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On the Zariski topology over an L-module M

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Abstract: Let L be a multiplicative lattice and M be an L-module. In this study, we present a topology said to be the Zariski topology over $\sigma(M)$, the collection of all prime elements of an L-module M. We research some results on the Zariski topology over $\sigma(M)$. We show that the topology is a T_0-space and a T_1-space under some conditions. Some properties and results are studied for the topology over $\sigma(L)$, the collection of all prime elements of a multiplicative lattice L.

Key words: Lattice, lattice module, prime element, Zariski topology

1. Introduction

A complete lattice L is called a multiplicative lattice if there exists a commutative, associative, completely join distributive product on the lattice with compact greatest element 1_L, which is the multiplicative identity, and least element 0_L. Note that $L/a = \{b \in L | a \leq b\}$ is a multiplicative lattice with product $x \circ y = xy \bigvee a$ where L is multiplicative lattice and $a \in L$. Several authors have studied multiplicative lattices in a series of articles [1–3,6–9].

Throughout this study, we suppose that L is multiplicative lattice and L_* is the collection of all compact elements of L.

An element a in L is said to be proper if $a < 1_L$. A proper element p in L is called prime if whenever $x, y \in L$ with $xy \leq p$, then $x \leq p$ or $y \leq p$. A proper element m of L is called a maximal element if $m < x \leq 1_L$ implies $x = 1_L$. The residual of a by b for any $a, b \in L$, denoted by $(a :_L b)$, is defined as the join of all $c \in L$ with $cb \leq a$. The radical of an element $a \in L$ is $\sqrt{a} = \bigvee\{x \in L | x^n \leq a \ \ for \ \ some \ n \in \mathbb{Z}^+\}$.

An element a of L is called compact if the following condition is satisfied: $a \leq \bigvee b_\alpha$ implies $a \leq b_{\alpha_1} \bigvee b_{\alpha_2} \bigvee ... \bigvee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$. A complete multiplicative (but not necessarily modular) lattice L is called a C-lattice if a multiplicatively closed subset C of L_* generates L under joins and L has least element 0_L and compact greatest element 1_L. The ideal lattice $L(R)$ of a commutative ring R with identity is an example for a C-lattice. Since 1_L is a compact element, then maximal elements exist in L. Let $\sigma(L)$ be the collection of all prime elements of a multiplicative lattice L. It is easily shown that $\sqrt{a} = \bigwedge\{p \in \sigma(L) | a \leq p\}$ for any element a of a C-lattice L (see [15, Theorem 3.6] and [3]).

In the literature, there are many distinct generalizations of the Zariski topology over the set of all prime submodules of a ring R-module M. In [10], the Zariski topology was introduced on the prime spectrum of a

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module over a commutative ring. In [14], the Zariski topology was studied on the prime spectrum of a module over a noncommutative ring. In [16], the Zariski topology was investigated over the prime spectrum of a module over an arbitrary associative ring. In [5], the authors introduced a new class of modules over R, called X-injective R-modules, where X is the prime spectrum of M. This class contains the family of top modules and that of weak multiplication modules properly. Some conditions under which the prime spectrum of M is a spectral space for its Zariski topology over a top module M were also studied. In [4], the authors specified the differences of topological properties of these Zariski topologies and investigated them in terms of spectral space.

In this study, we give some topological properties for the topology over $\sigma(L)$, which was introduced in [15]. Additionally, we investigate irreducible closed subsets. It is also considered that the topology is a T_0-space and a T_1-space under some conditions.

A complete lattice M is called a lattice module (or an L-module) on the multiplicative lattice L provided that there exists a product among elements of L and M that satisfies the following properties:

1. $(ka)\,K = k\,(aK)\,,$
2. $\left(\bigvee\limits_{\alpha} k_\alpha\right)\left(\bigvee\limits_{\beta} K_\beta\right) = \bigvee\limits_{\alpha,\beta} k_\alpha K_\beta,$
3. $1_L K = K,$
4. $0_L K = 0_M$

for every $k, k_\alpha, a \in L$ and $K, K_\beta \in M$.

Let M be an L-module. We denote the greatest element of M with 1_M. K in M is called a proper element if $K < 1_M$. An element $K < 1_M$ in M is prime if $aA \leq K$ for some $a \in L, and\ A \in M$ implies either $A \leq K$ or $a1_M \leq K$. We know that $(K :_L 1_M)$ is prime in L in the case that K is prime in M (see [14]). An element $K \in M$ with $K < 1_M$ is a maximal if $K < A \leq 1_M$ implies $A = 1_M$. If 1_M is compact, then M has a maximal element by [12]. If N and K belong to M, $(N :_L K)$ is the join of all $a \in L$ such that $aK \leq N$. For any $a \in L$, $(0_M :_M a)$ is the join of all $H \in M$ such that $aH = 0_M$. We say that M is a faithful L-module when $Ann\,(M) = (0_M :_L 1_M) = 0_L$.

An element K of an L-module M is compact if $K \leq \bigvee A_i$ implies $N \leq A_{i_1} \bigvee A_{i_2} \bigvee ... \bigvee A_{i_n}$ for some subset $\{i_1, i_2, ..., i_n\}$.

In this study, we define a topology that we call the Zariski topology over $\sigma(M)$, the family of all prime elements of an L-module M. For this topology, we assume that a closed set is a variety $V(K) = \{P \in \sigma(M) | (K :_L 1_M) \leq (P :_L 1_M)\}$ for any $K \in M$. We investigate some topological properties of this topology. We also show that the topology is a T_0-space and a T_1-space under some conditions.

