let $M = M_R$ be a projective module. If $N = N_R$ is a cyclic $M - c-$injective module, then every quotient of $N$ is again cyclic $M - c-$injective if and only if every closed and cyclic submodule of $M$ is projective.*

## 3. On cyclic quasi$-c-$injective modules

In this section, properties of cyclic $c-$injectivity that are similar to principal injectivity in [12] are proved. By Theorem 1, we have the following corollaries.

**Corollary 2** *Let $M$ be a projective right $R-$module that is cyclic quasi$-c-$injective. Then the following statements are equivalent:*

*(1) Every factor of $M$ is cyclic $M - c-$injective;*

*(2) $f(M)$ is cyclic $M - c-$injective for every endomorphism $f$ of $M$;*

*(3) $mR$ is projective for every closed element $m \in M$.*

**Corollary 3** *For a right cyclic $c-$injective ring $R$, the following conditions are equivalent:*

*(1) $xR$ is cyclic $R_R - c-$injective for every $x \in R$;*

*(2) $R/I$ is cyclic $R_R - c-$injective for every right ideal $I$;*

*(3) $aR$ is projective for every right closed element $a \in R$.*

**Example 3** *Let $F$ be a field with only two elements and $R$ be a $F-$algebra having basis $\{e_1, e_2, e_3, n_1, n_2, n_3, n_4\}$ with the following multiplication table:*

| | $e_1$ | $e_2$ | $e_3$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $e_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $e_2$ | 0 | $e_2$ | 0 | $n_1$ | 0 | 0 | $n_4$ |
| $e_3$ | 0 | 0 | $e_3$ | 0 | 0 | 0 | 0 |
| $n_1$ | $n_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $n_2$ | $n_2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $n_3$ | 0 | 0 | $n_3$ | 0 | 0 | 0 | 0 |
| $n_4$ | 0 | 0 | $n_4$ | 0 | 0 | 0 | 0 |

*Considering the right $R-$ module $M = e_2R$. Since $e_2R$ is a direct summand of $R$, $M$ is projective. Closed and cyclic submodules of $M$ are precisely $n_1R$ and $n_4R$. We observe that $M$ is cyclic quasi$-c-$ injective (but not CCS) and that $\mathbf{r}_R(n_1)$ and $\mathbf{r}_R(n_4)$ are direct summands of $R$. Thus, Corollary 2 is applied.*

Now we are going to prove a characterization of cyclic quasi$-c-$injective modules.

**Theorem 2** *Let $M$ be a right $R-$module and its endomorphism ring $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent:*

*(1) $M$ is a cyclic quasi$-c-$ injective module;*

*(2) $\mathbf{l}_M\mathbf{r}_R(m) = Sm = \{f(m)|f \in S\}$ for every closed element $m \in M$;*

*(3) $\mathbf{r}_R(m) \subseteq \mathbf{r}_R(n)$ implies $Sn \subseteq Sm$, for every closed element $m \in M$ and for any $n \in M$;*

**Proof** Let $m$ be an arbitrary closed element of $M$.

*(1)⇒(2)* For every $f \in S$, we have $f(m)\mathbf{r}_R(m) = 0$ and so $Sm \subseteq \mathbf{l}_M\mathbf{r}_R(m)$. For $x \in \mathbf{l}_M\mathbf{r}_R(m)$, it is clear that $\mathbf{r}_R(m) \subseteq \mathbf{r}_R(x)$; thus the mapping $\beta : mR \to M, ma \mapsto xa, a \in R$, is well defined. It is easy to see that $\beta$ is a homomorphism. Since $M$ is a cyclic quasi$-c-$injective module, $\beta$ extends to an endomorphism $\alpha$ of $M$; hence $x = \beta(m) = \alpha(m) \in Sm$. This shows that $\mathbf{l}_M\mathbf{r}_R(m) \subseteq Sm$; thus $\mathbf{l}_M\mathbf{r}_R(m) = Sm$.

