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Maximal subsemigroups and finiteness conditions on transformation semigroups with fixed sets

Yanisa CHAIYA[1], **Preeyanuch HONYAM, Jintana SANWONG***
Department of Mathematics, Chiang Mai University, Chiang Mai, Thailand

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Abstract: Let Y be a fixed subset of a nonempty set X and let $Fix(X,Y)$ be the set of all self maps on X which fix all elements in Y. Then under the composition of maps, $Fix(X,Y)$ is a regular monoid. In this paper, we prove that there are only three types of maximal subsemigroups of $Fix(X,Y)$ and these maximal subsemigroups coincide with the maximal regular subsemigroups when $X \setminus Y$ is a finite set with $|X \setminus Y| \geq 2$. We also give necessary and sufficient conditions for $Fix(X,Y)$ to be factorizable, unit-regular, and directly finite.

Key words: Transformation semigroup with fixed set, maximal subsemigroup, maximal regular subsemigroup, factorizable, unit-regular, directly finite

1. Introduction

Let X be a nonempty set and let $T(X)$ be the full transformation semigroup, that is the semigroup of all mappings from X into itself under the composition of maps. It is well known that $T(X)$ is a regular monoid and every semigroup can be embedded in $T(Z)$ for some nonempty set Z ([6], Exercises 15 and Theorem 1.1.2).

Let Y be a fixed subset of X and define

$$Fix(X,Y) = \{\alpha \in T(X) : a\alpha = a \text{ for all } a \in Y\}.$$

In 2013, Honyam and Sanwong [5] proved that $Fix(X,Y)$ is a regular semigroup and they also determined its Green's relations and ideals. Moreover, they proved that $Fix(X,Y)$ is never isomorphic to $T(Z)$ for any set Z, and every semigroup S is isomorphic to a subsemigroup of $Fix(X',Y')$ for some appropriate sets X' and Y' with $Y' \subseteq X'$.

Let S be a semigroup. $x \in S$ is *regular* if $x = xyx$ for some $y \in S$, and S is a *regular semigroup* if all of its elements are regular.

A proper subset M of a semigroup (regular semigroup) S is called a *maximal* (*maximal regular*) *subsemigroup* if M is a semigroup (regular semigroup), and any subsemigroup (regular subsemigroup) of S properly containing M must be S.

Let X be a set. The *symmetric group on* X is the set $\mathcal{S}(X)$ of all permutations of X and is the group of units of $T(X)$. In the case that $X = \{1,\ldots,n\}$, we will write $T(X) = T_n$ and $\mathcal{S}(X) = \mathcal{S}_n$.

*Correspondence: jintana.s@cmu.ac.th
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For an arbitrary integer r such that $1 \leq r \leq n$, define

$$K(n,r) = \{\alpha \in T_n : |X\alpha| \leq r\},$$

and so $K(n,r)$ is an ideal of T_n and $K(n,n) = T_n$.

In 1966, Baĭramov [2] characterized the maximal subsemigroups of T_n, which is of the form $K(n,n-1) \cup M$, where M is a maximal subgroup of $\mathcal{S}_n$, or $K(n,n-2) \cup \mathcal{S}_n$. In 2001, Yang [11] described the maximal subsemigroups of the finite singular transformation semigroup $K(n,n-1)$. In 2002, You [12] determined all the maximal regular subsemigroups of T_n, and those maximal regular subsemigroups coincide with the maximal subsemigroups that first appeared in [2]. Moreover, You described all the maximal regular subsemigroups of $K(n,r)$. Later, in 2004, Yang and Yang [10] completely described the maximal subsemigroups of the semigroup $K(n,r)$. For an infinite set X, in 1965 Gavrilov [4] proved that there are five maximal subsemigroups of $T(X)$ containing $\mathcal{S}(X)$ when X is countable and in 1995 Pinsker [8] extended Gavrilov's results to an arbitrary set. Recently, East et al. [3] classified the maximal subsemigroups of the full transformation semigroup on an infinite set X containing one of the following subgroups of $\mathcal{S}(X)$: the pointwise stabilizer of a nonempty finite subset of X, the stabilizer of an ultrafilter on X, or the stabilizer of a partition of X into finitely many subsets of equal cardinality.

A semigroup S is said to be *factorizable* if $S = GE$ for some subgroup G of S and some set E of idempotents of S. We note that if a semigroup S is factorizable as GE, then $S = GE(S)$.

In 1979, Tirasupa [9] proved that: if a semigroup S is factorizable as GE, then G is a maximal subgroup of S. If S has an identity, then G is a group of units of S. Moreover, the author showed that $T(X)$ is factorizable if and only if X is finite.

A monoid S with identity 1 is called *unit-regular* if, for every element x of S, there is a unit u with $x = xux$. S is called *directly finite*, if for any x and y in S, $xy = 1$ implies that $yx = 1$.

In 1980, Alarcao [1] characterized when a monoid S is unit-regular and when it is directly finite. Moreover, he gave a relationship between a unit-regular semigroup and a directly finite semigroup.

In this paper, we prove that there are only three types of maximal subsemigroups of $Fix(X,Y)$ when $X \setminus Y$ is a finite set with $|X \setminus Y| \geq 2$ in Section 3. In Section 4, we show that the maximal subsemigroups and the maximal regular subsemigroups of $Fix(X,Y)$ coincide when $X \setminus Y$ is finite. Moreover, in Section 5, we give necessary and sufficient conditions for $Fix(X,Y)$ to be factorizable, unit-regular, and directly finite.

