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Dynamic Shum inequalities

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Abstract: Recently, various forms and improvements of Opial dynamic inequalities have been given in the literature. In this paper, we give refinements of Opial inequalities on time scales that reduce in the continuous case to classical inequalities named after Beesack and Shum. These refinements are new in the important discrete case.

Key words: Opial's inequality, Beesack's inequality, Shum's inequality, time scales

1. Introduction

In 1960, Olech [13] extended an inequality of Opial [14] and proved the following result.

Theorem 1.1 (Opial Inequality) *If* $f \in \mathrm{C}^1([0,h],\mathbb{R})$ *with* $h>0$ *satisfies* $f(0)=0$*, then*

$$\int_0^h |f(t)f'(t)|\,\mathrm{d}t \le \frac{h}{2}\int_0^h (f'(t))^2\,\mathrm{d}t. \tag{1.1}$$

This inequality has generated a lot of research, both in the continuous and the discrete cases (see the monograph [3] and the references therein). In 1962, Beesack [4] refined Theorem 1.1 as follows (see also [15, Theorem 3]).

Theorem 1.2 (Beesack Inequality) *If* $f \in \mathrm{C}^1([0,h],\mathbb{R})$ *with* $h>0$ *satisfies* $f(0)=0$*, then*

$$\int_0^h |f(t)f'(t)|\,\mathrm{d}t + \frac{h}{2}\int_0^h \frac{g(t)}{t^2}\mathrm{d}t \le \frac{h}{2}\int_0^h (f'(t))^2\,\mathrm{d}t, \tag{1.2}$$

where

$$g(t) := 2\int_0^t |f(\tau)f'(\tau)|\,\mathrm{d}\tau - (f(t))^2 \ge 0.$$

We note that Theorem 1.2 implies Theorem 1.1.

In 2001, Bohner and Kaymakçalan [6] extended Theorem 1.1 to an arbitrary time scale and proved the following.

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Theorem 1.3 (Dynamic Opial Inequality) *Let $\mathbb{T}$ be a time scale with $0 \in \mathbb{T}$. If $f \in \mathrm{C}^1_{\mathrm{rd}}([0,h]_{\mathbb{T}},\mathbb{R})$ with $h > 0$ and $h \in \mathbb{T}$ satisfies $f(0) = 0$, then*

$$\int_0^h \left|(f^2)^{\Delta}(t)\right| \Delta t \leq h \int_0^h \left(f^{\Delta}(t)\right)^2 \Delta t. \tag{1.3}$$

For extensions and generalizations of (1.3), we refer the reader to the recent monograph [2]. Here we will not give an introduction to time scales calculus but instead refer the reader to [9, 10]. We only remark that the delta derivative is the usual derivative if $\mathbb{T} = \mathbb{R}$ and the forward difference if $\mathbb{T} = \mathbb{Z}$, and the delta integral is the usual integral if $\mathbb{T} = \mathbb{R}$ and a sum if $\mathbb{T} = \mathbb{Z}$, and that the theory can be applied to any nonempty closed set $\mathbb{T} \subset \mathbb{R}$, the so-called underlying time scale. We note that plugging $\mathbb{T} = \mathbb{R}$ in (1.3) results in (1.1).

Below, in Section 2, we prove the following generalization of Theorem 1.3.

Theorem 1.4 (Dynamic Beesack Inequality) *Let $\mathbb{T}$ be a time scale with $0 \in \mathbb{T}$. If $f \in \mathrm{C}^1_{\mathrm{rd}}([0,h]_{\mathbb{T}},\mathbb{R})$ with $h > 0$ and $h \in \mathbb{T}$ satisfies $f(0) = 0$, then*

$$\int_0^h \left|(f^2)^{\Delta}(t)\right| \Delta t + h \int_{\sigma(0)}^h \frac{g(t)}{t\sigma(t)} \Delta t \leq h \int_0^h \left(f^{\Delta}(t)\right)^2 \Delta t, \tag{1.4}$$

where

$$g(t) := \int_0^t \left|(f^2)^{\Delta}(\tau)\right| \Delta \tau - (f(t))^2 \geq 0.$$

We note that Theorem 1.4 implies Theorem 1.3. We also note that Theorem 1.4 implies Theorem 1.2 since, for $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$ for all $t \in \mathbb{R}$.

The main objective of this paper is to present a time scales version of the following 1972 extension of Theorem 1.2 due to Shum (see [15, Theorem 4 and (15)] and also [12, Theorem F and Theorem G]).

