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Sampling theorem by Green's function in a space of vector-functions

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Abstract: In this paper we give a sampling expansion for integral transforms whose kernels arise from Green's function of differential operators in a space of vector-functions. The differential operators are in a space of dimension m and consist of systems of m equations in m unknowns. We assume the simplicity of the eigenvalues.

Key words: Sampling theory, vector-functions, Green's function, boundary value problems

1. Introduction

The use of Green's function in sampling theory is due to the work of Zayed [8], where a sampling theorem for integral transform whose kernel includes Green's function of not necessarily self-adjoint problems of the form

$$\begin{aligned} \sum_{k=0}^{n} p_k(x) y^{(n-k)}(x) = \lambda y, \qquad a \leqslant x \leqslant b, \quad \lambda \in \mathbb{C}, \\ \sum_{j=1}^{n} \alpha_{ji} y^{(j-1)}(a) + \beta_{ji} y^{(j-1)}(b) = 0, \qquad i = 1, 2, \cdots, n, \end{aligned} \tag{1.1}$$

is introduced. This theorem can be stated as follows.
Let $H(x,\xi,\lambda)$ be the Green's function of the problem (1.1) and

$$P(\lambda) = \begin{cases} \prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right), & \text{if all } \lambda_k \neq 0, \\ \lambda \prod_{k=2}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right), & \text{if one of } \lambda_k, \text{ say } \lambda_1 = 0, \end{cases} \tag{1.2}$$

where $\{\lambda_k\}_{k=1}^{\infty}$ are the eigenvalues of (1.1). If this product is not convergent, then a multiplication by $\exp(\lambda/\lambda_k)$ is needed. For some fixed $\xi_0 \in [a,b]$, put

$$\Phi(x,\lambda) = P(\lambda)\, H(x,\xi_0,\lambda).$$

Hence, the sampling theorem of [8] reads:

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Theorem 1.1 *Let*

$$f(\lambda) = \int_a^b \overline{g}(x)\Phi(x,\lambda)\,dx, \qquad g \in L^2(a,b). \tag{1.3}$$

Then $f(\lambda)$ *can be represented as*

$$f(\lambda) = \sum_{k=1}^{\infty} f(\lambda_k)\frac{P(\lambda)}{(\lambda-\lambda_k)P'(\lambda_k)}. \tag{1.4}$$

If the problem (1.1) is self-adjoint or g *satisfies the boundary conditions, then the series converges uniformly on compact subsets of* $\mathbb{C}$.

For more details one may refer to [8]. In [4], certain modifications for the above result were given. In [2], Green's function for deriving sampling theorems associated with discontinuous eigenvalue problems was also given. In [1], the authors used Green's function associated with differential operators on a space of vector-function of dimension two to derive the sampling theorem. They considered the Dirac system of differential equations

$$y_2' - p(x)\,y_1 = \lambda\, y_1, \qquad y_1' + r(x)\,y_2 = -\lambda\, y_2, \tag{1.5}$$

together with the boundary conditions

$$\begin{aligned} y_1(0)\,\sin\alpha + y_2(0)\cos\,\alpha = 0,\\ y_1(\pi)\,\sin\beta + y_2(\pi)\,\cos\beta = 0, \end{aligned} \tag{1.6}$$

where $p(\cdot)$ and $r(\cdot)$ are continuous functions on $[0,\pi]$. Problem (1.5)–(1.6) [5, 6] has a countable number of real and simple eigenvalues $\{\lambda_n\}_{n=-\infty}^{\infty}$, and the solution $y(x,\lambda) = (y_1(x,\lambda), y_2(x,\lambda))^\top$, where $\top$ stands for the transpose, of (1.5), which satisfies $y(0,\lambda) = (\cos\alpha, -\sin\alpha)^\top$, generates all eigenfunctions. A vector-function $f(x) = (f_1(x), f_2(x))^\top$ is square-integrable if each component is square-integrable. Let $\phi(x,\lambda) = (\phi_1(x,\lambda), \phi_2(x,\lambda))^\top$, $\psi(x,\lambda) = (\psi_1(x,\lambda), \psi_2(x,\lambda))^\top$ be two solutions of (1.5) satisfying

$$\phi(0,\lambda) = \begin{pmatrix}\phi_1(0,\lambda)\\ \phi_2(0,\lambda)\end{pmatrix} = \begin{pmatrix}\cos\alpha\\ -\sin\alpha\end{pmatrix}, \quad \psi(\pi,\lambda) = \begin{pmatrix}\psi_1(\pi,\lambda)\\ \psi_2(\pi,\lambda)\end{pmatrix} = \begin{pmatrix}\cos\beta\\ -\sin\beta\end{pmatrix}, \tag{1.7}$$

and let $w(\lambda) = \psi_1(0)\,\sin\alpha + \psi_2(0)\,\cos\alpha$. Thus, Green's matrix of problem (1.5)–(1.6) [5, 6] is:

$$G(x,\xi,\lambda) = \frac{1}{w(\lambda)}\begin{cases} \phi(x,\lambda)\psi^\top(\xi,\lambda), & 0 \leqslant \xi \leqslant x \leqslant \pi,\\ \psi(x,\lambda)\phi^\top(\xi,\lambda), & 0 \leqslant x \leqslant \xi \leqslant \pi. \end{cases} \tag{1.8}$$