*(2)⇒(3)* For every $n \in M$, if $\mathbf{r}_R(m) \subseteq \mathbf{r}_R(n)$, then $n \in \mathbf{l}_M\mathbf{r}_R(m)$. By (2), $Sm = \mathbf{l}_M\mathbf{r}_R(m)$, and so $n = \beta(m)$ for some $\beta \in S$. Thus $Sn = S\beta(m) \subseteq Sm$.

*(3)⇒(1)* We will show that for every closed element $m \in M$, if $\gamma : mR \to M$ is an $R-$homomorphism, then $\gamma(m) \in Sm$. Assuming that $\gamma(m) = x$, for some $x \in M$, we have $\mathbf{r}_R(m) \subseteq \mathbf{r}_R(x)$, and so $Sx \subseteq Sm$ by (3). This implies $x = f(m)$ for some $f \in S$, showing $\gamma(m) \in Sm$. Therefore, $M$ is cyclic quasi$-c-$injective. □

We obtain a characterization of right cyclic $c-$injective rings by the following corollary.

**Corollary 4** *(also see [3, Corollary 3.9])*

*The following conditions are equivalent for a ring $R$:*

*(1) $R$ is right cyclic $c-$ injective;*

*(2) $\mathbf{lr}(a) = Ra$ for every right closed element $a \in R$;*

*(3) $\mathbf{r}(a) \subseteq \mathbf{r}(b)$ implies $Rb \subseteq Ra$, for every right closed element $a \in R$ and for every $b \in R$;*

*(4) For every right closed element $a \in R$, $\mathbf{l}(kR \cap \mathbf{r}(a)) = \mathbf{l}(k) + Ra$, for any $k \in R$ such that $akR$ is closed.*

**Proof** *(1)⇔(2)⇔(3)⇔(4)* This follows from Theorem 2.

*(3)⇒(4)* Let $k$ be any element of $R$ such that $akR$ is closed in $R_R$. If $x \in \mathbf{l}(kR \cap \mathbf{r}(a))$ then $\mathbf{r}(ak) \subseteq \mathbf{r}(xk)$, and so $xk = yak, y \in R$, by *(3)*. Thus, $x - ya \in \mathbf{l}(k)$; hence $x = z + ya, z \in \mathbf{l}(k)$. This implies that $\mathbf{l}(kR \cap \mathbf{r}(a)) \subseteq \mathbf{l}(k) + Ra$. The other inclusion always holds.

*(4)⇒(2)* It is routine by taking $k = 1$. □

**Proposition 4** *For a cyclic quasi−c−injective module $M = M_R$, and for every closed element $m \in M$, if $mR \cong A$ and $A$ is a direct summand and then so is $mR$.*

*In particular, if $R$ is right cyclic c−injective, then for every closed element $a \in R, aR \cong eR, e = e^2 \in R$ implies $aR = bR$, for some $b = b^2 \in R$.*

**Proof** Observe that $A$ is cyclic $M - c-$injective. Since $mR \cong A$, $mR$ is cyclic $M - c-$injective. Thus $mR$ is a direct summand of $M$. The case of the ring is clear. □

We will conclude this section by giving conditions in which the endomorphism ring of a module, in particular, a self-generator, a self-cogenerator, is right (left) cyclic $c-$injective. By analogous way, a ring $R$ is *left cyclic c−injective* if and only if for every left closed element $s \in R$ (i.e. $Rs$ is closed in ${}_RR$) and any $R-$homomorphism $\alpha : Rs \to R$ extends to a right multiplication, $\alpha = \cdot t$, for some $t \in R$. It is routine to obtain the following lemma, which is similar to Corollary 4.