Theorem 1.5 (Shum Inequality) *Assume that $a, b \in \mathbb{R}$, $a < b$, $p > 0$, and $s \in \mathrm{C}([a,b],(0,\infty))$ is nonincreasing. If $f \in \mathrm{C}^1([a,b],\mathbb{R})$ satisfies $f(a) = 0$, then*

$$\int_a^b s(t)\, |f(t)|^p\, |f'(t)|\, \mathrm{d}t + \frac{(b-a)^p}{p+1} \int_a^b \frac{pg(t)}{(t-a)^{p+1}} \mathrm{d}t \leq \frac{(b-a)^p}{p+1} \int_a^b s(t)\, |f'(t)|^{p+1}\, \mathrm{d}t, \tag{1.5}$$

where

$$g(t) := (p+1) \int_a^t s(\tau)\, |f(\tau)|^p\, |f'(\tau)|\, \mathrm{d}\tau - s(t)\, |f(t)|^{p+1} \geq 0.$$

We note that Theorem 1.5 implies Theorem 1.2 by choosing

$$a = 0, \quad b = h, \quad p = 1, \quad \text{and} \quad s(t) = 1 \text{ for all } t \in [a,b].$$

The main result of this paper is the following Shum-type inequality, which improves Theorem 1.5 to an arbitrary time scale.

Theorem 1.6 (Dynamic Shum Inequality) *Let $\mathbb{T}$ be a time scale. Assume that $a, b \in \mathbb{T}$, $a < b$, $\alpha > 1$, and $s \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, (0,\infty))$ is nonincreasing. Let*

$$K(t) := (t-a)^{\alpha-1} \quad \textit{for all} \quad t \in [a,b]_{\mathbb{T}}.$$

If $f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$ satisfies $f(a) = 0$, then

$$\int_a^b s(t) \left|(f^\alpha)^\Delta(t)\right| \Delta t + K(b) \int_{\sigma(a)}^b \frac{K^\Delta(t) g(t)}{K(t)K(\sigma(t))} \Delta t \le K(b) \int_a^b s(t) \left|f^\Delta(t)\right|^\alpha \Delta t, \tag{1.6}$$

where

$$g(t) := \int_a^t s(\tau) \left|(f^\alpha)^\Delta(\tau)\right| \Delta \tau - s(t) \left|f(t)\right|^\alpha \ge 0.$$

We note that, with $p = \alpha - 1$, Theorem 1.6 implies Theorem 1.5 since, for $\mathbb{T} = \mathbb{R}$, we have

$$\frac{K^\Delta(t)}{K(t)K(\sigma(t))} = \frac{p(t-a)^{p-1}}{(t-a)^{2p}} = \frac{p}{(t-a)^{p+1}}.$$

We also note that Theorem 1.6 implies Theorem 1.4 by choosing

$$a = 0, \quad b = h, \quad \alpha = 2, \quad \text{and} \quad s(t) = 1 \text{ for all } t \in [a,b]_{\mathbb{T}}.$$

The proof of Theorem 1.6 is given in Section 3 below. While Shum [15] combined Beesack's method [4] with that of Benson [5] and employed some integral inequalities involving partial derivatives of a function of two variables, our method of proof follows the elementary approach of our proof of Theorem 1.4 without making it necessary to first develop Benson's integral inequalities involving partial delta derivatives of a function of two time scales variables.

The set up of this paper is as follows. Section 2 features an easy and elementary proof of Beesack's inequality on time scales, Theorem 1.4. Next, some recently established Opial-type inequalities are used in Section 3 to present the proof of Shum's inequality on time scales, Theorem 1.6. Section 4 contains discrete versions of Beesack's inequality and of Shum's inequality.

2. Beesack's inequality on time scales

In this section, we prove Beesack's inequality on time scales, Theorem 1.4. Throughout this section, we assume that the assumptions of Theorem 1.4 hold and put

$$K(t) := t \quad \text{and} \quad v(t) := \int_0^t \left|(f^2)^\Delta(t)\right| \Delta t, \quad t \in [0,h]_{\mathbb{T}}.$$

Lemma 2.1 *For any $c \in (0,h]_{\mathbb{T}}$, we have*

$$\frac{v(h)}{h} + \int_c^h \left(\frac{v(t)}{t\sigma(t)} - \frac{\left|(f^2)^\Delta(t)\right|}{\sigma(t)} + \left(f^\Delta(t)\right)^2 \right) \Delta t \le \int_0^h \left(f^\Delta(t)\right)^2 \Delta t. \tag{2.1}$$