Define the matrix

$$\Phi(\xi,\lambda) = w(\lambda)G(x_0,\xi,\lambda), \tag{1.9}$$

where $x_0 \in [0,\pi]$. $\Phi(\xi,\lambda)$ is an entire vector-function of λ for every $\xi \in [0,\pi]$. The sampling theorem for vector-valued transforms whose kernels arise from Green's matrix of [1] states:

Theorem 1.2 *Let* $f(\cdot)$ *be a square-integrable vector-function on* $[0,\pi]$. *Let* $F(\lambda) = \begin{pmatrix}F_1(\lambda)\\ F_2(\lambda)\end{pmatrix}$ *be the vector-valued transform*

$$F(\lambda) = \int_0^\pi \Phi(\xi,\lambda)f(\xi)\,d\xi. \tag{1.10}$$

Then $F(\lambda)$ is a vector entire function of order one and exponential type at most π that admits the vector-valued sampling representation:

$$F(\lambda) = \sum_{k=-\infty}^{\infty} F(\lambda_k) \frac{w(\lambda)}{(\lambda - \lambda_k) w'(\lambda_k)}. \tag{1.11}$$

The vector-valued series (1.11) converges uniformly on any compact subset of $\mathbb{C}$ and absolutely on $\mathbb{C}$.

Also, in [3], two-dimensional sampling theorems associated with first- and second-order two-parameter differential equations via Green's function were given.

The aim of this article is to introduce a sampling theorem for integral transforms whose kernels arise from Green's function of differential operators in a space of vector-functions of dimension m and consist of systems of m equations in m unknowns. This will generalize the result of [4], where a system of differential equations of arbitrary order with boundary conditions that may be of mixed type is included.

Some main definitions for the problem and the formula of Green's function are given in the following section as well as some results that are used in the sequel. Most of these definitions and the results have been taken from [7]. Section 3 contains the sampling theorem in which the integral transform is a vector of dimension m involving Green's function in its kernel. Some illustrative examples are provided in the last section.

2. Preliminaries

Let $\mathbb{C}^m$ be the m-dimensional complex space consisting of all vectors $y^T = (y_1, \cdots, y_m)$, where T denotes the matrix transpose. Here $y(\cdot)$ is a vector-valued function in $\mathbb{C}^m$, with the inner product $\langle y, z \rangle = z^* y = \sum_{k=1}^{m} y_k(x) \overline{z_k(x)}$, where z^* denotes the conjugate transpose of z. We mean by an operator function A a square matrix $A(x) = [a_{ij}(x)]$ of order m, whose entries are scalar functions. By differentiability or continuity of vector-functions or operator functions we mean that those properties hold for their entries. Let $C^{(n)}[a,b]$ be the set of all vector-functions $y(\cdot)$ that have continuous derivatives up to nth order on $[a,b]$. Let $P_0(\cdot), \cdots, P_n(\cdot)$ be continuous operator functions on $[a,b]$, and let $\det(P_0(x)) \neq 0$ in $[a,b]$. Consider the linear differential expression in $\mathbb{C}^m$,

$$\ell(y) = P_0(x) y^{(n)} + \cdots + P_n(x) y, \tag{2.1}$$

where $y \in C^{(n)}[a,b]$ and the n boundary conditions

$$U_i(y) = A_{i,1} y_a + A_{i,2} y'_a \cdots + A_{i,n} y_a^{(n-1)} + B_{i,1} y_b + B_{i,2} y'_b + \cdots + B_{i,n} y_b^{(n-1)} = 0, \tag{2.2}$$

$i = 1, 2, \cdots, n,$ where $A_{i,1}, A_{i,2}, \cdots, A_{i,n}, B_{i,1}, B_{i,2}, \cdots, B_{i,n}$ are fixed (constant) operator functions. We assume that these boundary conditions are linearly independent, i.e. the rank of

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} & B_{1,1} & \cdots & B_{1,n} \\ . & \cdots & . & . & \cdots & . \\ A_{n,1} & \cdots & A_{n,n} & B_{i,1} & \cdots & B_{n,n} \end{pmatrix}$$

equals nm. Integrating by parts yields

$$\int_a^b \langle \ell(y), z \rangle \, dx = P(\eta, \zeta) + \int_a^b \langle y, \ell^*(z) \rangle \, dx, \tag{2.3}$$

where

$$\ell^*(z) = (-1)^n (P_0^* z)^{(n)} + (-1)^{(n-1)} (P_1^* z)^{(n-1)} + \cdots + P_n^* z, \tag{2.4}$$

and $P(\eta, \zeta)$ is a bilinear form in

$$\eta = (y_a, y_a', \cdots, y_a^{(n-1)}, y_b, y_b', \cdots, y_b^{(n-1)}),\ \zeta = (z_a, z_a', \cdots, z_a^{(n-1)}, z_b, z_b', \cdots, z_b^{(n-1)}).$$

The differential expression $\ell^*(z)$ is called the adjoint differential expression of $\ell(y)$, and (2.3) is called Lagrange's formula for vector-functions. The differential expressions $\ell(y)$ and $\ell^*(y)$ are mutually adjoint. A differential expression is said to be self-adjoint if $\ell^* = \ell$. A general form of self-adjoint differential expressions is given in the following.