2. Preliminaries and notations

For all undefined notions, the reader is referred to [6].

Let X be a set and Y a fixed subset of X. Then $Fix(X,Y)$ is a regular subsemigroup of $T(X)$. We note that $Fix(X,Y)$ contains 1_X, the identity map on X. If $Y = \emptyset$, then $Fix(X,Y) = T(X)$; and if $|X| = 1$ or $X = Y$, then $Fix(X,Y)$ consists of one element, 1_X. Hence, throughout this paper we will consider the case $Y \subsetneq X$ and $|X| > 1$.

Green's relations and ideals on $Fix(X,Y)$ are used in this paper. For convenience, we present them here.

Theorem 2.1 [5] *Let $\alpha, \beta \in Fix(X,Y)$. Then the following statements hold.*

(1) *$\alpha \mathcal{R} \beta$ in $Fix(X,Y)$ if and only if $\pi_\alpha = \pi_\beta$;*

(2) $\alpha\mathcal{L}\beta$ *in* $Fix(X,Y)$ *if and only if* $X\alpha\backslash Y = X\beta\backslash Y$ *;*

(3) $\alpha\mathcal{D}\beta$ *in* $Fix(X,Y)$ *if and only if* $|X\alpha\backslash Y| = |X\beta\backslash Y|$ *and* $\mathcal{D} = \mathcal{J}$ *.*

Here $\pi_\gamma = \{x\gamma^{-1} : x \in X\gamma\}$.

Let p be any cardinal number and let $p' = min\{q : q > p\}$.

Theorem 2.2 [5] *The following statements hold.*

(1) $Fix_k = \{\alpha \in Fix(X,Y) : |X\alpha\backslash Y| < k\}$, *where* $1 \leq k \leq |X\backslash Y|'$ *is an ideal of* $Fix(X,Y)$.

(2) *If* I *is an ideal of* $Fix(X,Y)$, *then* $I = Fix_k$ *for some* $1 \leq k \leq |X \setminus Y|'$.

For convenience, throughout this paper, unless otherwise stated, let $Y = \{y_i : i \in I\}$.

For each $\alpha \in Fix(X,Y)$, let $X\alpha = Y \cup \{b_j : j \in J\}, y_i\alpha^{-1} = A_i$ and $b_j\alpha^{-1} = B_j$. Then we can write α as follows:

$$\alpha = \begin{pmatrix} A_i & B_j \\ y_i & b_j \end{pmatrix}.$$

In this notation $A_i \cap Y = \{y_i\}$, $B_j \subseteq X \setminus Y$ and $\{b_j : j \in J\} \subseteq X \setminus Y$. Here J can be an empty set.

If S is a semigroup and $a \in S$, then D_a and H_a denote the equivalence class of $\mathcal{D}$ containing a and the equivalence class of $\mathcal{H}$ containing a, respectively, that is

$$D_a = \{x \in S : x\mathcal{D}a\} \text{ and } H_a = \{x \in S : x\mathcal{H}a\}.$$

In [5] the authors showed that H_{1_X} is the group of units of $Fix(X,Y)$. In this case,

$$H_{1_X} = \left\{ \begin{pmatrix} y_i & b_j \\ y_i & b_j\sigma \end{pmatrix} : \sigma \in \mathcal{S}(X \setminus Y) \right\}$$

where $X \setminus Y = \{b_j : j \in J\}$, is isomorphic to $\mathcal{S}(X \setminus Y)$ where $\mathcal{S}(X \setminus Y)$ is the permutation group on $X \setminus Y$. Thus H_{1_X} is the set of all bijections in $Fix(X,Y)$.

An idempotent e of a semigroup S is said to be *minimal* if e has property: $f \in E(S)$ and $f \leq e$ implies $f = e$.

The authors in [5] described the set of all minimal idempotents in $Fix(X,Y)$ as follows:

$$E_m = \left\{ \begin{pmatrix} A_i \\ y_i \end{pmatrix} : \{A_i : i \in I\} \text{ is a partition of } X \text{ with } y_i \in A_i \right\}.$$

We note that: α is an idempotent in $Fix(X,Y)$ if and only if $x\alpha = x$ for all $x \in X\alpha \setminus Y$. Moreover, E_m is a left zero semigroup.

3. Maximal subsemigroups of $Fix(X,Y)$

Throughout this section, let $X \backslash Y$ be a finite set with n elements such that $\emptyset \neq Y \subsetneq X$. In this case $Fix(X,Y)$ has $n+1$ $\mathcal{J}$-classes. Let

$$J_k = \{\alpha \in Fix(X,Y) : |X\alpha \setminus Y| = k\},$$

where $0 \leq k \leq n$. Since $\alpha_{|_Y} = 1_Y$ and $\alpha_{|_{X\setminus Y}}$ is a permutation on the set $X \setminus Y$ for each $\alpha \in J_n$, we obtain J_n is isomorphic to $\mathcal{S}_n$ the symmetric group on the set of n elements.