Proof Let $0 < c \leq h$ with $c \in \mathbb{T}$. Note that Theorem 1.3 asserts

$$\frac{v(c)}{c} \leq \int_0^c \left(f^\Delta(t)\right)^2 \Delta t.$$

Using the time scales quotient rule [9, Theorem 1.20 (v)] (note that the notation K^σ means $K \circ \sigma$) and the additivity property of time scales integrals [9, Theorem 1.77 (iv)], we obtain

$$\begin{aligned}
\frac{v(h)}{h} &= \left(\frac{v}{K}\right)(h) - \left(\frac{v}{K}\right)(c) + \frac{v(c)}{c} \\
&\leq \left(\frac{v}{K}\right)(h) - \left(\frac{v}{K}\right)(c) + \int_0^c \left(f^\Delta(t)\right)^2 \Delta t \\
&= \int_c^h \left(\frac{v}{K}\right)^\Delta (t)\Delta t + \int_0^c \left(f^\Delta(t)\right)^2 \Delta t \\
&= \int_c^h \left(\frac{v^\Delta K - K^\Delta v}{KK^\sigma}\right)(t)\Delta t + \int_0^c \left(f^\Delta(t)\right)^2 \Delta t \\
&= \int_c^h \left(\frac{v^\Delta(t)}{\sigma(t)} - \frac{v(t)}{t\sigma(t)}\right)\Delta t + \int_0^h \left(f^\Delta(t)\right)^2 \Delta t - \int_c^h \left(f^\Delta(t)\right)^2 \Delta t \\
&= \int_c^h \left(\frac{\left|(f^2)^\Delta(t)\right|}{\sigma(t)} - \left(f^\Delta(t)\right)^2 - \frac{v(t)}{t\sigma(t)}\right)\Delta t + \int_0^h \left(f^\Delta(t)\right)^2 \Delta t,
\end{aligned}$$

which proves (2.1). □

Lemma 2.2 *We have*

$$-\frac{f^2}{KK^\sigma} \leq -\frac{\left|(f^2)^\Delta\right|}{K^\sigma} + \left(f^\Delta\right)^2 \quad \text{on} \quad (0,h]_\mathbb{T}. \tag{2.2}$$

Proof Using the time scales product rule [9, Theorem 1.20 (iii)] and the "simple useful formula" [9, Theorem 1.16 (iv)], we get

$$\left(f^2\right)^\Delta = f^\Delta f + f^\Delta f^\sigma = 2f^\Delta f + \mu \left(f^\Delta\right)^2,$$

where $\mu = \sigma - K$ is the graininess of the time scale. Thus, we have

$$\begin{aligned}
K\left|(f^2)^\Delta\right| - f^2 - KK^\sigma \left(f^\Delta\right)^2 &= K\left|2ff^\Delta + \mu\left(f^\Delta\right)^2\right| - f^2 - \sigma K\left(f^\Delta\right)^2 \\
&\leq 2K\left|ff^\Delta\right| + \mu K\left(f^\Delta\right)^2 - f^2 - \sigma K\left(f^\Delta\right)^2 \\
&= 2K\left|ff^\Delta\right| - f^2 - K^2\left(f^\Delta\right)^2 \\
&= -\left(|f| - K\left|f^\Delta\right|\right)^2 \leq 0
\end{aligned}$$

on $(0,h]_\mathbb{T}$, which proves (2.2). □

Using Lemmas 2.1 and 2.2, we may now complete the proof of Theorem 1.4.

Proof [Proof of Theorem 1.4] First, using the definition of the time scales integral [9, Definition 1.71] and the triangle inequality for time scales integrals [9, Theorem 1.77 (viii)], we obtain

$$v(t) = \int_0^t \left|(f^2)^\Delta(\tau)\right| \Delta\tau \geq \left|\int_0^t (f^2)^\Delta(\tau)\Delta\tau\right| = \left|f^2(t) - f^2(0)\right| = f^2(t),$$

which shows that

$$g = v - f^2 \geq 0 \quad \text{on} \quad [0,h]_{\mathbb{T}}.$$

Now let $c \in (0,h]_{\mathbb{T}}$ be arbitrary. Using (2.2) in (2.1), we find

$$\frac{v(h)}{h} + \int_c^h \frac{g(t)}{t\sigma(t)}\Delta t \leq \int_0^h \left(f^\Delta(t)\right)^2 \Delta t. \tag{2.3}$$

If $0 \in \mathbb{T}$ is a right-scattered point of $\mathbb{T}$, i.e. $\sigma(0) > 0$, then (2.3) holds for $c = \sigma(0)$. If 0 is a right-dense point of $\mathbb{T}$, i.e. $\sigma(0) = 0$, then, by passing in (2.3) to the limit as $c \to 0^+$, (2.3) holds for $c = 0 = \sigma(0)$ and

$$0 \leq \int_0^h \frac{g(t)}{t\sigma(t)}\Delta t < \infty$$

since the other two occurring integrals in (2.3) are finite and do not depend on c. Hence, in either case, (2.3) holds for $c = \sigma(0)$. This proves (1.4). □