Lemma 2.1 *Any self-adjoint differential expression is a sum of differential expressions of the form*

$$\ell_{2\nu} = (Py^{(\nu)})^{(\nu)}, \quad \ell_{2\nu-1} = \frac{1}{2}[(iPy^{(\nu-1)})^{(\nu)} + (iPy^{(\nu)})^{(\nu-1)}],$$

where $P = P(x)$ is an operator function whose values are Hermitian matrices.

We extend $U_1, \cdots, U_n$ with other forms $U_{n+1}, \cdots, U_{2n}$ to obtain a linearly independent system of $2n$ forms $U_1, U_2, \cdots, U_{2n}$. In this case Lagrange's formula takes the form

$$\int_a^b \langle \ell(y), z \rangle \, dx = \langle U_1, V_{2n} \rangle + \langle U_2, V_{2n-1} \rangle + \cdots + \langle U_{2n}, V_1 \rangle + \int_a^b \langle y, \ell^*(z) \rangle \, dx, \tag{2.5}$$

where V_1, $V_2, \cdots, V_{2n}$ are linearly independent forms in the variables

$$z_a, z_a', \cdots, z_a^{(n-1)}, z_b, z_b', \cdots, z_b^{(n-1)}.$$

The boundary conditions

$$V_1 = 0, V_2 = 0, \cdots, V_n = 0, \tag{2.6}$$

are said to be the adjoint conditions to the original boundary conditions

$$U_1 = 0, U_2 = 0, \cdots, U_n = 0. \tag{2.7}$$

Boundary conditions are said to be self-adjoint if they are equivalent to their adjoint boundary conditions. Let L be the operator generated by the expression $\ell(y)$ and the boundary conditions (2.7). The operator generated by $\ell^*(y)$ and the boundary conditions (2.6) is denoted by L^* and is called the adjoint operator of L. If λ is an eigenvalue of L, then $\overline{\lambda}$ is an eigenvalue of L^*. An operator L is said to be self-adjoint if $L = L^*$, i.e. if and only if it is generated by a self-adjoint differential expression and self-adjoint boundary conditions.

Now we look for a special form of the boundary conditions (2.2), which are called *regular* boundary conditions. This definition can be stated as follows. Assume that the boundary conditions (2.2) have the form

$$U_\nu(y) = U_{\nu a}(y) + U_{\nu b}(y) = 0, \tag{2.8}$$

where

$$U_{\nu a}(y) = A_\nu \, y_a^{(k_\nu)} + \sum_{j=0}^{k_\nu - 1} A_{\nu j} \, y_a^{(j)}, \qquad U_{\nu b}(y) = B_\nu \, y_b^{(k_\nu)} + \sum_{j=0}^{k_\nu - 1} B_{\nu j} \, y_b^{(j)}, \tag{2.9}$$

$$n-1 \geqslant k_1 \geqslant k_2 \geqslant \cdots \geqslant k_n \geqslant 0, \quad k_{\nu+2} < k_\nu,$$

and for each ν, $\nu = 1, \cdots, n$, at least one of the matrices A_ν, B_ν is different from zero. The conditions (2.8) are called *normalized*. It is always possible to put boundary conditions in a normalized form [7, p. 120]. The definition of regularity of (2.8) [7] depends on whether n is even or odd as follows:

<u>When n is odd:</u> $n = 2\mu - 1$, conditions (2.8) are said to be regular if the numbers θ_0 and θ_m defined by

$$\theta_0 + \theta_1\, s + \cdots + \theta_m\, s^m =$$
$$\begin{vmatrix} A_1 w_1^{k_1} & \cdots & A_1 w_{\mu-1}^{k_1} & (A_1 + sB_1) w_\mu^{k_1} & B_1 w_{\mu+1}^{k_1} & \cdots & B_1 w_n^{k_1} \\ A_2 w_1^{k_2} & \cdots & A_2 w_{\mu-1}^{k_2} & (A_2 + sB_2) w_\mu^{k_2} & B_2 w_{\mu+1}^{k_2} & \cdots & B_2 w_n^{k_2} \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ A_n w_1^{k_n} & \cdots & A_n w_{\mu-1}^{k_n} & (A_n + sB_n) w_\mu^{k_n} & B_n w_{\mu+1}^{k_n} & \cdots & B_n w_n^{k_n} \end{vmatrix} \tag{2.10}$$

are both different from zero, where w_k are the different n roots of -1.

