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Some results on the $\mathcal{P}_{v,2n}$, $\mathcal{K}_{v,n}$, and $\mathcal{H}_{v,n}$-integral transforms

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Abstract: In this paper, the authors consider the $\mathcal{P}_{v,2n}$-transform, the $\mathcal{G}_n$-transform, and the $\mathcal{K}_{v,n}$-transform as generalizations of the Widder potential transform, the Glasser transform, and the $\mathcal{K}_v$-transform, respectively. Many identities involving these transforms are given. A number of new Parseval–Goldstein type identities are obtained for these and many other well-known integral transforms. Some useful corollaries for evaluating infinite integrals of special functions are presented. Illustrative examples are given for the results.

Key words: Laplace transforms, $\mathcal{L}_{2n}$-transforms, Widder potential transforms, Glasser transforms, Hankel transforms, $\mathcal{K}_v$ transforms, $\mathcal{P}_{v,2n}$-transforms, $\mathcal{G}_n$-transforms, Parseval-Goldstein type theorems

1. Introduction, definitions, and preliminaries

The Laplace-type integral transform called the $\mathcal{L}_2$-transform was introduced by Yürekli and Sadek [19] and is denoted as follows:

$$\mathcal{L}_2\{f(x);y\} = \int_0^\infty x\exp(-x^2y^2)f(x)dx. \tag{1.1}$$

In [5] Dernek and Aylıkçı introduced the $\mathcal{L}_n$ $(n \in \mathbb{N})$ and $\mathcal{L}_{2n}$ transforms as generalizations of the Laplace transform, respectively:

$$\mathcal{L}_n\{f(x);y\} = \int_0^\infty x^{n-1}\exp(-x^ny^n)f(x)dx, \tag{1.2}$$

$$\mathcal{L}_{2n}\{f(x);y\} = \int_0^\infty x^{2n-1}\exp(-x^{2n}y^{2n})f(x)dx. \tag{1.3}$$

The $\mathcal{L}_n$-transform and the $\mathcal{L}_{2n}$-transform are related to the Laplace transform with

$$\mathcal{L}_n\{f(x);y\} = \frac{1}{n}\mathcal{L}\{f(x^{1/n});y^n\}, \tag{1.4}$$

$$\mathcal{L}_{2n}\{f(x);y\} = \frac{1}{2n}\mathcal{L}\{f(x^{1/2n});y^{2n}\}. \tag{1.5}$$

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The Widder transform was introduced by Widder [13, 17] as follows:

$$\mathcal{P}\{f(x);y\} = \int_0^\infty \frac{xf(x)}{x^2+y^2}dx. \tag{1.6}$$

Glasser [10] defined the Glasser transform as:

$$\mathcal{G}\{f(x);y\} = \int_0^\infty \frac{f(x)}{\sqrt{x^2+y^2}}dx. \tag{1.7}$$

The $\mathcal{P}_{v,2}$-transform,

$$\mathcal{P}_{v,2}\{f(x);y\} = \int_0^\infty \frac{xf(x)}{(x^2+y^2)^v}dx, \tag{1.8}$$

was introduced by Dernek et al. [6] as a generalization of the Widder-potential transform and the Glasser transform. If we put $v = 1$ and $v = \frac{1}{2}$ in (1.8), we obtain the Widder potential transform (1.6) and the Glasser transform (1.7), respectively.
The Hankel transform is defined by

$$\mathcal{H}_v\{f(x);y\} = \int_0^\infty \sqrt{xy}J_v\{xy\}f(x)dx, \tag{1.9}$$

where $J_v(x)$ is the Bessel function of the first kind of order v.
The $\mathcal{K}$-transform is defined by

$$\mathcal{K}_v\{f(x);y\} = \int_0^\infty \sqrt{xy}K_v(xy)f(x)dx, \tag{1.10}$$

where K_v is the Bessel function of the second kind of order v.
In this article, we introduce new generalizations of the Widder potential transform and the Glasser transform as follows:

$$\mathcal{P}_{v,2n}\{f(x);y\} = \int_0^\infty \frac{x^{2n-1}f(x)}{(x^{2n}+y^{2n})^v}dx \tag{1.11}$$

and

$$\mathcal{G}_n\{f(x);y\} = \int_0^\infty \frac{f(x)}{\sqrt{x^{2n}+y^{2n}}}dx, n \in \mathbb{N}, \tag{1.12}$$

respectively. If we put $v = 1$ in definition (1.11), we obtain

$$\mathcal{P}_{1,2n}\{f(x);y\} = \mathcal{P}_{2n}\{f(x);y\}. \tag{1.13}$$

$\mathcal{P}_{2n}\{f(x);y\}$ was defined in [5] by

$$\mathcal{P}_{2n}\{f(x);y\} = \int_0^\infty \frac{x^{2n-1}f(x)}{x^{2n}+y^{2n}}dx. \tag{1.14}$$