Example 2.3 *Opial's inequality has many nice applications to differential and difference equations, e.g., uniqueness of solutions of initial value problems or boundary value problems, upper bounds for solutions, disfocality, and disconjugacy. So far, no such applications of Shum's inequality or Beesack's inequality have been studied. For disfocality/disconjugacy, two different weight functions are required, while Shum's inequality is only valid for the same weight function on each side. Here, we present a simple example of an application of Beesack's inequality. Note again that this is the first such example in the literature, even for* $\mathbb{T} = \mathbb{R}$. *This example is related to the application of Opial's inequality in [3, Example 6.3.1]. Define*

$$\mathcal{E}(t,y) := \int_{\sigma(0)}^t \frac{\int_0^s \left|(y^2)^\Delta(\tau)\right| \Delta\tau - (y(s))^2}{s\sigma(s)}\Delta s$$

and consider the initial value problem

$$y^\Delta = t^2 + t\mathcal{E}(t,y) + y^2 \quad \text{on} \quad [0,1] \cap \mathbb{T}, \quad y(0) = 0,$$

where we assume that $0, 1 \in \mathbb{T}$ *and* $(0,1) \cap \mathbb{T} \neq \emptyset$. *Then, for all* $0 \leq t \leq T \leq 1$, *using Theorem 1.4, we have*

$$\begin{aligned}
\left|y^\Delta(t)\right| &= y^\Delta(t) = t^2 + t\int_{\sigma(0)}^t \frac{g(s)}{s\sigma(s)}\Delta s + \int_0^t (y^2)^\Delta(s)\Delta s \\
&\leq t^2 + t\int_{\sigma(0)}^t \frac{g(s)}{s\sigma(s)}\Delta s + \int_0^t \left|(y^2)^\Delta(s)\right| \Delta s \\
&\leq t^2 + t\int_0^t (y^\Delta(s))^2 \Delta s \\
&\leq T\left\{t + \int_0^t (y^\Delta(s))^2\Delta s\right\} =: R(t),
\end{aligned}$$

and thus

$$R^{\Delta}(t) \le T\left\{1 + (y^{\Delta}(t))^2\right\} \le T\left(1 + (R(t))^2\right).$$

From here, we may derive an upper bound for R and hence for y, e.g., if $\mathbb{T} = \mathbb{R}$, as in [3, Example 6.3.1],

$$y(t) \le \int_0^t \tan(s^2)\mathrm{d}s.$$

3. Shum's inequality on time scales

In this section, we prove Shum's inequality on time scales, Theorem 1.6. Throughout this section, we assume that the assumptions of Theorem 1.6 hold and put

$$K(t) := (t-a)^{\alpha-1} \quad \text{and} \quad v(t) := \int_a^b s(t)\left|(f^{\alpha})^{\Delta}(t)\right|\Delta t, \quad t \in [a,b]_{\mathbb{T}}.$$

First, the generalization of Lemma 2.1 is needed for the current situation. However, in the first line of the proof of Lemma 2.1, Theorem 1.3 was used, so we first need to establish a generalization of Theorem 1.3 for the current situation. To do so, we use a recently published result [8, Corollary 3.2] (see also [1, 7]), which we recall as follows.

Theorem 3.1 (See [8, Corollary 3.2]) *Let $\mathbb{T}$ be a time scale. Assume that $a, b \in \mathbb{T}$, $a < b$, $\alpha > 1$,*

$$r, s \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C^1_{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

If $f(a) = 0$, then

$$\int_a^b s(t)\left|(f^{\alpha})^{\Delta}(t)\right|\Delta t \le M \int_a^b r(t)\left|f^{\Delta}(t)\right|^{\alpha}\Delta t, \tag{3.1}$$

where

$$M = \left\{\int_a^b (s(t))^{\frac{\alpha}{\alpha-1}}\left(R^{\alpha}\right)^{\Delta}(t)\Delta t\right\}^{\frac{\alpha-1}{\alpha}}$$

with

$$R(t) = \int_a^t \frac{\Delta\tau}{(r(\tau))^{\frac{1}{\alpha-1}}}.$$

Using Theorem 3.1 with $r = s$, we now prove the following new result, which will be used in place of Theorem 1.3.