Rings, semirings, graded rings, and polynomial rings are examples of multiplicative lattices. In this study, our aim is to extend the topology on rings to multiplicative lattices.

2. A topology on $\sigma(L)$ over L

By $V(a)$, we denote the set of all prime elements p of L with $a \leq p$ for any $a \in L$, that is, $V(a) = \{p \in \sigma(L) | a \leq p\}$. A topology over $\sigma(L)$ is introduced since the varieties satisfy the axioms for the closed sets. Now we prove the following proposition given in [15] without proof.

Proposition 1 *The following axioms hold by means of the above definition.*

1. $V(0_L) = \sigma(L)$ *and* $V(1_L) = \emptyset$.
2. $\bigcap_{i\in\Delta} V(a_i) = V(\bigvee_{i\in\Delta} a_i)$ *for any index set* $\triangle$.
3. $V(a) \cup V(b) = V(a \wedge b) = V(ab)$.

Proof *1.* It is clear.

2. $p \in \bigcap_{i\in\Delta} V(a_i) \Leftrightarrow p \in V(a_i)$ for any $i \in \triangle \Leftrightarrow a_i \leq p$ for every $i \in \Delta \Leftrightarrow \bigvee_{i\in\Delta} a_i \leq p \Leftrightarrow p \in V(\bigvee_{i\in\Delta} a_i)$.

3. Let $p \in V(a) \cup V(b)$. Then $a \leq p$ or $b \leq p$, so $a \wedge b \leq p$. Thus, $p \in V(a \wedge b)$ and so $V(a) \cup V(b) \subseteq V(a \wedge b)$. Since $ab \leq a \wedge b$, then $V(a \wedge b) \subseteq V(ab)$. Let $p \in V(ab)$. Then $ab \leq p$, so $a \leq p$ or $b \leq p$. Hence, $p \in V(a) \cup V(b)$. □

Any open set in $\sigma(L)$ is denoted by $\sigma(L)\backslash V(a)$ for some $a \in L$ since $V(a)$ is a closed set in $\sigma(L)$. Let $D_a = \sigma(L)\backslash V(a)$ for any $a \in L$.

Recall that an element $a \in L$ is called a nilpotent element if $a^n = 0$ for some $n \in \mathbb{Z}^+$.

Proposition 2 *Let* $D_a = \sigma(L)\backslash V(a)$ *for any* $a \in L$. *The following hold:*

1. *For any* $a, b \in L$, $D_a \cap D_b = D_{ab}$.

Suppose that L *is a* C*-lattice for 2., 3., and 4.*

2. $D_a = \emptyset \Leftrightarrow a$ *is nilpotent.*
3. $D_a = D_b \Leftrightarrow \sqrt{a} = \sqrt{b}$.
4. $\sigma(L)$ *is a quasi-compact space.*

Proof *1.* $p \in D_a \cap D_b$ for any $a, b \in L. \Leftrightarrow$ $a \nleq p$ and $b \nleq p \Leftrightarrow ab \nleq p \Leftrightarrow$ $p \in D_{ab}$ for any $a, b \in L$.

2. Let $D_a = \emptyset$. Then $V(a) = \sigma(L)$ and so $a \leq p$ for all elements $p \in \sigma(L)$. Then $a \leq \sqrt{0}$. Thus, $a^n = 0$ for any $n \in \mathbb{Z}^+$, that is, a is a nilpotent element. Conversely, if a is nilpotent, then $a^n = 0$ for some $n \in \mathbb{Z}^+$ and so $a^n \leq p$ for all elements $p \in \sigma(L)$. Thus, $a \leq p$ for all elements $p \in \sigma(L)$. Then $V(a) = \sigma(L)$, that is, $D_a = \emptyset$.

3. $D_a = D_b$ for any $a, b \in L. \Leftrightarrow V(a) = V(b) \Leftrightarrow \sqrt{a} = \sqrt{b}$ for any $a, b \in L$.

4. Let $\{G_i | i \in \nabla\}$ be an open cover of $\sigma(L)$. Without loss of generality, we may assume that $G_i = D_{c_i}$ for any $i \in \nabla$, $c_i \in L_*$. Then $\sigma(L) = \bigcup_{\substack{c_i\in L_*\\ i\in\nabla}} G_i = \bigcup_{\substack{c_i\in L_*\\ i\in\nabla}} D_{c_i} = \bigcup_{\substack{c_i\in L_*\\ i\in\nabla}} (\sigma(L)\backslash V(c_i)) = \sigma(L)\backslash \bigcap_{\substack{c_i\in L_*\\ i\in\nabla}} V(c_i) = \sigma(L)\backslash V(\bigvee_{\substack{c_i\in L_*\\ i\in\nabla}} c_i)$ so $V(\bigvee_{\substack{c_i\in L_*\\ i\in\nabla}} c_i) = \emptyset$, that is, $\bigvee_{\substack{c_i\in L_*\\ i\in\nabla}} c_i$ is not contained by any prime element. Thus, $\bigvee_{\substack{c_i\in L_*\\ i\in\nabla}} c_i = 1_L$. Since 1_L is compact element, then there is a finite subset I of ∇ such that $\bigvee_{\substack{c_i\in L_*\\ j\in I}} c_j = 1_L$. Then $V(\bigvee_{\substack{c_i\in L_*\\ j\in I}} c_j) = \emptyset$ and so $\sigma(L) = \sigma(L)\backslash V(\bigvee_{\substack{c_i\in L_*\\ j\in I}} c_j) = \sigma(L)\backslash \bigcap_{\substack{c_i\in L_*\\ j\in I}} V(c_j) = \bigcup_{\substack{c_i\in L_*\\ j\in I}} (\sigma(L)\backslash V(c_j)) = \bigcup_{\substack{c_i\in L_*\\ j\in I}} D_{c_j}$. Hence, $\sigma(L)$ is a quasi-compact space since $\sigma(L)$ is covered by finite number D_{c_i}. □

By the following proposition, we have a basis for the topology with D_a for some $a \in L_*$.