In this article the $\mathcal{K}_{v,n}$-transform is defined by

$$\mathcal{K}_{v,n}\{f(x);y\} = \int_0^\infty x^{n-1}(x^n y^n)^{1/2}K_v(x^n y^n)f(x)dx, \tag{1.15}$$

where $K_v(x^n y^n)$ is the Macdonald function. It is also known as the Bessel function of the second kind.
The $\mathcal{K}_{v,n}$-transform is related to the $\mathcal{K}_v$-transform and the $\mathcal{K}_{v,2}$ [7, p. 329, Eq. (20)] transform with the identities respectively

$$n\mathcal{K}_{v,n}\{f(x);y\} = \mathcal{K}_v\{f(x^{1/n});y^n\}, \tag{1.16}$$

$$n\mathcal{K}_{v,n}\{f(x);y\} = 2\mathcal{K}_{v,2}\{f(x^{2/n});y^{n/2}\}. \tag{1.17}$$

The generalized Hankel transform is defined by

$$\mathcal{H}_{v,n}\{f(x);y\} = \int_0^\infty x^{n-1}(x^n y^n)^{1/2}J_v(x^n y^n)f(x)dx, \tag{1.18}$$

which is related to the Hankel transform, the $\mathcal{H}_{v,2}$-transform [7, p. 329, Eq. (19)] with the following identities:

$$n\mathcal{H}_{v,n}\{f(x);y\} = \mathcal{H}_v\{f(x^{1/n});y^n\}, \tag{1.19}$$

$$n\mathcal{H}_{v,n}\{f(x);y\} = 2\mathcal{H}_{v,2}\{f(x^{2/n});y^{n/2}\}. \tag{1.20}$$

The Fourier sine-transform and Fourier cosine-transform are defined as, respectively,

$$\mathcal{F}_s\{f(x);y\} = \int_0^\infty f(x)\sin(xy)dx, \tag{1.21}$$

$$\mathcal{F}_c\{f(x);y\} = \int_0^\infty f(x)\cos(xy)dx. \tag{1.22}$$

We define the $\mathcal{F}_{s,n}$-transform and the $\mathcal{F}_{c,n}$-transform as follows:

$$\mathcal{F}_{s,n}\{f(x);y\} = \int_0^\infty x^{n-1}\sin(x^n y^n)f(x)dx, \tag{1.23}$$

$$\mathcal{F}_{c,n}\{f(x);y\} = \int_0^\infty x^{n-1}\cos(x^n y^n)f(x)dx,\ n\in\mathbb{N}, \tag{1.24}$$

which are related to the Fourier sine-transform and Fourier cosine-transform by means of the following relations:

$$n\mathcal{F}_{s,n}\{f(x);y\} = \mathcal{F}_s\{f(x^{1/n});y^n\}, \tag{1.25}$$

$$n\mathcal{F}_{c,n}\{f(x),y\} = \mathcal{F}_c\{f(x^{1/n});y^n\}. \tag{1.26}$$

Dernek et al. [6] gave the Parseval–Goldstein type theorem,

$$\int_0^\infty y^{2v-1}\mathcal{L}_2\{f(x);y\}\mathcal{L}_2\{g(u);y\}dy = \frac{\Gamma(v)}{2}\int_0^\infty xf(x)\mathcal{P}_{v,2}\{g(u);x\}dx, \tag{1.27}$$

for the $\mathcal{L}_2$-transform and the $\mathcal{P}_{v,2}$-transform. Various Parseval–Goldstein type identities were given (for example in [5–7,16,18]) for the $\mathcal{L}_2$-transform and the $\mathcal{L}_{2n}$-transform and the Widder potential transform.

In Section 2 of this paper, we show that the $\mathcal{P}_{v,2n}$-transform (1.11) is an iteration of the $\mathcal{L}_{2n}$-transform (1.3). The main theorem is shown to yield new identities for the integral transforms introduced above. In Section 3, some illustrative examples are given.

2. The main theorem

Lemma 2.1. *The identity*

$$\mathcal{L}_{2n}\{u^{2n(v-1)}\mathcal{L}_{2n}\{g(x);u\};y\} = \frac{\Gamma(v)}{2n}\mathcal{P}_{v,2n}\{g(x);y\} \tag{2.1}$$

holds true, provided that $Re(v) > 0$ and the integrals involved converge absolutely.