Theorem 3.2 *Let $\mathbb{T}$ be a time scale. Assume that $a, b \in \mathbb{T}$, $a < b$, $\alpha > 1$, and $s \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, (0,\infty)$ is nonincreasing. If $f \in \mathrm{C^1_{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$ satisfies $f(a) = 0$, then*

$$\int_a^b s(t)\left|\left(f^{\alpha}\right)^{\Delta}(t)\right|\Delta t \le (b-a)^{\alpha-1}\int_a^b s(t)\left|f^{\Delta}(t)\right|^{\alpha}\Delta t. \tag{3.2}$$

Proof Using the time scales chain rule [9, Theorem 1.90] and the nonincreasing character of $r = s$, we obtain

$$
\begin{aligned}
\left(R^{\alpha}\right)^{\Delta}(t) &= \alpha R^{\Delta}(t)\int_0^1 \left(R(t)+h\mu(t)R^{\Delta}(t)\right)^{\alpha-1}\mathrm{d}h \\
&= \frac{\alpha}{(r(t))^{\frac{1}{\alpha-1}}}\int_0^1\left(\int_a^t \frac{\Delta\tau}{(r(\tau))^{\frac{1}{\alpha-1}}}+\frac{h\mu(t)}{(r(t))^{\frac{1}{\alpha-1}}}\right)^{\alpha-1}\mathrm{d}h \\
&\leq \frac{\alpha}{(r(t))^{\frac{1}{\alpha-1}}}\int_0^1\left(\int_a^t \frac{\Delta\tau}{(r(t))^{\frac{1}{\alpha-1}}}+\frac{h\mu(t)}{(r(t))^{\frac{1}{\alpha-1}}}\right)^{\alpha-1}\mathrm{d}h \\
&= \frac{\alpha}{(r(t))^{\frac{1}{\alpha-1}}}\int_0^1 \frac{1}{r(t)}\left(t-a+h\mu(t)\right)^{\alpha-1}\mathrm{d}h \\
&= \frac{\alpha}{(r(t))^{\frac{\alpha}{\alpha-1}}}\int_0^1 \left(t-a+h\mu(t)\right)^{\alpha-1}\mathrm{d}h.
\end{aligned}
$$

Putting now $L(t)=t-a$ and using the time scales chain rule once more, we find

$$
\begin{aligned}
(r(t))^{\frac{\alpha}{\alpha-1}}\left(R^{\alpha}\right)^{\Delta}(t) &\leq \alpha\int_0^1 \left(t-a+h\mu(t)\right)^{\alpha-1}\mathrm{d}h \\
&= \alpha L^{\Delta}(t)\int_0^1\left(L(t)+h\mu(t)L^{\Delta}(t)\right)^{\alpha-1}\mathrm{d}h \\
&= \left(L^{\alpha}\right)^{\Delta}(t).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
M &= \left\{\int_a^b (s(t))^{\frac{\alpha}{\alpha-1}}\left(R^{\alpha}\right)^{\Delta}(t)\Delta t\right\}^{\frac{\alpha-1}{\alpha}} \\
&\leq \left\{\int_a^b \left(L^{\alpha}\right)^{\Delta}(t)\Delta t\right\}^{\frac{\alpha-1}{\alpha}} \\
&= \left(L^{\alpha}(b)-L^{\alpha}(a)\right)^{\frac{\alpha-1}{\alpha}} \\
&= \left(L^{\alpha}(b)\right)^{\frac{\alpha-1}{\alpha}} \\
&= L^{\alpha-1}(b)=(b-a)^{\alpha-1}.
\end{aligned}
$$

Hence, (3.1) implies (3.2). □

We note that Theorem 3.2 implies Theorem 1.3 by choosing

$$a=0,\quad b=h,\quad \alpha=2,\quad \text{and}\quad s(t)=1 \text{ for all } t\in[a,b]_{\mathbb{T}}.$$

We now may continue with the generalization of Lemma 2.1.

Lemma 3.3 *For any $c\in(a,b]_{\mathbb{T}}$, we have*

$$\frac{v(b)}{K(b)}+\int_c^b\left(\frac{K^{\Delta}(t)v(t)}{K(t)K(\sigma(t))}-\frac{s(t)\left|(f^{\alpha})^{\Delta}(t)\right|}{K(\sigma(t))}+s(t)\left|f^{\Delta}(t)\right|^{\alpha}\right)\Delta t\leq\int_a^b s(t)\left|f^{\Delta}(t)\right|^{\alpha}\Delta t. \tag{3.3}$$