**Lemma 1** *The following conditions are equivalent for a ring $R$:*

*(1) $R$ is left cyclic c−injective;*

*(2) $\mathbf{rl}(a) = aR$ for every left closed element $a \in R$;*

*(3) $\mathbf{l}(a) \subseteq \mathbf{l}(b)$ implies $bR \subseteq aR$, for every left closed element $a \in R$ and for every $b \in R$;*

**Theorem 3** *Let $M$ be a right $R-$module and $S = \mathrm{End}(M_R)$.*

*(1) We suppose that $M$ generates $\ker(f)$ for every right closed element $f \in S$. Then, $S$ is right cyclic c−injective if and only if $\ker(f) \subseteq \ker(g)$ implies $g \in Sf$, where $g$ is an arbitrary element of $S$.*

*(2) Assuming that $M$ cogenerates $M/f(M)$ for every left closed element $f \in S$. Then $S$ is left cyclic c−injective if and only if $g(M) \subseteq f(M)$ implies $g \in fS$, where $g$ is an arbitrary element of $S$.*

**Proof** *(1)* Let $f$ be an arbitrary right closed element of $S$. We assume that $S$ is right cyclic $c-$injective and $\ker(f) \subseteq \ker(g)$, where $g \in S$. For every $s \in S, fs = 0$ implies $sM \subseteq \ker(f) \subseteq \ker(g)$; thus $gsM = 0$. This means that $gs = 0$, and hence $\mathbf{r}(f) \subseteq \mathbf{r}(g)$. By Corollary 4, $Sg \subseteq Sf$ and thus $g = sf \in Sf$ for some $s \in S$.

Conversely, for $h \in \mathbf{lr}(f)$, we will show that $\ker(f) \subseteq \ker(h)$. For any $x \in \ker(f)$, since $\ker(f)$ is generated by $M$, we have $x = \sum_I \gamma_i(m_i), m_i \in M, \gamma_i : M \to \ker(f)$, for some finite index set $I$. Then $f\gamma_i = 0$ for each $i \in I$, and so $\gamma_i \in \mathbf{r}(f)$. Therefore, $\mathbf{lr}(f) \subseteq \mathbf{l}(\gamma_i), i \in I$, and thus $h\gamma_i = 0$. As a consequence, $h(x) = 0$, so $x \in \ker(h)$ and hence $\ker(f) \subseteq \ker(h)$. By the hypothesis, $h \in Sf$ and hence $\mathbf{lr}(f) \subseteq Sf$. This implies $\mathbf{lr}(f) = Sf$, since the other inclusion always holds. By Corollary 4, $S$ is a right cyclic $c-$injective ring.

*(2)* Let $f$ be an arbitrary left closed element of $S$. We suppose that $S$ is left cyclic $c-$injective and $g(M) \subseteq f(M)$, where $g \in S$. For every $s \in S, sf = 0$ implies $sf(M) = 0$; thus $sg(M) = 0$. This means that $sg = 0$ and hence $\mathbf{l}(f) \subseteq \mathbf{l}(g)$. By Lemma 1, $gS \subseteq fS$ and thus $g = fs \in fS$ for some $s \in S$.

Conversely, if $h \in \mathbf{rl}(f)$, then $\mathbf{l}(f) \subseteq \mathbf{l}(h)$. We will show that $h(M) \subseteq f(M)$. In contrast, assume that there exists some $m_0 \in M$ such that $h(m_0) \notin f(M)$. Thus the homomorphism $\alpha : M \to M/f(M), m \mapsto h(m) + f(M)$ is nonzero, because at least $\alpha(m_0) = h(m_0) + f(M) \neq 0$. Then there is an $R-$homomorphism $\gamma : M/f(M) \to M$ satisfying $\gamma\alpha(m_0) = \gamma(h(m_0)+f(M)) \neq 0$, since $M$ cogenerates $M/f(M)$. We have $\beta f = 0$, where $\beta : M \to M$ is defined by $\beta(m) = \gamma(m+f(M)), m \in M$. Then $\beta h(m_0) = \gamma(h(m_0)+f(M)) \neq 0, \beta h \neq 0$, contradicting $\mathbf{l}(f) \subseteq \mathbf{l}(h)$. Therefore, $h(M) \subseteq f(M)$ must hold. Thus, by assumption, $h = fs \in fS$ and so $\mathbf{rl}(f) \subseteq fS$. This says that $\mathbf{rl}(f) = fS$, since the other inclusion always holds. Consequently, by Lemma 1, $S$ is left cyclic $c-$injective. □

## 4. Applications of weak forms of principal injectivity

This section aims to answer our question stated in the introductory part. We show that cyclic $c-$injectivity is useful for characterizing commutative Noetherian rings that are Dedekind domains; principal injectivity, simple injectivity, and mininjectivity help us to obtain right V-rings, right GV-rings, and WV-rings. The next theorems can be regarded as the most important ones in this paper. Throughout this section, $R$ is an associative ring with identity and all modules are unitary, right over $R$.