Proof Using definition (1.3) of the $\mathcal{L}_{2n}$-transform and then changing the order of integration, which is permissible by absolute convergence of the integrals involved, we get

$$\mathcal{L}_{2n}\{u^{2n(v-1)}\mathcal{L}_{2n}\{g(x);u\};y\} = \int_0^\infty u^{2nv-1}\exp(-u^{2n}y^{2n})\int_0^\infty \frac{x^{2n-1}g(x)}{\exp(x^{2n}u^{2n})}dxdu$$

$$= \int_0^\infty x^{2n-1}g(x)\mathcal{L}_{2n}\{u^{2n(v-1)};(x^{2n}+y^{2n})^{1/2n}\}. \tag{2.2}$$

Utilizing the relation (1.5) and the formula [8, p. 133, Entry (3)], we arrive at identity (2.1).

Corollary 2.1. *We have*

$$2n\mathcal{L}_{2n}\{\mathcal{L}_{2n}\{f(x);u\};y\} = \mathcal{P}_{2n}\{g(x);y\}. \tag{2.3}$$

Proof Setting $v = 1$ in (2.1) of Lemma 2.1 and using relation (1.13), we obtain identity (2.3). (2.3) was previously obtained in [5].

Corollary 2.2. *We have for $-1 < Re(\mu) < Re(2v - \frac{1}{2})$*

$$\mathcal{P}_{v,2n}\{x^{n\mu}J_\mu(z^nx^n);y\} = \frac{z^{n(v-1)}}{n\Gamma(v)2^{v-1}}y^{n(\mu-v+1)}K_{v-\mu-1}(z^ny^n). \tag{2.4}$$

Proof Substituting

$$g(x) = x^{n\mu} J_\mu(z^n x^n) \tag{2.5}$$

in (2.1) of Lemma 2.1 and using relation (1.5), then the known formula [8, p. 185, Entry (30)], we get

$$\mathcal{L}_{2n}\{x^{n\mu} J_\mu(z^n x^n); u\} = \frac{1}{2n}\left(\frac{z^n}{2}\right)^{\mu} (u^{2n})^{-\mu-1} \exp(-z^{2n}/4u^{2n}), \tag{2.6}$$

where $Re(\mu) > -1$. Multiplying both sides of equation (2.6) with $u^{2n(v-1)}$, then applying the $\mathcal{L}_{2n}$-transform and using relation (1.5) once again and the well-known formula [8, p. 146, Entry (29)], we have

$$\mathcal{L}_{2n}\{u^{2n(v-\mu-2)} \exp(-z^{2n}/4u^{2n}); y\} = \frac{1}{n}\frac{z^{n(v-\mu-1)}}{2^{v-\mu-1} y^{n(v-\mu-1)}} K_{v-\mu-1}(z^n y^n). \tag{2.7}$$

Substituting (2.7) into identity (2.1) of Lemma 2.1, we obtain assertion (2.4).

Remark 2.1. *Using definition (1.11) of the $\mathcal{P}_{v,2n}$-transform, we obtain the following relation from formula (2.4) of Corollary 2.2:*

$$\int_0^\infty \frac{x^{n(\mu+2)-1} J_\mu(z^n x^n)}{(x^{2n}+y^{2n})^v} dx = \frac{z^{n(v-1)}}{n\Gamma(v) 2^{v-1}} y^{n(\mu-v+1)} K_{v-\mu-1}(z^n y^n). \tag{2.8}$$

Remark 2.2. *Setting $v = \mu + \frac{3}{2}$ in (2.8) and using the formula*

$$K_{1/2}(x) = K_{-1/2}(x) = \left(\frac{\pi}{2x}\right)^{1/2} \exp(-x), \tag{2.9}$$

we obtain

$$\int_0^\infty \frac{x^{n(\mu+2)-1} J_\mu(z^n x^n)}{(x^{2n}+y^{2n})^{\mu+3/2}} dx = \frac{\sqrt{\pi} z^{n\mu} y^{-n}}{n\Gamma(\mu+3/2) 2^{\mu+1}} \exp(-z^n y^n). \tag{2.10}$$

Similarly, setting $v = \mu + \frac{1}{2}$ in (2.8) and using formula (2.9), we get

$$\int_0^\infty \frac{x^{n(\mu+2)-1} J_\mu(z^n x^n)}{(x^{2n}+y^{2n})^{\mu+1/2}} dx = \frac{\sqrt{\pi} z^{n(\mu-1)}}{n\Gamma(\mu+1/2) 2^{\mu}} \exp(-z^n y^n), \tag{2.11}$$

where $Re(\mu) > -\frac{1}{2}$.