Proof Let $0 < c \le h$ with $c \in \mathbb{T}$. Note that Theorem 3.2 asserts

$$\frac{v(c)}{K(c)} \le \int_a^c s(t)\left|f^\Delta(t)\right|^\alpha \Delta t.$$

Proceeding similarly as in the proof of Lemma 2.1, we have

$$\begin{aligned}
\frac{v(b)}{K(b)} &= \left(\frac{v}{K}\right)(b) - \left(\frac{v}{K}\right)(c) + \frac{v(c)}{K(c)} \\
&\le \left(\frac{v}{K}\right)(b) - \left(\frac{v}{K}\right)(c) + \int_a^c s(t)\left|f^\Delta(t)\right|^\alpha \Delta t \\
&= \int_c^b \left(\left(\frac{v}{K}\right)^\Delta(t) - s(t)\left|f^\Delta(t)\right|^\alpha\right)\Delta t + \int_a^b s(t)\left|f^\Delta(t)\right|^\alpha \Delta t \\
&= \int_c^b \left(\frac{v^\Delta(t)}{K(\sigma(t))} - s(t)\left|f^\Delta(t)\right|^\alpha - \frac{K^\Delta(t)v(t)}{K(t)K(\sigma(t))}\right)\Delta t + \int_a^b s(t)\left|f^\Delta(t)\right|^\alpha \Delta t,
\end{aligned}$$

which proves (3.3). □

The main difficulty in the proof of Theorem 1.6 now is in establishing the following generalization of Lemma 2.2.

Lemma 3.4 *We have*

$$-\frac{K^\Delta\left|f^\alpha\right|}{KK^\sigma} \le -\frac{\left|(f^\alpha)^\Delta\right|}{K^\sigma} + \left|f^\Delta\right|^\alpha \quad \text{on} \quad (a,b]_{\mathbb{T}}. \tag{3.4}$$

Proof We consider two cases. First, suppose $t \in (a,b]_{\mathbb{T}}$ is such that $f(t) = 0$ holds. Then

$$\begin{aligned}
\left(f^\alpha\right)^\Delta(t) &= \alpha f^\Delta(t)\int_0^1 \left(f(t) + h\mu(t)f^\Delta(t)\right)^{\alpha-1}\mathrm{d}h \\
&= \alpha f^\Delta(t)\left(\mu(t)f^\Delta(t)\right)^{\alpha-1}\int_0^1 h^{\alpha-1}\mathrm{d}h \\
&= (\mu(t))^{\alpha-1}\left(f^\Delta(t)\right)^\alpha
\end{aligned}$$

and thus

$$\begin{aligned}
&\left|(f^\alpha)^\Delta(t)\right| - \frac{K^\Delta(t)}{K(t)}\left|f(t)\right|^\alpha - K(\sigma(t))\left|f^\Delta(t)\right|^\alpha \\
&= (\mu(t))^{\alpha-1}\left|f^\Delta(t)\right|^\alpha - K(\sigma(t))\left|f^\Delta(t)\right|^\alpha \\
&= \left\{(\sigma(t)-t)^{\alpha-1} - (\sigma(t)-a)^{\alpha-1}\right\}\left|f^\Delta(t)\right|^\alpha \le 0.
\end{aligned}$$

Hence, the inequality in (3.4) holds at t. Now suppose $t \in (a,b]_{\mathbb{T}}$ is such that $f(t) \ne 0$ holds. Put

$$y := \left|\frac{f^\Delta(t)}{f(t)}\right| \ge 0.$$

Then

$$\begin{aligned}
&\left|(f^\alpha)^\Delta(t)\right| - \frac{K^\Delta(t)}{K(t)}|f(t)|^\alpha - K(\sigma(t))\left|f^\Delta(t)\right|^\alpha \\
&= \alpha\left|f^\Delta(t)\int_0^1 \left(f(t)+h\mu(t)f^\Delta(t)\right)^{\alpha-1}\mathrm{d}h\right| - \frac{K^\Delta(t)}{K(t)}|f(t)|^\alpha - K(\sigma(t))\left|f^\Delta(t)\right|^\alpha \\
&= |f(t)|^\alpha\left\{\alpha y\left|\int_0^1\left(1+h\mu(t)\frac{f^\Delta(t)}{f(t)}\right)^{\alpha-1}\mathrm{d}h\right| - \frac{K^\Delta(t)}{K(t)} - y^\alpha K(\sigma(t))\right\} \\
&\le |f(t)|^\alpha\left\{\alpha y\int_0^1 (1+h\mu(t)y)^{\alpha-1}\mathrm{d}h - \frac{K^\Delta(t)}{K(t)} - y^\alpha K(\sigma(t))\right\}.
\end{aligned}$$

Hence, the inequality in (3.4) holds at t provided we can show

$$\begin{aligned}
F(x) &:= \alpha x\int_0^1 (1+h\mu(t)x)^{\alpha-1}\mathrm{d}h - \frac{K^\Delta(t)}{K(t)} - x^\alpha K(\sigma(t)) \\
&\le 0 \quad \text{for all} \quad x\ge 0.
\end{aligned} \tag{3.5}$$

We now show the following facts about F:

(a) $F(0) < 0$,

(b) $F'(x) > 0$ if $0 \le x < \frac{1}{t-a}$,

(c) $F\left(\frac{1}{t-a}\right) = F'\left(\frac{1}{t-a}\right) = 0$,

(d) $F'(x) < 0$ if $x > \frac{1}{t-a}$.