Consider the case when $|X \setminus Y| = 1$. Hence there are only two $\mathcal{J}$-classes of $Fix(X,Y)$, J_1 and J_0. Here J_1 has only one element 1_X and $J_0 = E_m$ the set of all minimal idempotents in $Fix(X,Y)$. In this case

$$M_\alpha = Fix(X,Y) \setminus \{\alpha\},$$

where $\alpha \in Fix(X,Y)$ are the only maximal regular subsemigroups of $Fix(X,Y)$.

In what follows, we assume that $|X \setminus Y| = n \geq 2$ and define two subsets of J_{n-1} playing an essential role in maximal subsemigroups of $Fix(X,Y)$. Let

$$J^*_{n-1} = \{\alpha \in J_{n-1} : |y\alpha^{-1}| > 1 \text{ for some } y \in Y\},$$

and for each $y \in Y$, define

$$J^y_{n-1} = \{\alpha \in J_{n-1} : |y\alpha^{-1}| = 1\}.$$

We observe that $J^*_{n-1} \cup J^y_{n-1} = J_{n-1}$ for all $y \in Y$.

We begin with the following simple theorem.

Theorem 3.1 *Let M be a maximal subgroup of J_n. Then $M \cup Fix_n$ is a maximal subsemigroup of $Fix(X,Y)$.*

Proof It is clear that $\emptyset \neq M \cup Fix_n \subsetneq Fix(X,Y)$. Since M is a group and Fix_n is an ideal, we obtain $M \cup Fix_n$ is a subsemigroup of $Fix(X,Y)$. Let S be a subsemigroup of $Fix(X,Y)$ such that $M \cup Fix_n \subsetneq S$. Then there exists $\gamma \in S \setminus (M \cup Fix_n)$, and so $\gamma \in (J_n \setminus M) \cap S$. Since M is a maximal subgroup of J_n, the subgroup of J_n generated by $M \cup \{\gamma\}$ is J_n. Thus

$$S = J_n \cup Fix_n = Fix(X,Y).$$

Therefore, $M \cup Fix_n$ is a maximal subsemigroup of $Fix(X,Y)$. □

We note that the maximal subgroups of J_n were completely characterized by Liebeck et al. (see [7] for details).

The following lemma is needed in proving Theorem 3.3 and Theorem 3.4.

Lemma 3.2 *Let S be a subsemigroup of $Fix(X,Y)$, $J_n \subseteq S$, and $\alpha \in S \cap J_{n-1}$.*

(1) *If $|y\alpha^{-1}| > 1$ for some $y \in Y$, then $\{\gamma \in J_{n-1} : |y\gamma^{-1}| > 1\} \subseteq S$.*

(2) *If $|y\alpha^{-1}| = 1$ for all $y \in Y$, then $\{\gamma \in J_{n-1} : |y\gamma^{-1}| = 1 \text{ for all } y \in Y\} \subseteq S$.*

Proof (1) Suppose that there is $i_0 \in I$ such that $|y_{i_0}\alpha^{-1}| = 2$. Let $y_{i_0}\alpha^{-1} = \{y_{i_0}, x\}$ for some $x \in X \setminus Y$. Let $X \setminus (Y \cup \{x\}) = \{a_1, \ldots, a_{n-1}\}$, $J = \{1, \ldots, n-1\}$ and $I' = I \setminus \{i_0\}$. Then we can write

$$\alpha = \begin{pmatrix} y_{i'} & \{y_{i_0}, x\} & a_j \\ y_{i'} & y_{i_0} & b_j \end{pmatrix}, \qquad (*)$$

where $b_j \in X \setminus Y$ for all $j \in J$. Let $\beta \in \{\gamma \in J_{n-1} : |y_{i_0}\gamma^{-1}| > 1\}$. As α, there is $x' \in X \setminus Y$ such that $y_{i_0}\alpha^{-1} = \{y_{i_0}, x'\}$. Therefore, we can write

$$\beta = \begin{pmatrix} y_{i'} & \{y_{i_0}, x'\} & c_j \\ y_{i'} & y_{i_0} & d_j \end{pmatrix},$$

where $c_j, d_j \in X \setminus Y$ for all $j \in J$. Now choose

$$\theta = \begin{pmatrix} y_i & x' & c_j \\ y_i & x & a_j \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} y_i & u & b_j \\ y_i & v & d_j \end{pmatrix},$$

where $u \in X \setminus X\alpha$ and $v \in X \setminus X\beta$. Then $\theta, \eta \in J_n$ and $\beta = \theta\alpha\eta \in S$.

(2) Assume that $|y\alpha^{-1}| = 1$ for all $y \in Y$. Since $\alpha \in J_{n-1}$, there exists $b \in X \setminus Y$ such that $|b\alpha^{-1}| = 2$. Let $b\alpha^{-1} = \{x, z\} \subseteq X \setminus Y$, $X \setminus (Y \cup \{x, z\}) = \{a_1, \ldots, a_{n-2}\}$ and $J = \{1, \ldots, n-2\}$. Thus we can write

$$\alpha = \begin{pmatrix} y_i & \{x, z\} & a_j \\ y_i & b & b_j \end{pmatrix}, \qquad (**)$$

where $b_j \in X \setminus Y$ for all $j \in J$. Let $\beta \in \{\gamma \in J_{n-1} : |y\gamma^{-1}| = 1 \text{ for all } y \in Y\}$. As before, we can write

$$\beta = \begin{pmatrix} y_i & \{x', z'\} & c_j \\ y_i & d & d_j \end{pmatrix},$$

where $\{x', z', d\} \subseteq X \setminus Y$ and $c_j, d_j \in X \setminus Y$ for all $j \in J$. Choose

$$\theta = \begin{pmatrix} y_i & x' & z' & c_j \\ y_i & x & z & a_j \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} y_i & b & b_j & u \\ y_i & d & d_j & v \end{pmatrix},$$

where $u \in X \setminus X\alpha$ and $v \in X \setminus X\beta$. Thus $\theta, \eta \in J_n$ and $\beta = \theta\alpha\eta \in S$. □