<u>When n is even:</u> $n = 2\mu$, conditions (2.8) are said to be regular if the numbers θ_{-m} and θ_m defined by

$$\theta_{-m}\, s^{-m} + \theta_{-m+1}\, s^{-m+1} + \cdots + \theta_m\, s^m =$$
$$\begin{vmatrix} A_1 w_1^{k_1} & \cdots & A_1 w_{\mu-1}^{k_1} & (A_1 + sB_1) w_\mu^{k_1} & \left(A_1 + \dfrac{B_1}{s}\right) w_{\mu+1}^{k_1} & B_1 w_{\mu+2}^{k_1} & \cdots & B_1 w_n^{k_1} \\ A_2 w_1^{k_2} & \cdots & A_2 w_{\mu-1}^{k_2} & (A_2 + sB_2) w_\mu^{k_2} & \left(A_2 + \dfrac{B_2}{s}\right) w_{\mu+1}^{k_2} & B_2 w_{\mu+2}^{k_2} & \cdots & B_2 w_n^{k_2} \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ A_n w_1^{k_n} & \cdots & A_n w_{\mu-1}^{k_n} & (A_n + sB_n) w_\mu^{k_n} & \left(A_n + \dfrac{B_n}{s}\right) w_{\mu+1}^{k_n} & B_n w_{\mu+2}^{k_n} & \cdots & B_n w_n^{k_n} \end{vmatrix} \tag{2.11}$$

are both different from zero.

In the following we investigate the eigenvalues of the operator L and Green's function of the operator $L - \lambda I$. For this task, we need an operator form of equation (2.1). Consider the equation in an operator function $Y(x)$:

$$\ell(Y) = P_0(x) Y^{(n)} + \cdots + P_n(x) Y = \lambda Y. \tag{2.12}$$

Solutions $Y_1(x), \cdots, Y_n(x)$ of (2.12) are linearly independent if $C_1 Y_1 + \cdots + C_n Y_n = 0$, for any constant operators $C_1, \cdots, C_n$, holds only when $C_1 = \cdots = C_n = 0$. This is equivalent to that the matrix

$$\begin{pmatrix} Y_1 & \cdots & Y_n \\ \cdot & \cdots & \cdot \\ Y_1^{(n-1)} & \cdots & Y_n^{(n-1)} \end{pmatrix}$$

is nonsingular. Any solution of $\ell(y) = \lambda y$ has the form $y = Y_1 c_1 + \cdots + Y_n c_n$, where $c_1, \cdots, c_n$ are constant vectors. The eigenvalues of $Ly = \lambda y$ are the zeros of the characteristic determinant

$$\Delta(\lambda) = \begin{vmatrix} U_1(Y_1) & \cdots & U_1(Y_n) \\ \cdot & \cdots & \cdot \\ U_n(Y_1) & \cdots & U_n(Y_n) \end{vmatrix}. \tag{2.13}$$

The solutions $Y_1, \cdots, Y_n$ can be chosen to be analytic operator functions in the parameter λ, and hence $\Delta(\lambda)$ also is an analytic function. If this determinant is not identically zero, then these are zeros; the eigenvalues of the operator L are at most a countable set with no finite limit point. If λ_0 is a zero of $\Delta(\lambda)$ with multiplicity ν, then the multiplicity of the eigenvalue λ_0 cannot be greater than ν. Hence, if λ_0 is a simple zero of $\Delta(\lambda)$, then the multiplicity of the eigenvalue λ_0 is also unity. In this case λ_0 is called a simple eigenvalue.

In the following we give the Green's function expression for the operator $L - \lambda I$. This is guaranteed if $L - \lambda I = 0$ has only a trivial solution; in other words, λ is different from the eigenvalues of L.

Let $|W|$ be the determinant of the matrix

$$W = \begin{pmatrix} Y_1^{(n-1)} & \cdots & Y_n^{(n-1)} \\ Y_1^{(n-2)} & \cdots & Y_n^{(n-2)} \\ \cdot & \cdots & \cdot \\ Y_1 & \cdots & Y_n \end{pmatrix},$$

and denote by W_ν, $\nu = 1, \cdots, n$, the transpose of the mth order matrices consisting of the cofactors of the elements of Y_ν in W. Put

$$Z_\nu = \frac{1}{|W|} W_\nu,$$

$$g(x, \xi, \lambda) = \begin{cases} \frac{1}{2} \sum_{\nu=1}^{n} Y_\nu(x) Z_\nu(\xi), & \xi < x, \\ \\ -\frac{1}{2} \sum_{\nu=1}^{n} Y_\nu(x) Z_\nu(\xi), & \xi > x, \end{cases}$$

$$U = \begin{pmatrix} U_1(Y_1) & \cdots & U_1(Y_n) \\ \cdot & \cdots & \cdot \\ U_n(Y_1) & \cdots & U_n(Y_n) \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} W_{11} & \cdots & W_{1n} \\ \cdot & \cdots & \cdot \\ W_{n1} & \cdots & W_{nn} \end{pmatrix},$$

where $W_{j\nu}$ are $m \times m$ matrices.