Note that (a)–(d) implies (3.5) and thus immediately completes the proof.

Now, to show (a)–(d), let us put, as in the proof of Theorem 3.2, $L(t) = t-a$. Then $L^\Delta = 1$ and $K = L^{\alpha-1}$. Clearly,

$$\begin{aligned}
F(0) &= -\frac{K^\Delta(t)}{K(t)} = -(\alpha-1)\frac{L^\Delta(t)}{L^{\alpha-1}(t)}\int_0^1\left(L(t)+h\mu(t)L^\Delta(t)\right)^{\alpha-2}\mathrm{d}h \\
&= -\frac{\alpha-1}{L(t)}\int_0^1\left(1+h\frac{\mu(t)}{L(t)}\right)^{\alpha-2}\mathrm{d}h < 0,
\end{aligned}$$

so (a) holds. Next, since

$$\begin{aligned}
\frac{\alpha}{L(t)}\int_0^1\left(1+h\frac{\mu(t)}{L(t)}\right)^{\alpha-1}\mathrm{d}h &= \frac{\alpha L^\Delta(t)}{L^\alpha(t)}\int_0^1\left(L(t)+h\mu(t)L^\Delta(t)\right)^{\alpha-1}\mathrm{d}h \\
&= \frac{(L^\alpha)^\Delta(t)}{L^\alpha(t)} = \frac{\left(LL^{\alpha-1}\right)^\Delta(t)}{L^\alpha(t)} \\
&= \frac{L^{\alpha-1}(\sigma(t)) + L(t)\left(L^{\alpha-1}\right)^\Delta(t)}{L^\alpha(t)} \\
&= \frac{K(\sigma(t))}{L^\alpha(t)} + \frac{K^\Delta(t)}{K(t)},
\end{aligned}$$

we obtain $F\left(\frac{1}{L(t)}\right) = 0$ so that one part of (c) is established. Next, we calculate F':

$$\begin{aligned} F'(x) &= \alpha \int_0^1 (1+h\mu(t)x)^{\alpha-1}\,\mathrm{d}h + \alpha \int_0^1 (\alpha-1)h\mu(t)x\,(1+h\mu(t)x)^{\alpha-2}\,\mathrm{d}h \\ &\quad -\alpha x^{\alpha-1} K(\sigma(t)) \\ &= \alpha \int_0^1 \frac{\mathrm{d}}{\mathrm{d}h}\left[h\,(1+h\mu(t)x)^{\alpha-1}\right]\mathrm{d}h - \alpha x^{\alpha-1}\left(L(t)+\mu(t)L^{\Delta}(t)\right)^{\alpha-1} \\ &= \alpha\left\{(1+\mu(t)x)^{\alpha-1} - (xL(t)+\mu(t)x)^{\alpha-1}\right\}, \end{aligned}$$

from which the remaining part of (c) as well as (b) and (d) follow. $\square$

Using Lemmas 3.3 and 3.4, we may now complete the proof of Theorem 1.6.

Proof [Proof of Theorem 1.6] First, using similar estimates as in the proof of Theorem 1.4 and the nonincreasing character of s, we find

$$\begin{aligned} v(t) &= \int_a^t s(\tau)\left|(f^{\alpha})^{\Delta}(\tau)\right|\Delta\tau \geq s(t)\int_a^t \left|(f^{\alpha})^{\Delta}(\tau)\right|\Delta\tau \\ &\geq s(t)\left|\int_a^t (f^{\alpha})^{\Delta}(\tau)\right| = s(t)\left|f^{\alpha}(t) - f^{\alpha}(0)\right| \\ &= s(t)\left|f^{\alpha}(t)\right|, \end{aligned}$$

which shows

$$g = v - s\,|f^{\alpha}| \geq 0 \quad \text{on} \quad [a,b]_{\mathbb{T}}.$$

Next, let $c \in (0,h]_{\mathbb{T}}$ be arbitrary. Using (3.4) in (3.3), we find

$$\frac{v(b)}{K(b)} + \int_c^b \frac{K^{\Delta}(t)g(t)}{K(t)K(\sigma(t))}\Delta t \leq \int_a^b s(t)\left(f^{\Delta}(t)\right)^{\alpha}\Delta t. \tag{3.6}$$

Now, exactly as in the proof of Theorem 1.4 in Section 2, (3.6) holds for $c = \sigma(a)$. This proves (1.6). $\square$

We summarize the important case $s(t) = 1$ for all $t \in [a,b]_{\mathbb{T}}$ in the following corollary.