Theorem 3.3 *$J_n \cup J^y_{n-1} \cup Fix_{n-1}$ is a maximal subsemigroup of $Fix(X, Y)$.*

Proof Let $A = J_n \cup J^y_{n-1} \cup Fix_{n-1}$. We first prove that A is a subsemigroup of $Fix(X, Y)$. Let $\alpha, \beta \in A$. If $\alpha \in Fix_{n-1}$ or $\beta \in Fix_{n-1}$, then $\alpha\beta \in Fix_{n-1} \subseteq A$ since Fix_{n-1} is an ideal of $Fix(X, Y)$. If $\alpha, \beta \in J_n$, then $\alpha\beta \in J_n$ since J_n is a group. Now we consider the case $\alpha, \beta \in J^y_{n-1}$. Hence, we have

$$|X\alpha\beta \setminus Y| = |(X\alpha)\beta \setminus Y| \leq |X\beta \setminus Y| = n - 1.$$

The case $|X\alpha\beta \setminus Y| < n - 1$ gives $\alpha\beta \in Fix_{n-1}$. For the case $|X\alpha\beta \setminus Y| = n - 1$, we let $x \in X \setminus Y$. Thus $x\alpha \neq y \neq x\beta$. If $x\alpha \in Y$, then $x\alpha\beta = x\alpha \neq y$. If $x\alpha \in X \setminus Y$, then $x\alpha\beta \neq y$. That is $\alpha\beta \in J^y_{n-1}$. For $\alpha \in J^y_{n-1}$ and $\beta \in J_n$, we have $|X\alpha \setminus Y| = n - 1$ and β is bijective. Therefore,

$$n - 1 = |X\alpha \setminus Y| = |(X\alpha \setminus Y)\,\beta| = |X\alpha\beta \setminus Y\beta| = |X\alpha\beta \setminus Y|,$$

and $|X\alpha \setminus Y| = |(X\beta)\alpha \setminus Y| = |X\beta\alpha \setminus Y|$. Thus $\alpha\beta,\ \beta\alpha \in J_{n-1}$. Let $a \in X \setminus Y$. We have $a\alpha \neq y$. If $a\alpha \in X \setminus Y$, then $(a\alpha)\beta \neq y$ since $y\beta = y$ and β is injective. If $a\alpha \in Y$, then $(a\alpha)\beta = a\alpha \neq y$. That is $\alpha\beta \in J^y_{n-1} \subseteq A$. Since $a\beta \in X \setminus Y$, we get $a\beta\alpha \neq y$, that is $\beta\alpha \in J^y_{n-1} \subseteq A$. We obtain that A is a subsemigroup of $Fix(X, Y)$.

Let S be a subsemigroup of $Fix(X, Y)$ such that $A \subsetneq S$. Then there exists $\theta \in S \setminus A \subseteq J_{n-1}$, and so $x\theta = y$ for some $x \in X \setminus Y$. By Lemma 3.2(1), we have $\{\gamma \in J_{n-1} : |y\gamma^{-1}| > 1\} \subseteq S$. Since

$$Fix(X, Y) = A \cup \{\gamma \in J_{n-1} : |y\gamma^{-1}| > 1\} \subseteq S \subseteq Fix(X, Y),$$

we obtain $S = Fix(X, Y)$. Therefore, A is a maximal subsemigroup of $Fix(X, Y)$. □

Theorem 3.4 *$J_n \cup J^*_{n-1} \cup Fix_{n-1}$ is a maximal subsemigroup of $Fix(X,Y)$.*

Proof Let $A = J_n \cup J^*_{n-1} \cup Fix_{n-1}$. To prove that A is a subsemigroup of $Fix(X,Y)$, we consider the case $\alpha, \beta \in J^*_{n-1}$ and $|X\alpha\beta \setminus Y| = n-1$ and the case $\alpha \in J^*_{n-1}$ and $\beta \in J_n$. For the case $\alpha, \beta \in J^*_{n-1}$ and $|X\alpha\beta \setminus Y| = n-1$, we have $x\alpha = y$ for some $x \in X \setminus Y$ and $y \in Y$. Then $x\alpha\beta = (x\alpha)\beta = y\beta = y$, that is $\alpha\beta \in J^*_{n-1} \subseteq A$. Now consider the case $\alpha \in J^*_{n-1}$ and $\beta \in J_n$. Then there exists $x \in X \setminus Y$ such that $x\alpha = y$ for some $y \in Y$. Hence, $x\alpha\beta = y\beta = y$, that is $\alpha\beta \in J^*_{n-1} \subseteq A$. Since β is surjective, there exists $x' \in X \setminus Y$ such that $x'\beta = x$, and so $x'\beta\alpha = x\alpha = y$, that is $\beta\alpha \in J^*_{n-1} \subseteq A$. Hence A is a subsemigroup of $Fix(X,Y)$.