The Green's function for the operator $L - \lambda I$ is given by

$$G(x, \xi, \lambda) = g(x, \xi, \lambda) - \sum_{j,\nu=1}^{n} Y_j(x)\, W_{j\nu}\, U_\nu(g). \tag{2.14}$$

The function $G(x, \xi, \lambda)$ is a meromorphic matrix-function of the parameter λ, and only eigenvalues of L can be poles of this function [7, p. 117]. Hence, the solution of $(L - \lambda)y = f(x)$ is

$$y(x) = \int_a^b G(x, \xi, \lambda)\, f(\xi)\, d\xi. \tag{2.15}$$

The following theorem gives a significant characterization of the Green's function $G(x, \xi, \lambda)$. From now on we assume that all the eigenvalues of the operator L generated by regular boundary conditions are simple zeros of $\Delta(\lambda)$. Assume also that $\{\lambda_k\}_{k=1}^{\infty}$, and $\{\overline{\lambda_k}\}_{k=1}^{\infty}$ are the eigenvalues of L and L^* associated with the eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$, respectively, where ϕ_k and ψ_k are normalized so that

$$\int_a^b \langle \phi_k, \psi_k \rangle\, dx = 1.$$

Theorem 2.2 *Green's function can be represented as the uniformly convergent expansion*

$$G(x, \xi, \lambda) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\psi_k^*(\xi)}{\lambda_k - \lambda}, \qquad x, \xi \in [a, b], \quad \lambda \neq \lambda_k. \tag{2.16}$$

Proof Green's function of the operator L has the following uniform convergent series [7, p. 128]:

$$G(x, \xi) = \sum_{k=1}^{\infty} \frac{\phi_k(x)\psi_k^*(\xi)}{\lambda_k}.$$

Since $\lambda_k - \lambda$, $k = 1, 2, \cdots$, are the eigenvalues of the operator $L - \lambda I$ with the corresponding eigenfunctions $\phi_k(x)$ and $\overline{\lambda_k - \lambda}$ are the eigenvalues of the operator $L^* - \overline{\lambda} I$ with the corresponding eigenfunctions $\psi_k(x)$, then replacing λ_k by $\lambda_k - \lambda$ in the former formula, one gets (2.16). □

3. The sampling theorem

In this section we introduce the main result for a sampling theorem of vector-valued transform of dimension m with a kernel defined via Green's function as follows:

Let $\xi_0 \in [a, b]$ such that $G(x, \xi_0, \lambda) \not\equiv 0$ on $[a, b]$. Define the entire function

$$\Psi(x, \lambda) = \Delta(\lambda)\, G(x, \xi_0, \lambda). \tag{3.1}$$

Let $\mathfrak{L}_m^2(a, b)$ be the Hilbert space:

$$\mathfrak{L}_m^2(a, b) = \left\{ y(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_m(x) \end{pmatrix} : \, y_i(x) \in L^2(a, b) \right\}, \tag{3.2}$$

where the inner product and the norm are

$$\begin{aligned} \left(y, z\right)_{\mathfrak{L}_m^2(a,b)} &= \int_a^b \langle y, z \rangle \, dx = \int_a^b z^* y \, dx, \\ \|y\|_{\mathfrak{L}_m^2(a,b)} &= \sqrt{\left(y, y\right)_{\mathfrak{L}_m^2(a,b)}}. \end{aligned} \tag{3.3}$$

Theorem 3.1 *For* $f \in \mathfrak{L}_m^2(a, b)$, *let*

$$F(\lambda) = \int_a^b f^*(x)\Psi(x, \lambda)\, dx, = \left(\Psi, f\right)_{\mathfrak{L}_m^2(a,b)}, \tag{3.4}$$

be an m*-dimensional vector-valued transform, and then* $F(\lambda)$ *can be represented as*

$$F(\lambda) = \sum_{k=1}^{\infty} F(\lambda_k) \frac{\Delta(\lambda)}{(\lambda - \lambda_k)\Delta'(\lambda_k)}. \tag{3.5}$$

This series converges uniformly on a compact subset of $\mathbb{C}$ *if:*

1. f is in the domain of L when it is not self-adjoint, or

2. the operator L is self-adjoint.

Proof From (3.4) and (2.16), one has

$$\begin{aligned} F(\lambda) &= \int_a^b f^*(x)\Delta(\lambda)\sum_{k=1}^{\infty}\frac{\phi_k(x)\psi_k^*(\xi)}{\lambda_k-\lambda}\,dx \\ &= \sum_{k=1}^{\infty}\frac{\Delta(\lambda)}{\lambda_k-\lambda}\left(\int_a^b f^*(x)\phi_k(x)\,dx\right)\,\psi_k^*(\xi). \end{aligned} \tag{3.6}$$

This leads to

$$F(\lambda_k) = -\Delta'(\lambda_k)\left(\int_a^b f^*(x)\phi_k(x)\,dx\right)\,\psi_k^*(\xi). \tag{3.7}$$

Combining (3.6) and (3.7), one gets (3.5).

For the uniform convergence, we apply the same technique used in [9, p. 176]. Let K be a compact subset of the complex λ-plane. Let $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ and $\overline{\Lambda} = \{\lambda_{K1}, \cdots, \lambda_{Kp}\}$ be the set of eigenvalues that lie in K, which is finite since Λ has no finite limit point. Put $\rho = distance(K, \Lambda - \overline{\Lambda})$; thus, for all $\lambda_k \in \Lambda - \overline{\Lambda}$, one gets

$$\sup_{\lambda\in K}\left|\frac{\Delta(\lambda)}{\lambda_k-\lambda}\right| \leqslant \frac{1}{\rho}\sup_{\lambda\in K}|\Delta(\lambda)| = \frac{1}{\rho}\,||\Delta(\lambda)||_K,$$

where $\sup\limits_{\lambda\in K}|\Delta(\lambda)| = ||\Delta(\lambda)||_K$. For $\lambda_k \in \overline{\Lambda}$, since $\Delta(\lambda)$ has zeros at $\lambda_{K1}, \cdots, \lambda_{Kp}$, it follows that $h_{Ki} = D(\lambda)/(\lambda_{Ki} - \lambda)$ is an analytic function in K. Let

$$C(K) = \max\left\{\frac{1}{\rho}\,||\Delta(\lambda)||_K, ||h_{K1}||, \cdots, ||h_{Kp}||\right\}.$$