Proposition 3 *The set* $\{D_a | a \in L_*\}$ *is a basis of the topology over* $\sigma(L)$ *where* L *is a* C*-lattice.*

Proof Let G be an open set. Then there is an element $a \in L$ such that $G = \sigma(L)\backslash V(a)$. Then $G = \sigma(L)\backslash V(a) = \sigma(L)\backslash V(\bigvee_{c \in L_*} c) = \sigma(L)\backslash \bigcap_{c \in L_*} V(c) = \bigcup_{c \in L_*} (\sigma(L)\backslash V(c)) = \bigcup_{c \in L_*} D_c$ since L is a C-lattice. As G is any open set, $\{D_a | a \in L_*\}$ is a basis of the topology on $\sigma(L)$. □

Definition 4 *A topological space F is called irreducible space if $F \neq \emptyset$ and we have $F = F_1$ or $F = F_2$ for any decomposition $F = F_1 \cup F_2$ where there are nonempty closed subsets F_1 and F_2 of F. This statement is equal to $G_1 \cap G_2 \neq \emptyset$ for any two nonempty open sets G_1 and G_2 of F. A nonempty subset K of F is called irreducible if K is irreducible as a subspace of F [13].*

Lemma 5 *Let L be a C-lattice. Then $\sigma(L)$ is irreducible if and only if $\sqrt{0}$ is a prime element of L.*

Proof $(\Longrightarrow)$: Assume that $\sqrt{0}$ is not a prime element of L. Then there are some elements $a, b \in L$ with $ab \leq \sqrt{0}$ but $a, b \nleq \sqrt{0}$. Since $a \nleq \sqrt{0}$, then we get $V(a) \neq \sigma(L)$. Thus, $\sigma(L)\backslash V(a) = G_a \neq \emptyset$. Similarly, $\sigma(L)\backslash V(b) = G_b \neq \emptyset$. For open sets G_a and G_b, we have that $G_a \cap G_b = (\sigma(L)\backslash V(a)) \cap (\sigma(L)\backslash V(b)) = \sigma(L)\backslash(V(a) \cup V(b)) = \sigma(L)\backslash V(ab)$. Hence, $G_a \cap G_b = \sigma(L)\backslash V(ab) = \emptyset$ since $ab \leq \sqrt{0} = \bigwedge_{p \in \sigma(L)} p$ and $V(ab) = \sigma(L)$. Consequently, $\sigma(L)$ is not irreducible.

$(\Longleftarrow)$: Suppose that $\sqrt{0}$ is a prime element of L. Let G_1 and G_2 be two nonempty open sets of $\sigma(L)$. Let $p \in G_1$, $q \in G_2$ and $G_1 = \sigma(L)\backslash V(a)$ for any $a \in L$. Since $p \notin V(a)$ and $\sqrt{0} \leq p$, then we get $a \nleq p$ and $\sqrt{0} \notin V(a)$. Therefore, $\sqrt{0} \in \sigma(L)\backslash V(a)$, that is, $\sqrt{0} \in G_1$. Similarly, $\sqrt{0} \in G_2$. Thus, $\sqrt{0} \in G_1 \cap G_2$ and so $G_1 \cap G_2 \neq \emptyset$. Hence, $\sigma(L)$ is irreducible. □

Remark 6 *Since any singleton is irreducible, so is its closure.*

Let S be a subset of $\sigma(L)$. The meet of all elements of S will be represented by $\xi^*(S)$ and the closure of S will be represented by $cl(S)$ for the Zariski topology.

Proposition 7 *The following hold:*

1. $cl(S) = V(\xi^*(S))$.
2. *S is a closed set if and only if $V(\xi^*(S)) = S$.*
3. *S is irreducible if and only if $\xi^*(S)$ is prime.*

Proof *1.* If S is contained by a closed set $V(a)$, then $a \leq p$ for each element $p \in S$, and hence $a \leq \xi^*(S)$. As a result of this, $V(\xi^*(S)) \subseteq V(a)$, and since $S \subseteq V(\xi^*(S))$, then the smallest closed set of $\sigma(L)$ containing S is $V(\xi^*(S))$.

2. It is clear from (*1*).

3. Let us denote $\xi^*(S) = p$. Assume that S is irreducible and $ab \leq p$ for some $a, b \in L$. Then $S \subseteq V(ab) = V(a) \cup V(b)$. Since S is irreducible and $V(a)$ and $V(b)$ are closed sets, then $S \subseteq V(a)$ or $S \subseteq V(b)$. Therefore, $a \leq p$ or $b \leq p$. On the contrary, suppose that p is prime. By (*1*), we obtain $cl(S) = V(\xi^*(\{p\})) = cl(\{p\})$ as $p = \xi^*(\{p\})$. Thus, S is irreducible because a single point set is irreducible. □

Let (F, τ) be a topological space. We denote $cl(f) = cl(\{f\})$ for all $f \in F$.