Now let S be a subsemigroup of $Fix(X,Y)$ with $A \subsetneq S$. Then there exists $\theta \in S \setminus A$, so $\theta \in J_{n-1}$ and $|y\theta^{-1}| = 1$ for all $y \in Y$. By Lemma 3.2(2), we have $\{\gamma \in J_{n-1} : |y\gamma^{-1}| = 1 \text{ for all } y \in Y\} \subseteq S$. Since

$$Fix(X,Y) = A \cup \{\gamma \in J_{n-1} : |y\gamma^{-1}| = 1 \text{ for all } y \in Y\} \subseteq S,$$

we obtain $S = Fix(X,Y)$. Therefore, A is a maximal subsemigroup of $Fix(X,Y)$. □

Our final aim is to prove that there are only three types of maximal subsemigroups of $Fix(X,Y)$.

Lemma 3.5 *If S is a maximal subsemigroup of $Fix(X,Y)$, then either $J_n \subseteq S$ or $J_{n-1} \subseteq S$.*

Proof Let S be a maximal subsemigroup of $Fix(X,Y)$ and $J_n \nsubseteq S$. Since S is maximal, we have $S \cap J_n \neq \emptyset$; otherwise $S \subsetneq M \cup Fix_n$ where M is a maximal subgroup of J_n, which contradicts the maximality of S. Moreover, $S \cap J_n = H$ is a maximal subgroup of J_n. For if H is not a maximal subgroup of J_n, then H is contained in a maximal subgroup M of J_n. Thus $S \subsetneq M \cup Fix_n$ where $M \cup Fix_n$ is a maximal subsemigroup of $Fix(X,Y)$ and this contradicts the maximality of S. Hence $S \subseteq H \cup Fix_n$. Since S is maximal, we obtain $S = H \cup Fix_n$ and that $J_{n-1} \subseteq S$ as required. □

Lemma 3.6 *Let $\alpha \in J_k$ where $0 \leq k \leq n-2$. Then α can be written as a product of β, γ for some $\beta, \gamma \in J_{k+1}$.*

Proof Let $J = \{1, \ldots, k\}$ and write

$$\alpha = \begin{pmatrix} A_i & B_j \\ y_i & b_j \end{pmatrix},$$

where $B_j \subseteq X \setminus Y, b_j \in X \setminus Y$ for all $j \in J$. Since $k \leq n-2$, we have $|A_{i_0}| \geq 2$ for some $i_0 \in I$ or $|B_{j_0}| \geq 2$ for some $j_0 \in J$.

Case 1: $|A_{i_0}| \geq 2$ for some $i_0 \in I$. Choose $u \in A_{i_0} \setminus Y$ and $v \in X \setminus X\alpha$. Let $I' = I \setminus \{i_0\}$ and define $\beta, \gamma \in Fix(X,Y)$ by

$$\beta = \begin{pmatrix} A_{i'} & A_{i_0} \setminus \{u\} & B_j & u \\ y_{i'} & y_{i_0} & b_j & v \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} y_{i'} & \{y_{i_0}, v\} & b_j & X \setminus (X\alpha \cup \{v\}) \\ y_{i'} & y_{i_0} & b_j & v \end{pmatrix}.$$

Hence $\beta, \gamma \in J_{k+1}$ and $\alpha = \beta\gamma$.

Case 2: $|B_{j_0}| \geq 2$ for some $j_0 \in J$. Choose $u \in B_{j_0}$ and $v \in X \setminus X\alpha$. Let $J' = J \setminus \{j_0\}$ and define $\beta, \gamma \in Fix(X,Y)$ by

$$\beta = \begin{pmatrix} A_i & B_{j'} & B_{j_0} \setminus \{u\} & u \\ y_i & b_{j'} & b_{j_0} & v \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} y_i & b_{j'} & \{b_{j_0}, v\} & X \setminus (X\alpha \cup \{v\}) \\ y_i & b_{j'} & b_{j_0} & v \end{pmatrix}.$$

So $\beta, \gamma \in J_{k+1}$ and $\alpha = \beta\gamma$. □

Lemma 3.7 *Let S be a subsemigroup of $Fix(X,Y)$. If $S \cap J_n = J_n$ and $S \cap J_{n-1} = J_{n-1}$, then $S = Fix(X,Y)$.*

Proof Assume that $S \cap J_n = J_n$ and $S \cap J_{n-1} = J_{n-1}$. Let $\alpha \in Fix(X,Y)$. It is clear that if $\alpha \in J_n \cup J_{n-1}$, then $\alpha \in S$. Now consider when $\alpha \in J_k$, where $0 \leq k \leq n-2$. By Lemma 3.6, we have α can be written as a product of β, γ for some $\beta, \gamma \in J_{n-1}$, that is $\alpha \in S$. Thus $S = Fix(X,Y)$. □

Theorem 3.8 *Let S be a maximal subsemigroup of $Fix(X,Y)$. Then S is one of the following forms:*

(1) *$M \cup Fix_n$, where M is a maximal subgroup of J_n;*

(2) *$J_n \cup J^y_{n-1} \cup Fix_{n-1}$ for some $y \in Y$;*

(3) *$J_n \cup J^*_{n-1} \cup Fix_{n-1}$.*

Proof Since S is a maximal subsemigroup, by Lemma 3.5 we have either $J_n \subseteq S$ or $J_{n-1} \subseteq S$.