Hence, for all λ_k, one obtains

$$\sup_{\lambda\in K}\left|\frac{\Delta(\lambda)}{\lambda_k-\lambda}\right| \leqslant C(K).$$

Now in view of (3.6), we have

$$\begin{aligned} &\left\|F(\lambda) - \sum_{k=1}^{m} F(\lambda_k)\frac{\Delta(\lambda)}{(\lambda-\lambda_k)\Delta'(\lambda_k)}\right\|_{\mathfrak{L}^2_m(a,b)} \\ &\qquad = \left\|\sum_{k=m+1}^{\infty}\frac{\Delta(\lambda)}{\lambda_k-\lambda}\left(\int_a^b f^*(x)\phi_k(x)\,dx\right)\,\psi_k^*(\xi)\right\|_{\mathfrak{L}^2_m(a,b)} \\ &\qquad \leqslant C(K)\sum_{k=m+1}^{\infty}\left|\left(\phi_k, f\right)_{\mathfrak{L}^2_m(a,b)}\right|\,\|\psi_k\|_{\mathfrak{L}^2_m(a,b)}. \end{aligned}$$

The last series is independent of λ and tends to zero as $m \to \infty$ in the following cases.

1. If f is in the domain of L when the operator L is not self-adjoint, then f has the uniform convergent expansion

$$f(\xi) = \sum_{k=1}^{\infty} \left(f, \phi_k\right)_{\mathfrak{L}^2_m(a,b)} \psi_k(\xi); \tag{3.8}$$

   see [7, p. 129].

2. If the operator L is self-adjoint, then (3.8) is valid for any $f \in \mathfrak{L}^2_m(a,b)$ with $\psi_k \equiv \phi_k$; see [7, pp. 124–125].

□

Remark 3.2 *We can replace $\Delta(\lambda)$ in (3.1) by the infinite product (1.2) in the kernel (3.1) to get a representation similar to (1.4). In fact, let*

$$\widetilde{\Psi}(x,\lambda) = P(\lambda)\, G(x,\xi_0,\lambda). \tag{3.9}$$

Then the transform

$$\widetilde{F}(\lambda) = \int_a^b f^*(x)\widetilde{\Psi}(x,\lambda)\, dx, \qquad f \in \mathfrak{L}^2_m(a,b), \tag{3.10}$$

will have the expansion

$$\widetilde{F}(\lambda) = \sum_{k=1}^{\infty} \widetilde{F}(\lambda_k) \frac{P(\lambda)}{(\lambda-\lambda_k)P'(\lambda_k)}. \tag{3.11}$$

Since $P(\lambda)$ and $\Delta(\lambda)$ have the same zeros, then

$$\Delta(\lambda) = R(\lambda)P(\lambda),$$

where $R(\lambda)$ is an entire function with no zeros.

4. Examples

In this section we give three examples for the above sampling theorem associated with boundary value problems of order one and two on a space of vector-functions that is two-dimensional.

Example 4.1 *Consider the boundary value problem*

$$i\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, 0 \leqslant x \leqslant \pi, \tag{4.1}$$

$$U(y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_1((\pi)) \\ y_2(\pi) \end{pmatrix} = \begin{pmatrix} y_1(0) - y_2(\pi) \\ y_2(0) + y_1(\pi) \end{pmatrix} = 0. \tag{4.2}$$

This problem is self adjoint since

$$V(z) = i\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z(0) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z(\pi)\right].$$

Any solution of $\ell(Y) = \lambda Y$ *will be*

$$Y(x,\lambda) = \begin{pmatrix} \mathrm{e}^{-i\lambda x}\cos x & -\mathrm{e}^{-i\lambda x}\sin x \\ \mathrm{e}^{-i\lambda x}\sin x & \mathrm{e}^{-i\lambda x}\cos x \end{pmatrix}.$$

Hence, any solution of (4.1) *is*

$$y(x,\lambda) = \begin{pmatrix} c_1\,\mathrm{e}^{-i\lambda x}\cos x - c_2\,\mathrm{e}^{-i\lambda x}\sin x \\ c_1\,\mathrm{e}^{-i\lambda x}\sin x + c_2\,\mathrm{e}^{-i\lambda x}\cos x \end{pmatrix}.$$