Proposition 8 *Let $p \in \sigma(L)$. Then the following hold:*

1. $cl(p) = V(p)$.
2. $\{p\}$ *is a closed set if and only if* p *is a maximal element of* L.

Proof *1.* It is obvious from Proposition 7 when $Y = \{p\}$.

2. $(\Longleftarrow)$: Let $p \in \sigma(L)$ be a maximal element of L. Then $\{p\} = V(p) = cl(p)$. Hence, $\{p\}$ is a closed set. $(\Longrightarrow)$: If $\{p\}$ is a closed set, then $\{p\} = cl(p) = V(p)$. Thus, $p \in \sigma(L)$ is a maximal element of L. □

The next propositions show that $\sigma(L)$ is a T_0-space and a T_1-space under some conditions.

Proposition 9 *Let L be a C-lattice. Then $\sigma(L)$ is a T_0-space.*

Proof Let $p, q \in \sigma(L)$. We suppose that $q \nleq p$. Since L is a C-lattice, then there is an element a in L_* with $a \leq p$ and $a \nleq q$, so $p \in V(a)$ and $q \notin V(a)$. Hence, $p \notin \sigma(L) \backslash V(a)$ and $q \in \sigma(L) \backslash V(a)$. Thus, $\sigma(L)$ is a T_0-space. □

It is known that a topological space F is a T_1-space if and only if every singleton subset of F is closed.

Proposition 10 *$\sigma(L)$ is a T_1-space if and only if $\max(L) = \sigma(L)$ with $\max(L) = \{p \in \sigma(L) | p$ is a maximal element of $L\}$.*

Proof $(\Longrightarrow)$: It is clear from Proposition 8(*2*).

$(\Longleftarrow)$: Assume that $\max(L) = \sigma(L)$ with $\max(L) = \{p \in \sigma(L) | p$ is a maximal element$\}$. If $\{p\} \neq cl(\{p\})$, there is an element $q \in V(p) \backslash \{p\}$. Then $p \lneqq q$. This contradicts the hypothesis, so $\{p\} = cl(\{p\})$. Thus, $\sigma(L)$ is a T_1-space. □

Definition 11 *A space (X, τ) is an R_0-space if for every $U \in \tau$ and $x \in U$, we have $cl(x) \subseteq U$ [13].*

Theorem 12 *Let L be a C-lattice. Then the following are equivalent:*

1. $\max(L) = \sigma(L)$ with $\max(L) = \{p \in \sigma(L) | p$ is a maximal element of L$\}$.
2. $\sigma(L)$ is a T_1-space.
3. $\sigma(L)$ is an R_0-space.

Proof (1) ⇔ (2) : It is easily shown by Proposition 10.

(2) ⇔ (3) : It is clear from $T_1 = T_0 + R_0$ (see [13]). □

3. A topology on $\sigma(M)$ over M

In this section, our aim is to introduce a topology over $\sigma(M)$ the set of all prime elements of an L-module M, so we define a variety of any element K of M as the set $V^*(K) = \{P \in \sigma(M) | K \leq P\}$. Then we get the following proposition.

Proposition 13 *The following hold by means of the above definition:*

1. $V^*(0) = \sigma(M)$ *and* $V^*(1_M) = \emptyset$.
2. $\bigcap_{i\in\Delta} V^*(K_i) = V^*(\bigvee_{i\in\Delta} K_i)$ *for any index set* $\triangle$.
3. $V^*(N) \cup V^*(K) \subseteq V^*(N \wedge K)$.

Proof

1. Straightforward.
2. $P \in \bigcap_{i\in\Delta} V^*(K_i) \Leftrightarrow P \in V^*(K_i)$ for any $i \in \triangle \Leftrightarrow K_i \leq P$ for every $i \in \Delta \Leftrightarrow \bigvee_{i\in\Delta} K_i \leq P \Leftrightarrow P \in V^*(\bigvee_{i\in\Delta} K_i)$.
3. $P \in V^*(N) \cup V^*(K) \Rightarrow N \leq P$ or $K \leq P$ and so $N \wedge K \leq P \Rightarrow P \in V^*(N \wedge K)$.

□

According to the above proposition, the varieties do not satisfy the property that is to be closed under finite union, so we define a new variety of any element K of M with $V(K) = \{P \in \sigma(M) | (K :_L 1_M) \leq (P :_L 1_M)\}$. Then we have the next proposition.

Proposition 14 *Let M be an L-module. Then the following hold:*

1. $V(0_M) = \sigma(M)$ *and* $V(1_M) = \emptyset$.
2. $\bigcap_{i\in\triangle} V(K_i) = V(\bigvee_{i\in\triangle} (K_i :_L 1_M)1_M)$ *for any index set* $\triangle$.
3. $V(N) \cup V(K) = V(N \wedge K)$.

Proof

1. Is obvious.
2. $(\Longrightarrow)$: Let $P \in \bigcap_{i\in\triangle} V(K_i)$. Then $P \in V(K_i)$ for any $i \in \triangle$. We have $(K_i :_L 1_M) \leq (P :_L 1_M)$ for every $i \in \Delta$. Then $(K_i :_L 1_M)1_M \leq (P :_L 1_M)1_M$. We get $\bigvee_{i\in\triangle} (K_i :_L 1_M)1_M \leq P$. Clearly, $(\bigvee_{i\in\triangle} (K_i :_L 1_M)1_M :_L 1_M) \leq (P :_L 1_M)$. Thus, $P \in V(\bigvee_{i\in\triangle} (K_i :_L 1_M)1_M)$.