Case 1: $J_n \subseteq S$. Therefore, $S \cap J_{n-1} \subsetneq J_{n-1}$ by Lemma 3.7. We consider two subcases.

Subcase 1.1: $(X \setminus Y)\alpha \cap Y \neq \emptyset$ for all $\alpha \in S \cap J_{n-1}$. Let $\alpha \in S \cap J_{n-1}$. Then by assumption, we have $|y\alpha^{-1}| > 1$ for some $y \in Y$. That is $\alpha \in J^*_{n-1}$. Hence $S \cap J_{n-1} \subseteq J^*_{n-1}$. Since $J_n \subseteq S$, we obtain

$$S \subseteq J_n \cup J^*_{n-1} \cup Fix_{n-1}.$$

Since the right-hand side of the above expression is a maximal subsemigroup, it follows that $S = J_n \cup J^*_{n-1} \cup Fix_{n-1}$. Therefore, S is of the form (3).

Subcase 1.2: $(X \setminus Y)\alpha \cap Y = \emptyset$ for some $\alpha \in S \cap J_{n-1}$. Then $|y\alpha^{-1}| = 1$ for all $y \in Y$. By Lemma 3.2(2) we have

$$\{\gamma \in J_{n-1} : |y\gamma^{-1}| = 1 \text{ for all } y \in Y\} \subseteq S.$$

We prove that $S \cap J_{n-1} \subseteq J^{y_0}_{n-1}$ for some $y_0 \in Y$, by supposing that it is false. Therefore, for each $y \in Y$, there exists $\beta \in S \cap J_{n-1}$ such that $|y\beta^{-1}| > 1$. Thus by Lemma 3.2(1), $\{\gamma \in J_{n-1} : |y\gamma^{-1}| > 1\} \subseteq S$. Hence

$$\bigcup_{y \in Y} \{\gamma \in J_{n-1} : |y\gamma^{-1}| > 1\} \subseteq S,$$

and so $J_{n-1} \subseteq S$, which contradicts $S \cap J_{n-1} \subsetneq J_{n-1}$. Therefore,

$$S \cap J_{n-1} \subseteq J^{y_0}_{n-1}$$

for some $y_0 \in Y$. Again, since $J_n \subseteq S$, we obtain

$$S \subseteq J_n \cup J^{y_0}_{n-1} \cup Fix_{n-1},$$

and that S is of the form (2).

Case 2: $J_{n-1} \subseteq S$. Then $S \cap J_n \subsetneq J_n$ by Lemma 3.7. Since S is maximal, by the same proof as given for Lemma 3.5, we get that $S \cap J_n = M$, where M is a maximal subgroup of J_n. Thus $S \subseteq M \cup Fix_n$. By the maximality of S, we obtain that S is of the form (1). □

Example 3.9 *Let* $X = \{1, 2, 3\}$ *and* $Y = \{1\}$. *Then* $|X \setminus Y| = 2$ *and*

$$J_2 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\}.$$

Moreover, we have

$$J_1^1 = \left\{ \begin{pmatrix} 1 & \{2,3\} \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & \{2,3\} \\ 1 & 3 \end{pmatrix} \right\},$$

$$J_1^* = \left\{ \begin{pmatrix} \{1,2\} & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} \{1,2\} & 3 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} \{1,3\} & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} \{1,3\} & 2 \\ 1 & 3 \end{pmatrix} \right\} \text{ and}$$

$$Fix_1 = \left\{ \begin{pmatrix} X \\ 1 \end{pmatrix} \right\}.$$

Thus there are only three maximal subsemigroups of $Fix(X, Y)$, *namely*

$$M_1 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\} \cup Fix_2 = Fix(X, Y) \setminus \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\},$$

$$M_2 = J_2 \cup J_1^1 \cup Fix_1 \text{ and}$$

$$M_3 = J_2 \cup J_1^* \cup Fix_1.$$

4. Maximal regular subsemigroups of $Fix(X, Y)$

In general, if S is a regular semigroup and T is a maximal subsemigroup of S, then T may not be a maximal regular subsemigroup of S (see [10], Theorem 2 for example).

In this section, we prove that the maximal subsemigroups and the maximal regular subsemigroups of $Fix(X, Y)$ coincide.

Lemma 4.1 *The following statements hold.*

(1) *If* $\alpha \in J_{n-1}^*$, *then* $\alpha = \alpha\beta\alpha$ *for some* $\beta \in J_{n-1}^*$.

(2) *If* $y \in Y$ *and* $\alpha \in J_{n-1}^y$, *then* $\alpha = \alpha\beta\alpha$ *for some* $\beta \in J_{n-1}^y$.