The boundary condition is regular since (2.10) *will be*

$$\left|A_1 + sB_1\right| = \begin{vmatrix} 1 & -s \\ s & 1 \end{vmatrix} = 1 + s^2, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ B_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This means that $\theta_0 = 0,\ \theta_2 = 1$. *We have*

$$\Delta(\lambda) = 1 + \mathrm{e}^{-2i\lambda\pi},$$

and hence the eigenvalues are $\lambda_k = k - \frac{1}{2},\ k \in \mathbb{Z}$. *Green's function* $G(x,\xi.\lambda)$ *here has the form*

$$G(x,\xi,\lambda) = \frac{-i\,\mathrm{e}^{i\lambda(\xi-x)}}{1+\mathrm{e}^{-2i\lambda\pi}} \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} \times$$
$$\times \left\{ \begin{matrix} \begin{pmatrix} 1 & -\mathrm{e}^{-i\lambda\pi} \\ \mathrm{e}^{-i\lambda\pi} & 1 \end{pmatrix} & \xi < x \\ \mathrm{e}^{-i\lambda\pi}\begin{pmatrix} -\mathrm{e}^{-i\lambda\pi} & -1 \\ 1 & -\mathrm{e}^{-i\lambda\pi} \end{pmatrix} & \xi > x \end{matrix} \right\} \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix}.$$

Let $\Psi(x,\lambda) = \Delta(\lambda)\,G(x,\xi_0,\lambda)$; *then the transform* (3.4), *which is*

$$F(\lambda) = \int_0^\pi f^*(x)\Psi(x,\lambda)\,dx, \qquad f \in \mathfrak{L}_2^2(0,\pi),$$

has the expansion

$$F(\lambda) = \sum_{k=-\infty}^{\infty} F(k-1/2)\frac{1+\mathrm{e}^{-2i\lambda\pi}}{2i\pi(\lambda - k + \frac{1}{2})}. \tag{4.3}$$

Here $P(\lambda) = \prod_{k=1}^{\infty}\left(1 - \frac{\lambda^2}{(k-1/2)^2}\right) = \cos\pi\lambda$, $\left(R(\lambda) = 2\mathrm{e}^{-i\lambda\pi}\right)$. *For the transform*

$$\widetilde{F}(\lambda) = \int_0^\pi f^*(x)\widetilde{\Psi}(x,\lambda)\,dx, \qquad \widetilde{\Psi}(x,\lambda) = P(\lambda)\,G(x,\xi_0,\lambda),$$

one obtains

$$\widetilde{F}(\lambda) = \sum_{k=-\infty}^{\infty} \widetilde{F}(k-1/2)\frac{\sin\pi(\lambda - k + \frac{1}{2})}{\pi(\lambda - k + \frac{1}{2})}. \tag{4.4}$$

Example 4.2 *Consider the differential equation* (4.1) *with the boundary condition*

$$U(y)=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}y_1(0)\\y_2(0)\end{pmatrix}+\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}y_1((\pi))\\y_2(\pi)\end{pmatrix}=\begin{pmatrix}y_1(0)+y_2(\pi)\\y_2(0)+y_1(\pi)\end{pmatrix}=0. \tag{4.5}$$

This problem is self adjoint since $V(z)=i\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}z(0)+\begin{pmatrix}0&1\\1&0\end{pmatrix}z(\pi)\right]$. *The solution is the same as in the previous example and the boundary conditions are regular. The eigenvalues are the zeros of*

$$\Delta(\lambda)=1-e^{-2i\lambda\pi},$$

and hence the eigenvalues are $\lambda_k=k$, $k\in\mathbb{Z}$. *Here, we have*

$$\begin{aligned}G(x,\xi,\lambda)&=\frac{-i\,\mathrm{e}^{-i\lambda(\xi-x)}}{1-\mathrm{e}^{-2i\lambda\pi}}\begin{pmatrix}\cos x&-\sin x\\\sin x&\cos x\end{pmatrix}\times\\&\times\begin{Bmatrix}\begin{pmatrix}1&\mathrm{e}^{-i\lambda\pi}\\\mathrm{e}^{-i\lambda\pi}&1\end{pmatrix}&\xi<x\\\mathrm{e}^{-i\lambda\pi}\begin{pmatrix}\mathrm{e}^{-i\lambda\pi}&1\\1&\mathrm{e}^{-i\lambda\pi}\end{pmatrix}&\xi>x\end{Bmatrix}\begin{pmatrix}\cos\xi&\sin\xi\\-\sin\xi&\cos\xi\end{pmatrix}.\end{aligned}$$

Let $\Psi(x,\lambda)=\Delta(\lambda)\,G(x,\xi_0,\lambda)$; *then the transform* (3.4) *has the expansion*

$$F(\lambda)=\sum_{k=-\infty}^{\infty}F(k)\frac{\mathrm{e}^{-i\pi\lambda}\sin\pi\lambda}{\pi(\lambda-k)}. \tag{4.6}$$

We have $P(\lambda)=\lambda\prod_{k=1}^{\infty}\left(1-\frac{\lambda^2}{k^2}\right)=\frac{\sin\pi\lambda}{\pi}$, $\left(R(\lambda)=2i\mathrm{e}^{-i\lambda\pi}\right)$. *For the transform*

$$\widetilde{F}(\lambda)=\int_0^{\pi}f^*(x)\widetilde{\Psi}(x,\lambda)\,dx,\qquad\widetilde{\Psi}(x,\lambda)=P(\lambda)\,G(x,\xi_0,\lambda),$$

one gets

$$\widetilde{F}(\lambda)=\sum_{k=-\infty}^{\infty}\widetilde{F}(k)\frac{\sin\pi(\lambda-k)}{\pi(\lambda-k)}.$$