   $(\Longleftarrow)$: Let $P \in V(\bigvee_{i\in\triangle} (K_i :_L 1_M)1_M)$. Then $(\bigvee_{i\in\triangle} (K_i :_L 1_M)1_M :_L 1_M) \leq (P :_L 1_M)$. Since $\bigvee_{i\in\triangle} (K_i :_L 1_M) \leq (\bigvee_{i\in\triangle} (K_i :_L 1_M)1_M :_L 1_M) \leq (P :_L 1_M)$, we have $\bigvee_{i\in\triangle} (K_i :_L 1_M)1_M \leq P$. Hence, $(K_i :_L 1_M)1_M \leq P$ for every $i \in \triangle$. We get $(K_i :_L 1_M) \leq (P :_L 1_M)$ for every $i \in \triangle$. Then $P \in V(K_i)$ for any $i \in \triangle$. Thus, $P \in \bigcap_{i\in\triangle} V(K_i)$.
3. $(\Longrightarrow)$: Let $P \in V(N) \cup V(K)$. Then $(N :_L 1_M) \leq (P :_L 1_M)$ or $(K :_L 1_M) \leq (P :_L 1_M)$. Thus, $(N :_L 1_M) \wedge (K :_L 1_M) \leq (P :_L 1_M)$ and so $((N \wedge K) :_L 1_M) \leq (P :_L 1_M)$. Therefore, $P \in V(N \wedge K)$.

   $(\Longleftarrow)$: Let $P \in V(N \wedge K)$. Then $(N \wedge K :_L 1_M)) \leq (P :_L 1_M)$ and we get $(N :_L 1_M) \wedge (K :_L 1_M) \leq (P :_L 1_M)$. Hence, $(N :_L 1_M) \leq (P :_L 1_M)$ or $(K :_L 1_M) \leq (P :_L 1_M)$ as $(P :_L 1_M)$ is a prime of element L. Thus, $P \in V(N) \cup V(K)$.

□

Proposition 13, it is clear that there is a topology, denoted by τ^*, over $\sigma(M)$, called the quasi-Zariski topology if and only if the family of all closed sets $\zeta^*(M) = \{V^*(N) | N \leq M\}$ is closed under finite union. If $\zeta^*(M)$ induces τ^*, then the L-module M is called a top L-module. See [11] for more information about top modules.

Lastly, we concentrate on $\zeta(M) = \{V(K) | K \leq M\}$, the collection of all closed sets. By Proposition 14, it is obvious that there is always a topology on $\sigma(M)$, denoted by τ, for any L-module M. The topology τ is said to be the Zariski topology over $\sigma(M)$. In this study, we especially study the structure of it.

Proposition 15 *Let M be an L-module. Let N and K be elements of M.*

1. If $(N :_L 1_M) = (K :_L 1_M)$, then $V(N) = V(K)$. Also, the converse holds when N and K are prime.

2. $V(K) = V((K :_L 1_M)1_M) = V^*((K :_L 1_M)1_M)$. Notably, $V(a1_M) = V^*(a1_M)$ for any $a \in L$.

Proof 1. If $(N :_L 1_M) = (K :_L 1_M)$, then it is obvious that $V(N) = V(K)$. Conversely, we assume that N and K are prime elements in M. Since $(N :_L 1_M) \leq (K :_L 1_M)$ and $(K :_L 1_M) \leq (N :_L 1_M)$, consequently we get $(N :_L 1_M) = (K :_L 1_M)$.

2. Let $P \in V(K)$. Then $(K :_L 1_M) \leq (P :_L 1_M)$, so $(K :_L 1_M)1_M \leq (P :_L 1_M)1_M \leq P$. As $((K :_L 1_M)1_M :_L 1_M) \leq (P :_L 1_M)$, we have $P \in V((K :_L 1_M)1_M)$. Conversely, let $P \in V((K :_L 1_M)1_M)$. Then $((K :_L 1_M)1_M :_L 1_M) \leq (P :_L 1_M)$, so $(K :_L 1_M) \leq (P :_L 1_M)$ and $P \in V(K)$. Thus, $V(K) = V((K :_L 1_M)1_M)$.

Now let us confirm that $V((K :_L 1_M)1_M) = V^*((K :_L 1_M)1_M)$. Let $P \in V((K :_L 1_M)1_M)$. Then $((K :_L 1_M)1_M :_L 1_M) \leq (P :_L 1_M)$ and so $(K :_L 1_M) \leq (P :_L 1_M)$. Hence, $(K :_L 1_M)1_M \leq P$. Therefore, $P \in V^*((K :_L 1_M)1_M)$. Conversely, let $P \in V^*((K :_L 1_M)1_M)$. Then $(K :_L 1_M)1_M \leq P$, so $((K :_L 1_M)1_M :_L 1_M) \leq (P :_L 1_M)$, and thus $P \in V((K :_L 1_M)1_M)$. Hence, $V((K :_L 1_M)1_M) = V^*((K :_L 1_M)1_M)$.