Corollary 3.5 *Let $\mathbb{T}$ be a time scale. Assume that $a, b \in \mathbb{T}$, $a < b$, and $\alpha > 1$.*

$$K(t) := (t-a)^{\alpha-1} \quad \textit{for all} \quad t \in [a,b]_{\mathbb{T}}.$$

If $f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R})$ satisfies $f(a) = 0$, then

$$\int_a^b \left|(f^{\alpha})^{\Delta}(t)\right|\Delta t + K(b)\int_{\sigma(a)}^b \frac{K^{\Delta}(t)g(t)}{K(t)K(\sigma(t))}\Delta t \leq K(b)\int_a^b \left|f^{\Delta}(t)\right|^{\alpha}\Delta t, \tag{3.7}$$

where

$$g(t) := \int_a^t \left|(f^{\alpha})^{\Delta}(\tau)\right|\Delta\tau - |f(t)|^{\alpha} \geq 0.$$

Remark 3.6 *It must be noted that the proof of the crucial Lemma 3.4 is very easy in the classical case* $\mathbb{T} = \mathbb{R}$. *Indeed, when* $p = \alpha - 1$, *the inequality in* (3.4) *is equivalent to*

$$p\,|f|^{p+1} + (L\,|f'|) \geq (p+1)\,|f|^{p}\,(L\,|f'|)^{p+1},$$

where we used $L(t) = t - a$, *and this follows from the well-known inequality (e.g., [11, Theorem 41])*

$$x^{p+1} + py^{p+1} \geq (p+1)xy^{p} \quad \text{for} \quad x, y \geq 0 \quad \text{and} \quad p > 0.$$

4. Discrete inequalities

In this section, for completeness, we give the discrete analogues of Theorem 1.4, Theorem 1.6, and Corollary 3.5. These special cases of the main results of this paper are new in the important discrete setting. Note that below we use the usual forward difference operator Δ defined for a sequence $g : \mathbb{N}_0 \to \mathbb{R}$ by $\Delta g(k) = g(k+1) - g(k)$.

Corollary 4.1 (Discrete Beesack Inequality) *Let* $N \in \mathbb{N}$. *If* $f : [0, N] \cap \mathbb{N}_0 \to \mathbb{R}$ *satisfies* $f(0) = 0$, *then*

$$\sum_{k=0}^{N-1} \left|\Delta f^{2}(k)\right| + N \sum_{k=1}^{N-1} \frac{g(k)}{k(k+1)} \leq N \sum_{k=0}^{N-1} (\Delta f(k))^{2}, \tag{4.1}$$

where

$$g(n) := \sum_{k=0}^{n-1} \left|\Delta f^{2}(k)\right| - f^{2}(n) \geq 0.$$

Corollary 4.2 (Discrete Shum Inequality) *Assume* $M, N \in \mathbb{N}_0$, $M < N$, $\alpha > 1$, *and* $s : [M, N] \cap \mathbb{N}_0 \to (0, \infty)$ *is nonincreasing. Let*

$$K(n) := (n - M)^{\alpha - 1} \quad \text{for all} \quad n \in [M, N] \cap \mathbb{N}_0.$$

If $f : [M, N] \cap \mathbb{N}_0 \to \mathbb{R}$ *satisfies* $f(M) = 0$, *then*

$$\sum_{k=M}^{N-1} s(k)\,|\Delta f^{\alpha}(k)| + K(N) \sum_{k=M+1}^{N-1} \frac{(\Delta K(k))g(k)}{K(k)K(k+1)} \leq K(N) \sum_{k=M}^{N-1} s(k)\,|\Delta f(k)|^{\alpha}, \tag{4.2}$$

where

$$g(n) := \sum_{k=M}^{n-1} s(k)\,|\Delta f^{\alpha}(k)| - s(n)\,|f(n)|^{\alpha} \geq 0.$$

Corollary 4.3 *Assume* $M, N \in \mathbb{N}_0$, $M < N$, *and* $\alpha > 1$. *Let*

$$K(n) := (n - M)^{\alpha - 1} \quad \text{for all} \quad n \in [M, N] \cap \mathbb{N}_0.$$

If $f : [M, N] \cap \mathbb{N}_0 \to \mathbb{R}$ *satisfies* $f(M) = 0$, *then*

$$\sum_{k=M}^{N-1} |\Delta f^{\alpha}(k)| + K(N) \sum_{k=M+1}^{N-1} \frac{(\Delta K(k))g(k)}{K(k)K(k+1)} \leq K(N) \sum_{k=M}^{N-1} |\Delta f(k)|^{\alpha}, \tag{4.3}$$

where

$$g(n) := \sum_{k=M}^{n-1} |\Delta f^{\alpha}(k)| - |f(n)|^{\alpha} \geq 0.$$

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