**Lemma 2** *The following conditions are equivalent for a simple module $M_R$ :*

*(1) $M$ is $c-$injective;*

*(2) $M$ is cyclic $c-$injective;*

*(3) $M$ is $c-$injective relative to minimal submodules, that is for every right $R-$module $N$, every homomorphism from any minimal, closed submodule of $N$ to $M$ extends to an element of* $\mathrm{Hom}_R(N, M)$.

**Proof** It is obvious for the implications *(1)⇒(2)⇒(3)*.

*(3)⇒(1)* Let $N$ be an arbitrary right $R-$module and $A$ a closed submodule of $N$. For any homomorphism $f : A \to M$, since $M$ is simple, $A/\ker(f)$ is a minimal submodule of $N/\ker(f)$. Moreover, $A/\ker(f)$ is closed in $N/\ker(f)$ (see [6]). We put $\bar{f} : A/\ker(f) \to M, \bar{f}(a + \ker(f)) = f(a), a \in A$. Since $M$ is $c-$injective relative to minimal submodules, $\bar{f}$ extends to a homomorphism $\varphi : N/\ker(f) \to M$. Let $\pi : N \to N/\ker(f)$ be the canonical projection. Then we observe that $\alpha = \varphi\pi$ is an extension of $f$. Indeed, for every $a \in A, \alpha(a) = \varphi\pi(a) = \varphi(a + \ker(f)) = \bar{f}(a + \ker(f)) = f(a)$. This implies that $M$ is $N-c-$injective. Consequently, $M$ is $c-$injective, completing the proof. □

**Theorem 4** *If $R$ is a commutative Noetherian domain, then the following statements are equivalent:*

*(1) $R$ is Dedekind;*

*(2) Every simple $R-$module is $c-$injective;*

*(3) Every simple $R-$module is cyclic $c-$injective;*

*(4) Every simple $R-$module is $c-$injective relative to minimal submodules;*

*(5) Every simple $R-$module $M$ is simple $N-$injective relative to closed submodules, for every $R-$module $N$, that is for any closed submodule $A$ of $N$, every $f \in \mathrm{Hom}_R(A, M)$ with a simple image extends to an element of* $\mathrm{Hom}_R(N, M)$.

**Proof** The implications *(2)⇔(3)⇔(4)* follow from Lemma 2.

*(1)⇔(2)* It is by [10, Corollary 5.5].

*(2)⇒(5)* It is straightforward.

*(5)⇒(4)* For every $R-$module $N$ and for any closed, minimal submodule $A$ of $N$, every $f : A \to M$ has a simple image. Since $M$ is simple $N-$injective relative to closed submodules, $f$ extends to an element of $\mathrm{Hom}_R(N, M)$. This implies that $M$ is $c-$injective relative to minimal submodules, as required. The proof is now complete. □

**Theorem 5** *The following conditions are equivalent for a ring $R$ :*

*(1) $R$ is a right V-ring (resp. right GV-ring);*

*(2) Every simple (resp. simple singular) module is injective;*

*(3) Every simple (resp. simple singular) module is principally $N-$injective for every module $N$;*

*(4) Every simple (resp. simple singular) module is principally $R/I-$injective for every right ideal $I$ of $R$;*

*(5) Every simple (resp. simple singular) module is $N-$mininjective for every module $N$;*

*(6) Every simple (resp. simple singular) module is $R/I-$mininjective for every right ideal $I$ of $R$;*

*(7) Every simple (resp. simple singular) module is simple $N-$injective for every module $N$;*

*(8) Every simple (resp. simple singular) module is simple $R/I-$injective for every right ideal $I$ of $R$;*

*(9) Every module is quasi-mininjective.*

**Proof** We will prove for the case of being right V-rings. The other one is induced similarly.

*(1)*$\Leftrightarrow$*(2)* It is by the definition of right V-rings.