Let $P \in V(a1_M)$. Then $(a1_M :_L 1_M) \leq (P :_L 1_M)$, so $a1_M \leq P$. Hence, $P \in V^*(a1_M)$. Conversely, let $P \in V^*(a1_M)$. Then $a1_M \leq P$ and so $(a1_M :_L 1_M) \leq (P :_L 1_M)$. Thus, $P \in V(a1_M)$. Hence, $V(a1_M) = V^*(a1_M)$ for any $a \in L$. □

Definition 16 *An L-module M is said to be a multiplication lattice module if there is an element $a \in L$ with $K = a1_M$ for each element $K \in M$ [6].*

It is true that M is a multiplication lattice module if and only if $K = (K :_L 1_M)1_M$ for any element K of M (see [6, Proposition 3]).

Theorem 17 *A multiplication L-module M is a top L-module where $\tau^* = \tau$.*

Proof Let N and K be any two elements in M and $P \in V^*(N \wedge K)$. Then $N \wedge K \leq P \Rightarrow (N \wedge K :_L 1_M) \leq (P :_L 1_M) \Rightarrow (N :_L 1_M) \wedge (K :_L 1_M) \leq (P :_L 1_M)$ and so $(N :_L 1_M) \leq (P :_L 1_M)$ or $(K :_L 1_M) \leq (P :_L 1_M)$ since $(P :_L 1_M)$ is a prime element in L. Then $(N :_L 1_M)1_M \leq (P :_L 1_M)1_M$ or $(K :_L 1_M)1_M \leq (P :_L 1_M)1_M$, and thus $N \leq P$ or $K \leq P$ since M is multiplication, and so $P \in V^*(N) \cup V^*(K)$. □

In the rest of this paper, we suppose that $\sigma(M)$ is nonempty unless indicated otherwise. Accordingly, the Zariski topology can be applied to $\sigma(M)$ for any L-module M. The set $\sigma(L / Ann(1_M))$ will be represented by $\sigma(\overline{L})$. Besides, the cardinality of any subset S of $\sigma(M)$ will be represented by $|S|$. Let $\psi : \sigma(M) \longrightarrow \sigma(L\diagup Ann(1_M))$ be a map defined by $\psi(P) = \overline{(P :_L 1_M)}$ for each element $P \in \sigma(M)$. The map is said to be the natural map of $\sigma(M)$.

Proposition 18 *The natural map* $\psi : \sigma(M) \longrightarrow \sigma(L\diagup Ann(1_M))$ *is continuous; particularly,* $\psi^{-1}(V^{\overline{L}}(\overline{a})) = V(a1_M)$ *for every element* a *in* L *where* $a \geq Ann(1_M)$.

Proof Let S be a closed set in $\sigma(\overline{L})$. Then we get $S = V^{\overline{L}}(\overline{a})$ for any element $\overline{a}$ of $\overline{L}$. Then for any $P \in \psi^{-1}(S)$, $\psi(P) = \overline{(P :_L 1_M)} \in V^{\overline{L}}(\overline{a})$ if and only if $\overline{a} \leq \overline{(P :_L 1_M)}$, if and only if $a \leq (P :_L 1_M)$, if and only if $a1_M \leq P$, if and only if $(a1_M :_L 1_M) \leq (P :_L 1_M)$, if and only if $P \in V(a1_M)$. Thus, $\psi^{-1}(S) = \psi^{-1}(V^{\overline{L}}(\overline{a})) = V(a1_M)$ is a closed set of $\sigma(M)$, so ψ is continuous. □

Proposition 19 *The following are equivalent for any* L*-module* M *and* $P, Q \in \sigma(M)$.

1. $\psi : \sigma(M) \longrightarrow \sigma(L\diagup Ann(1_M))$ *is an injective map.*
2. $V(P) = V(Q) \Rightarrow P = Q$.
3. $|\sigma_p(M)| \leq 1$ *for every* $p \in \sigma(L)$ *where* $\sigma_p(M) = \{P \in \sigma(M) | (P :_L 1_M) = p$ *where* $p \in \sigma(L)\}$.

Proof (1) $\Rightarrow$ (2) : Let $\psi : \sigma(M) \longrightarrow \sigma(L\diagup Ann(1_M))$ be injective. We assume that $V(P) = V(Q)$. Then $\overline{(P :_L 1_M)} = \overline{(Q :_L 1_M)}$, so $\psi(P) = \psi(Q)$. Thus, $P = Q$.

(2) $\Rightarrow$ (3) : If $(P :_L 1_M) = (Q :_L 1_M) = p$, then $V(P) = V(Q)$. Thus, $P = Q$.

(3) $\Rightarrow$ (1) : Let $\psi(P) = \psi(Q)$. Then $\overline{(P :_L 1_M)} = \overline{(Q :_L 1_M)} = cl(p)$. Therefore, $(P :_L 1_M) = (Q :_L 1_M) = p$. Thus, $P = Q$. □

Theorem 20 *Let* $\psi : \sigma(M) \longrightarrow \sigma(L\diagup Ann(1_M))$ *be the natural map of* $\sigma(M)$ *for an* L*-module* M. *If* ψ *is surjective, then it is a closed and open map.*