Example 4.3 *Consider the boundary value problem*

$$\ell(y)=\begin{pmatrix}0&1\\1&0\end{pmatrix}y''=\lambda y,\quad\text{or}\quad\left\{\begin{array}{l}y_2''=\lambda y_1,\\y_1''=\lambda y_2,\end{array}\right.\quad 0\leqslant x\leqslant\pi, \tag{4.7}$$

$$U_1(y)=y(0)=\begin{pmatrix}y_1(0)\\y_2(0)\end{pmatrix}=0,\quad U_2(y)=\begin{pmatrix}y_1(\pi)\\y_2(\pi)\end{pmatrix}=y(\pi)=0. \tag{4.8}$$

Lagrange's formula will be

$$\int_0^{\pi}z^*\ell(y)\,dx=V_4^*U_1+V_3^*U_2+V_2^*U_3+V_1^*U_4+\int_0^{\pi}\ell^*(z)y\,dx, \tag{4.9}$$

where

$$V_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z(\pi),\; V_2 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z(0). \tag{4.10}$$

Hence, ℓ *and the boundary conditions are self-adjoint. Two linearly solutions of* $\ell(Y) = \lambda Y$ *are*

$$Y_1(x,\lambda) = \begin{pmatrix} \cosh tx & \cos tx \\ \cosh tx & -\cos tx \end{pmatrix}, \qquad Y_2(x,\lambda) = \begin{pmatrix} \sinh tx/t & \sin tx/t \\ \sinh tx/t & -\sin tx/t \end{pmatrix},$$

where $\lambda = t^2$*. Therefore, any solution of* (4.7) *is given by*

$$y(x,\lambda) = \begin{pmatrix} c_1 \cosh tx + c_2 \cos tx + c_3 \sinh tx/t + c_4 \sin tx/t \\ c_1 \cosh tx - c_2 \cos tx + c_3 \sinh tx/t - c_4 \sin tx/t \end{pmatrix}.$$

The problem (4.7) $-$ (4.8) *is regular since we have*

$$A_1 = B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad k_1 = k_2 = 0,$$

and then (2.11) *will be*

$$\begin{vmatrix} (A_1 + sB_1) & \left(A_1 + \dfrac{B_1}{s}\right) \\ (A_2 + sB_2) & \left(A_2 + \dfrac{B_2}{s}\right) \end{vmatrix} = \frac{1}{s^2} - 2 + s^2,$$

i.e. $\theta_{-2} = \theta_2 = 1$*. The eigenvalues of* (4.7) $-$ (4.8) *are the zeros of*

$$\Delta(\lambda) = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \cosh \pi t & \cos \pi t & \sinh \pi t/t & \sin \pi t/t \\ \cosh \pi t & -\cos \pi t & \sinh \pi t/t & -\sin \pi t/t \end{vmatrix} = \frac{4 \sin \pi t \sinh \pi t}{t^2},$$

which are $\lambda_k = \pm k^2$, $k = 1, 2, \cdots$ *, and all of them are simple. Here,*

$$g(x,\xi,\lambda) = \frac{1}{4}\left\{ \begin{pmatrix} \cosh tx & \cos tx \\ \cosh tx & -\cos tx \end{pmatrix} \begin{pmatrix} \cosh t\xi & \cosh t\xi \\ \cos t\xi & -\cos t\xi \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} \sinh tx & \sin tx \\ \sinh tx & -\sin tx \end{pmatrix} \begin{pmatrix} -\sinh t\xi & -\sinh t\xi \\ \sin t\xi & -\sin t\xi \end{pmatrix} \right\} \begin{cases} 1 & \xi < x, \\ -1 & \xi > x, \end{cases}$$

$$U^{-1} = \frac{1}{2} \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -\frac{t}{\tanh \pi t} & -\frac{t}{\tanh \pi t} \\ -\frac{t}{\tan \pi t} & \frac{t}{\tan \pi t} \end{pmatrix} & \begin{pmatrix} \frac{t}{\sinh \pi t} & \frac{t}{\sinh \pi t} \\ \frac{t}{\sin \pi t} & -\frac{t}{\sin \pi t} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}.$$

Hence, $G(x,\xi,\lambda)$ *can be determined from* (2.14)*. For the transform* (3.4)*, we have*

$$F(\lambda) = \sum_{k=1}^{\infty} \left(\frac{F(k^2)}{\lambda - k^2} - \frac{F(-k^2)}{\lambda + k^2} \right) \frac{2k^3 \sin \pi(\sqrt{\lambda} - k) \sinh \pi\sqrt{\lambda}}{\pi\, \lambda \sinh \pi k}. \tag{4.11}$$

Here $P(\lambda) = \prod_{k=1}^{\infty} \left(1 - \frac{\lambda^2}{k^2}\right)\left(1 + \frac{\lambda^2}{k^2}\right) = \frac{\sin \pi t \sinh \pi t}{\pi^2\, t^2}$, $\left(R(\lambda) = 4\,\pi^2\right)$*. Thus, the transform* $\widetilde{F}$ *of Remark* 3.2 *has the same representation of the transform* F*.*

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