Proof (1) Let $\alpha \in J_{n-1}^*$. Then there are $x \in X \setminus Y$ and $y_{i_0} \in Y$ such that $x\alpha = y_{i_0}$. Let $X \setminus (Y \cup \{x\}) = \{a_1, \ldots a_{n-1}\}$, $J = \{1, \ldots, n-1\}$ and $I' = I \setminus \{i_0\}$. Then we can write α as $(*)$. Choose

$$\beta = \begin{pmatrix} y_{i'} & \{y_{i_0}, x'\} & b_j \\ y_{i'} & y_{i_0} & a_j \end{pmatrix},$$

where $x' \in X \setminus X\alpha$. Thus $\beta \in J_{n-1}^*$ and $\alpha = \alpha\beta\alpha$.

(2) Let $\alpha \in J_{n-1}^y$. Then $a\alpha \neq y$ for all $a \in X \setminus Y$. If $\alpha \in J_{n-1}^*$, there exist $y \neq y_{i_0} \in Y$ and $x \in X \setminus Y$ such that $x\alpha = y_{i_0}$. Define β as given in (1); then $\beta \in J_{n-1}^y$ since $y_{i_0} \neq y$ and $\alpha = \alpha\beta\alpha$. If $\alpha \notin J_{n-1}^*$, then $|y\alpha^{-1}| = 1$ for all $y \in Y$. Since $\alpha \in J_{n-1}$, there exists $b \in X \setminus Y$ such that $b\alpha^{-1} = \{x, z\} \subseteq X \setminus Y$. Let $X \setminus (Y \cup \{x, z\}) = \{a_1, \ldots, a_{n-2}\}$ and $J = \{1, \ldots, n-2\}$. Therefore, we can write α as $(**)$. Now choose

$$\beta = \begin{pmatrix} y_i & \{b, w\} & b_j \\ y_i & x & a_j \end{pmatrix},$$

where $w \in X \setminus X\alpha$. Thus $\beta \in J_{n-1}^y$ and $\alpha = \alpha\beta\alpha$. □

We note that if T is a maximal subsemigroup of S and T is regular, then T is a maximal regular subsemigroup of S.

Now we aim to characterize the maximal regular subsemigroups of $Fix(X,Y)$.

Theorem 4.2 *The following subsemigroups of $Fix(X,Y)$ are maximal regular subsemigroups.*

(1) $M \cup Fix_n$, *where M is a maximal subgroup of J_n;*

(2) $J_n \cup J^y_{n-1} \cup Fix_{n-1}$ *for some $y \in Y$;*

(3) $J_n \cup J^*_{n-1} \cup Fix_{n-1}$.

Proof The three subsemigroups above are maximal subsemigroups of $Fix(X,Y)$, and so by the previous note we only show that they are regular.

(1) Since M is a group, it is regular. Since $Fix(X,Y)$ is regular and Fix_n is an ideal of $Fix(X,Y)$, we obtain Fix_n is also regular. Hence $M \cup Fix_n$ is a regular subsemigroup of $Fix(X,Y)$.

Similar to (1), we have that J_n and Fix_{n-1} are regular, and for each $\alpha \in J^y_{n-1}$ (J^*_{n-1}) there exists $\beta \in J^y_{n-1}$ (J^*_{n-1}) by Lemma 4.1 such that $\alpha = \alpha\beta\alpha$. Therefore, (2) and (3) hold. □

By replacing the maximal subsemigroup by a maximal regular subsemigroup in the proof of Lemma 3.5 and using the results in Theorem 4.2, we obtain the following lemma.

Lemma 4.3 *If S is a maximal regular subsemigroup of $Fix(X,Y)$, then either $J_n \subseteq S$ or $J_{n-1} \subseteq S$.*

With some mild modifications of the proof given in Theorem 3.8 and the results in Theorem 4.2 and Lemma 4.3, we get that maximal subsemigroups and maximal regular subsemigroups of $Fix(X,Y)$ coincide.

Theorem 4.4 *Let S be a maximal regular subsemigroup of $Fix(X,Y)$. Then S is one of the following forms:*

(1) $M \cup Fix_n$, *where M is a maximal subgroup of J_n;*

(2) $J_n \cup J^y_{n-1} \cup Fix_{n-1}$ *for some $y \in Y$;*

(3) $J_n \cup J^*_{n-1} \cup Fix_{n-1}$.

5. Finiteness conditions on $Fix(X,Y)$

In 1980, Alarcao [1] characterized when a monoid S is unit-regular and when it is directly finite as follows:

Theorem 5.1 *Let S be a monoid having 1 as an identity.*

(1) *S is unit-regular if and only if it is factorizable.*

(2) *S is directly finite if and only if $H_1 = D_1$.*

For the semigroup $Fix(X,Y)$, the properties unit-regular, factorizable, and directly finite depend on the finiteness conditions on sets.

Theorem 5.2 *$Fix(X,Y)$ is unit-regular if and only if $X \setminus Y$ is finite.*

Proof Suppose that $Fix(X,Y)$ is unit-regular. Assume by contrary that $X \setminus Y$ is infinite. Let $a \in X \setminus Y$. Then $|X \setminus Y| = |(X \setminus Y) \setminus \{a\}| = |X \setminus (Y \cup \{a\})|$. Thus there is a bijection $\sigma : X \setminus Y \to X \setminus (Y \cup \{a\})$. Let $X \setminus Y = \{x_j : j \in J\}$ and define $\alpha \in Fix(X,Y)$ by

$$\alpha = \begin{pmatrix} y_i & x_j \\ y_i & x_j\sigma \end{pmatrix}.$$

Hence α is injective and $X\alpha = X \setminus \{a\}$. Since $Fix(X,Y)$ is unit-regular, there is a unit $\beta \in Fix(X,Y)$ such that $\alpha = \alpha\beta\alpha$. Assume that $a\beta = b$. We have $b\alpha = b\alpha\beta\alpha = (b\alpha\beta)\alpha$ and then $b = (b\alpha)\beta$ since α is injective. Since β is injective, $b\alpha = a \notin X\alpha$, a contradiction.