Proof By Proposition 18, $\psi : \sigma(M) \longrightarrow \sigma(L\diagup Ann(1_M))$ is continuous map such that $\psi^{-1}(V^{\overline{L}}(\overline{a})) = V(a1_M)$ for every element a in L with $a \geq Ann(1_M)$. Then for any $K \leq 1_M$, $\psi^{-1}(V^{\overline{L}}(\overline{(K :_L 1_M)})) = V((K :_L 1_M)1_M) = V(K)$ by Proposition 15. Since ψ is surjective, $\psi(V(K)) = V^{\overline{L}}(\overline{(K :_L 1_M)})$. Accordingly, $\psi(\sigma(M) \backslash V(K)) = \psi(\psi^{-1}(\sigma(L\diagup Ann(1_M))) \backslash \psi^{-1}(V^{\overline{L}}(\overline{(K :_L 1_M)}))) = \sigma(L\diagup Ann(1_M)) \backslash V^{\overline{L}}(\overline{(K :_L 1_M)})$. □

We know that the set $D_a = \sigma(L) \backslash V(a)$ is open set in $\sigma(L)$ for any element a of L and if L is a C-lattice, the set $\{D_a | a \in L_*\}$ is a basis of the topology over $\sigma(L)$.

We define $X_a = \sigma(M) \backslash V(a1_M)$ for any $a \in L$. It is obvious that every X_a is an open set in $\sigma(M)$.

Proposition 21 *Let* M *be an* L*-module and* $\psi : \sigma(M) \longrightarrow \sigma(L\diagup Ann(1_M))$ *be the natural map. Then the following hold:*

1. $\psi^{-1}(D_{\overline{a}}) = X_a$.
2. $\psi(X_a) \subseteq D_{\overline{a}}$.
3. *If* ψ *is surjective, then* $\psi(X_a) = D_{\overline{a}}$.

4. $X_{ab} = X_a \cap X_b$ *for any elements* a, b *of* L*.*

Proof 1. $\psi^{-1}(D_{\overline{a}}) = \psi^{-1}(\sigma(L/Ann(1_M))\backslash V^{\overline{L}}(\overline{a})) = \sigma(M)\backslash\psi^{-1}(V^{\overline{L}}(\overline{a})) = \sigma(M)\backslash V(a1_M) = X_a$ by Proposition 18.

2. $\psi(X_a) = \psi(\psi^{-1}(D_{\overline{a}})) \subseteq D_{\overline{a}}$.

3. It is clear from (2).

4. $X_{ab} = \psi^{-1}(D_{\overline{ab}}) = \psi^{-1}(D_{\overline{a}} \cap D_{\overline{b}}) = \psi^{-1}(D_{\overline{a}}) \cap \psi^{-1}(D_{\overline{b}}) = X_a \cap X_b$ for any elements a, b of L by Proposition 2. □

In the next proposition, it is proved that there is a basis for the Zariski topology over $\sigma(M)$ with X_a for some $a \in L_*$.

Theorem 22 *The set* $\{X_a | a \in L_*\}$ *is a basis of the Zariski topology over* $\sigma(M)$ *for an* L*-module* M *where* L *is a* C*-lattice.*

Proof Assume that G is an open set in $\sigma(M)$. For some element a of L, $G = \sigma(M)\backslash V(a1_M)$ by Proposition 15. Then $G = \sigma(M)\backslash V(a1_M) = \sigma(M)\backslash V((\bigvee_{a_i \in L_*} a_i)1_M) = \sigma(M)\backslash V(\bigvee_{a_i \in L_*} a_i 1_M) = \sigma(M)\backslash \bigcap_{a_i \in L_*} V(a_i 1_M) = \bigcup_{a_i \in L_*} (\sigma(M)\backslash V(a_i 1_M)) = \bigcup_{a_i \in L_*} X_{a_i}$. Hence, the set $\{X_a | a \in L_*\}$ is a basis of the Zariski topology over $\sigma(M)$. □

Theorem 23 *Let* L *be a* C*-lattice and* M *be an* L*-module. If the natural map of* $\sigma(M)$ *is surjective, then* $\sigma(M)$ *is quasi-compact.*

Proof Since the set $\{X_a | a \in L_*\}$ is a basis of the Zariski topology over $\sigma(M)$, then $\sigma(M) = \bigcup_{\substack{a_i \in L_* \\ i \in \Delta}} X_{a_i}$ for any open cover of $\sigma(M)$. Hence, $\sigma(L/Ann(1_M)) = \psi(\sigma(M)) = \psi(\bigcup_{\substack{a_i \in L_* \\ i \in \Delta}} X_{a_i}) = \bigcup_{\substack{a_i \in L_* \\ i \in \Delta}} \psi(X_{a_i}) = \bigcup_{\substack{a_i \in L_* \\ i \in \Delta}} D_{\overline{a_i}}$. It follows that there exists a finite subset Δ^* of Δ such that $\sigma(L/Ann(1_M)) = \bigcup_{\substack{a_i \in L_* \\ j \in \Delta^*}} D_{\overline{a_j}}$ as $\sigma(L/Ann(1_M))$ is quasi-compact by Proposition 2. As a result, we get $\sigma(M) = \bigcup_{\substack{a_i \in L_* \\ j \in \Delta^*}} X_{a_j}$ by Proposition 21. □

Let S be a subset of $\sigma(M)$ for an L-module M. The meet of all elements in S will be represented by $\xi(S)$ and the closure of S in $\sigma(M)$ will be represented by $cl(S)$ for the Zariski topology over $\sigma(M)$.