*(1)*$\Leftrightarrow$*(9)* It is by[14, Theorem 1.5].

*(2)*$\Leftrightarrow$*(6)* Let $M$ be an arbitrary simple module. If $M$ is injective, then it is obvious that $M$ is $R/I-$mininjective for every right ideal $I$ of $R$. Conversely, we assume that for every right ideal $J$ of $R$, $M$ is $R/J-$mininjective. For every right ideal $I$ of $R$ and for any homomorphism $f: I \to M$, we have $I/\ker(f) \cong M$ via the isomorphism $\bar{f}(a+\ker(f)) = f(a), \forall a \in I$. Moreover, $\ker(f)$ is a right ideal of $R$ and $I/\ker(f)$ is a minimal submodule of $R/\ker(f)$. Since $M$ is $R/\ker(f)-$mininjective, $\bar{f}$ extends to a homomorphism $\varphi: R/\ker(f) \to M$. Let $\pi: R \to R/\ker(f)$ be the canonical projection. Then $\alpha = \varphi\pi$ is an extension of $f$. Indeed, for every $a \in I, \alpha(a) = \varphi\pi(a) = \varphi(a+\ker(f)) = \bar{f}(a+\ker(f)) = f(a)$. This implies that $M$ is $R-$injective and hence is injective, as required.

*(2)*$\Rightarrow$*(3)*$\Rightarrow$*(4)*$\Rightarrow$*(6)*, *(2)*$\Rightarrow$*(5)*$\Rightarrow$*(6)*, and *(2)*$\Rightarrow$*(7)*$\Rightarrow$*(8)* are clear.

*(8)*$\Rightarrow$*(6)* Let $M$ be a simple module. For any right ideal $I$ of $R$, we put $N = R/I$. Let $A$ be an arbitrary minimal submodule of $N$. Then every homomorphic image of $A$ is simple. Since $M$ is simple $N-$injective, every homomorphism $f: A \to M$ extends to $N$. This shows that $M$ is $R/I-$mininjective. □

**Theorem 6** *The following conditions are equivalent for a ring $R$:*

*(1) $R$ is a WV-ring;*

*(2) Every simple module is $R/I-$injective for every right ideal $I$ of $R$ satisfying $R/I \ncong R$;*

*(3) Every simple module is principally $R/I-$injective for every right ideal $I$ of $R$ satisfying $R/I \ncong R$;*

*(4) Every simple module is $R/I-$mininjective for every right ideal $I$ of $R$ satisfying $R/I \ncong R$;*

*(5) Every simple module is simple $R/I-$injective for every right ideal $I$ of $R$ satisfying $R/I \ncong R$.*

**Proof** *(1)*$\Leftrightarrow$*(2)*, *(2)*$\Rightarrow$*(3)*$\Rightarrow$*(4)*, and *(2)*$\Rightarrow$*(5)* It is obvious.

*(4)*$\Rightarrow$*(2)* It is similar to the proof of *(2)*$\Leftrightarrow$*(6)* of Theorem 5 that if a simple module is $R/I-$mininjective for every right ideal $I$ of $R$ satisfying $R/I \ncong R$, then it is $R/I-$injective for every right ideal $I$ of $R$ satisfying $R/I \ncong R$.

*(5)*$\Rightarrow$*(2)* It is elementary. □

We see that our Theorem 5 agrees with Zhu and Tan [14, Theorem 1.5], in which right V-rings are characterized by minimal quasi-injective (or quasi-mininjective) modules. The proof of the following theorem is similar to such things of Theorem 5 and Theorem 6.

**Theorem 7** *The following conditions are equivalent for a ring $R$ :*

*(1) $R$ is a right $pV$-ring;*

*(2) Every simple module is principally $R_R-$ injective;*

*(3) Every simple module is simple $R_R-$ injective;*

*(4) Every simple module is $(R/I)_R-$ mininjective for every right ideal $I$ of $R$.*

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