Conversely, assume that $X \setminus Y$ is finite. Let $\alpha \in Fix(X,Y)$. We can write

$$\alpha = \begin{pmatrix} A_i & B_1 & \dots & B_n \\ y_i & b_1 & \dots & b_n \end{pmatrix},$$

where $B_j \subseteq X \setminus Y$, $b_j \in X \setminus Y$ for all $j \in \{1, \dots, n\}$. Let $C = X \setminus (Y \cup \{b_j\})$. For each $j \in \{1, \dots, n\}$, choose $b'_j \in B_j$ and let $C' = X \setminus (Y \cup \{b'_j\})$. Then $|C| = |C'|$ since $X \setminus Y$ is finite and thus there exists a bijection $\sigma : C \to C'$. Let $C = \{x_k : k \in K\}$ and define

$$\beta = \begin{pmatrix} y_i & b_j & x_k \\ y_i & b'_j & x_k\sigma \end{pmatrix}.$$

Then β is a unit in $Fix(X,Y)$ and $\alpha = \alpha\beta\alpha$. Thus $Fix(X,Y)$ is unit-regular. □

Combining Theorem 5.1 and Theorem 5.2, we obtain the following corollary.

Corollary 5.3 *The following statements are equivalent.*

(1) *$Fix(X,Y)$ is unit-regular;*

(2) *$Fix(X,Y)$ is factorizable;*

(3) *$X \setminus Y$ is a finite set.*

The following example shows that $X \setminus Y$ being a finite set is a sufficient condition for $Fix(X,Y)$ to be directly finite.

Example 5.4 *Let $X = \mathbb{N}$ be the set of all natural numbers and $Y = \{x \in \mathbb{N} : x > 3\}$. Then $X \setminus Y = \{1,2,3\}$. If $\alpha, \beta \in Fix(X,Y)$ such that $\alpha\beta = 1_X$, then α is injective and $y\alpha = y$ for all $y \in Y$. Thus $\{1,2,3\}\alpha = \{1,2,3\}$. Hence $1 = z\alpha$ for some $z \in \{1,2,3\}$, that is $1\beta\alpha = (z\alpha)\beta\alpha = (z\alpha\beta)\alpha = (z1_X)\alpha = z\alpha = 1$. Similarly, we have $2\beta\alpha = 2$ and $3\beta\alpha = 3$. Hence $\beta\alpha = 1_X$.*

Moreover, we have the following theorem.

Theorem 5.5 *Fix(X, Y) is directly finite if and only if $X \setminus Y$ is finite.*

Proof Suppose that $Fix(X,Y)$ is directly finite. By Theorem 5.1(2), we get $D_{1_X} = H_{1_X}$. Assume by contrary that $X \setminus Y$ is infinite. Choose $a \in X \setminus Y$ and $y_{i_0} \in Y$. Then $|X \setminus (Y \cup \{a\})| = |X \setminus Y|$. Therefore, there is a bijection

$$\sigma : X \setminus (Y \cup \{a\}) \to X \setminus Y.$$

Let $I' = I \setminus \{i_0\}$, $X \setminus (Y \cup \{a\}) = \{x_j : j \in J\}$ and define $\alpha \in Fix(X,Y)$ by

$$\alpha = \begin{pmatrix} y_{i'} & \{y_{i_0}, a\} & x_j \\ y_{i'} & y_{i_0} & x_j\sigma \end{pmatrix}.$$

Then α is surjective. Hence $X\alpha \setminus Y = X \setminus Y = X1_X \setminus Y$, that is $\alpha \in D_{1_X}$. However, $\alpha \notin H_{1_X}$ since α is not injective, a contradiction. Thus $X \setminus Y$ is finite.

Conversely, assume that $X \setminus Y$ is finite. Let $\alpha, \beta \in Fix(X,Y)$ be such that $\alpha\beta = 1_X$. Then α is injective and so $(X \setminus Y)\alpha \subseteq X \setminus Y$. Since $X \setminus Y$ is finite, we have $(X \setminus Y)\alpha = X \setminus Y$. Thus for each $x \in X \setminus Y$, there exists $z \in X \setminus Y$ such that $z\alpha = x$. Hence $x\beta = z\alpha\beta = z1_X = z$. Therefore, $x\beta\alpha = z\alpha = x$ for all $x \in X \setminus Y$ and so we conclude that $\beta\alpha = 1_X$. □

If $Y = \emptyset$, then $Fix(X,Y) = T(X)$, and we have the following corollary, which first appeared in [1] and [9].

Corollary 5.6 *The following statements are equivalent.*

(1) *$T(X)$ is unit-regular;*

(2) *$T(X)$ is factorizable;*

(3) *$T(X)$ is directly finite;*

(4) *X is a finite set.*

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