Proposition 24 *Let* M *be an* L*-module and* $S \subseteq \sigma(M)$*. Then* $V(\xi(S)) = cl(S)$*. Hence,* S *is closed if and only if* $V(\xi(S)) = S$*.*

Proof Let $P \in S$. Then $\xi(S) \leq P \Rightarrow (\xi(S) :_L 1_M) \leq (P :_L 1_M)$ and so $P \in V(\xi(S))$. Thus, $S \subseteq V(\xi(S))$. Then $cl(S) \subseteq V(\xi(S))$. Now, let us indicate that $V(\xi(S))$ is the smallest subset of $\sigma(M)$ containing S. Let $V(K)$ be a closed subset of $\sigma(M)$ where $S \subseteq V(K)$. For every $P \in S$, it is true that $(K :_L 1_M) \leq (P :_L 1_M)$ and so $(K :_L 1_M) \leq (\xi(S) :_L 1_M)$. Hence, we have $(K :_L 1_M) \leq (\xi(S) :_L 1_M) \leq (Q :_L 1_M)$ for each $Q \in V(\xi(S))$, namely $V(\xi(S)) \subseteq V(K)$. Therefore, $V(\xi(S))$ is the smallest closed set of $\sigma(M)$, which contains S. Then $V(\xi(S)) = cl(S)$. □

Proposition 25 *Let M be an L-module and $P \in \sigma(M)$. Then the following hold:*

1. *$cl(\{P\}) = V(P)$.*

2. *$Q \in cl(\{P\})$ for any $Q \in \sigma(M)$ iff $(P :_L 1_M) \leq (Q :_L 1_M)$ iff $V(Q) \subseteq V(P)$.*

3. *If M is a multiplication L-module, then the set $\{P\}$ is a closed set of $\sigma(M)$ if and only if P is a maximal element of M.*

Proof 1. $cl(\{P\}) = V(\xi(\{P\})) = V(P)$.

2. Clear.

3. Suppose that the set $\{P\}$ is closed in $\sigma(M)$. Then $V(P) = \{P\}$ by (1). Since every prime element J that satisfies $J \geq P$ must be in $V(P) = \{P\}$, then P is a maximal element of M. On the contrary, let $Q \in cl(\{P\})$. Then $(P :_L 1_M) \leq (Q :_L 1_M)$ by (2). As M is a multiplication L-module, then $P = (P :_L 1_M)1_M \leq (Q :_L 1_M)1_M = Q$ and so $P = Q$. Thus, $cl(\{P\}) = \{P\}$. □

Corollary 26 *$V(P)$ is an irreducible closed subset of $\sigma(M)$ for any $P \in \sigma(M)$.*

Proof It is clear from Proposition 25 and Remark 6. □

Proposition 27 *Let S be a subset of $\sigma(M)$ for an L-module M. If $\xi(S)$ is a prime element in M, then S is irreducible. On the contrary, if S is irreducible, then $\Im = \{(P :_L 1_M) | P \in S\}$ is irreducible such that $\xi^*(\Im) = (\xi(S) : 1_M)$ is a prime element in L.*

Proof Assume that $\xi(S) = Q$ is a prime element of M. It is clear that $V(Q) = V(\xi(S)) = cl(S)$ is irreducible from Corollary 26. Thus, S is irreducible. On the contrary, let S be irreducible. As the natural map ψ of $\sigma(M)$ is continuous, the image $\psi(S) = S^*$ of S is an irreducible subset of $\sigma(\overline{L})$. Consequently, by Proposition 7(3), $\xi^*(S^*) = \overline{(\xi(S) :_L 1_M)}$ is a prime element of $\overline{L}$. Hence, $\xi^*(\Im) = (\xi(S) :_L 1_M)$ is a prime element of L and so $\Im$ is an irreducible subset of $\sigma(L)$. □

Theorem 28 *Let S be a subset of $\sigma(M)$ for an L-module M and the natural map $\psi : \sigma(M) \longrightarrow \sigma(L/Ann(1_M))$ be surjective. Then S is an irreducible closed subset if and only if $S = V(P)$ for any $P \in \sigma(M)$.*

Proof Let $S = V(P)$ for any $P \in \sigma(M)$. Then $S = V(P)$ is an irreducible closed subset of $\sigma(M)$ from Corollary 26. In contrast, if S is an irreducible closed subset of $\sigma(M)$, then $S = V(K)$ for some element K of M such that $(\xi(V(K)) :_L 1_M) = (\xi(S) :_L 1_M)$ is a prime element of L by Proposition 27. By surjectivity of ψ, there is $P \in \sigma(M)$ with $(\xi(V(K)) :_L 1_M) = (P :_L 1_M)$, so $V(\xi(V(K))) = V(P)$ by Proposition 15(1). Thus, $V(K) = V(P)$ as $V(K)$ is closed by Proposition 24. □

Theorem 29 *The following are equivalent for an L-module M:*

1. $\sigma(M)$ is a T_0-space,

2. The natural map ψ is injective,

3. $V(P) = V(Q) \Rightarrow P = Q$ for all $P, Q \in \sigma(M)$,

4. $|\sigma_p(M)| \leq 1$ for every $p \in \sigma(L)$.

Proof (1) $\Leftrightarrow$ (3): It is obvious from Proposition 25 and the statement that a topological space is T_0 if and only if the closures of different points are distinct.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): It is obvious from Proposition 19. □

Proposition 30 *Let M be a multiplication L-module. Then $\sigma(M)$ is a T_1-space if and only if $Max(M) = \sigma(M)$ where $Max(M) = \{P | P$ is a maximal element of $M\}$.*

Proof Is obvious from Proposition 25(3). □

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