Remark 2.3. *If we set $n = 1$ in (2.10) and (2.11), we obtain the known formulas [12, p. 686, Entry 6.565 (3)] and [12, p. 686, Entry 6.565 (2)].*

Corollary 2.3. *We have the following identities:*

$$\mathcal{G}_n\{x^{n(\mu+2)-1} J_\mu(z^n x^n); y\} = \left(\frac{2}{\pi z^n}\right)^{1/2} \frac{y^{n(\mu+1/2)}}{n} K_{\mu+1/2}(z^n y^n), \tag{2.12}$$

where $-1 < Re(\mu) < \frac{1}{2}$,

$$\mathcal{G}_n\{x^{n(\mu+2)-1} J_\mu(z^n x^n); y\} = \mathcal{P}_{1/2,2n}\{x^{n\mu} J_\mu(z^n x^n); y\} \tag{2.13}$$

and

$$\mathcal{P}_{2n}\{x^{n\mu}J_\mu(z^n x^n);y\} = \frac{1}{n}y^{n\mu}K_\mu(z^n y^n). \tag{2.14}$$

Theorem 2.1. *If the conditions stated in Lemma 2.1 are satisfied, then the following Parseval–Goldstein type relations hold true:*

$$\int_0^\infty y^{2nv-1}\mathcal{L}_{2n}\{f(x),y\}\mathcal{L}_{2n}\{g(u);y\}dy = \frac{\Gamma(v)}{2n}\int_0^\infty x^{2n-1}f(x)\mathcal{P}_{v,2n}\{g(u);x\}dx, \tag{2.15}$$

$$\int_0^\infty y^{2nv-1}\mathcal{L}_{2n}\{f(x),y\}\mathcal{L}_{2n}\{g(u);y\}dy = \frac{\Gamma(v)}{2n}\int_0^\infty u^{2n-1}g(u)\mathcal{P}_{v,2n}\{f(x);u\}du, \tag{2.16}$$

and

$$\int_0^\infty x^{2n-1}f(x)\mathcal{P}_{v,2n}\{g(u);x\}dx = \int_0^\infty u^{2n-1}g(u)\mathcal{P}_{v,2n}\{f(x);u\}du. \tag{2.17}$$

Proof We only give a proof of (2.15), since the proof of (2.16) is similar. Assertion (2.17) follows from the identities (2.15) and (2.16).

Using definition (1.3) of the $\mathcal{L}_{2n}$-transform twice and changing the order of integration, which is permissible by absolute convergence of the integrals involved, we find

$$\int_0^\infty y^{2nv-1}\mathcal{L}_{2n}\{f(x),y\}\mathcal{L}_{2n}\{g(u);y\}dy$$

$$= \int_0^\infty y^{2nv-1}\mathcal{L}_{2n}\{g(u);y\}\int_0^\infty x^{2n-1}\exp(-x^{2n}y^{2n})f(x)dxdy$$

$$= \int_0^\infty x^{2n-1}f(x)\mathcal{L}_{2n}\{y^{2n(v-1)}\mathcal{L}_{2n}\{g(u);y\};x\}dx. \tag{2.18}$$

Now, using identity (2.1) of Lemma 2.1, we arrive at assertion (2.15).

Corollary 2.4. *If the conditions stated in Lemma 2.1 are satisfied, then the Parseval–Goldstein type relations,*

$$\int_0^\infty y^{2n-1}\mathcal{L}_{2n}\{f(x),y\}\mathcal{L}_{2n}\{g(u);y\}dy = \frac{1}{2n}\int_0^\infty \frac{f(x)}{x^{1-2n}}\mathcal{P}_{2n}\{g(u);x\}dx, \tag{2.19}$$

$$\int_0^\infty y^{2n-1}\mathcal{L}_{2n}\{f(x),y\}\mathcal{L}_{2n}\{g(u);y\}dy = \frac{1}{2n}\int_0^\infty u^{2n-1}g(u)\mathcal{P}_{2n}\{f(x);u\}du, \tag{2.20}$$

and

$$\int_0^\infty x^{2n-1} f(x) \mathcal{P}_{2n}\{g(u); x\} dx = \int_0^\infty u^{2n-1} g(u) \mathcal{P}_{2n}\{f(x); u\} du, \tag{2.21}$$

hold true.

Proof Setting $v = 1$ in identities (2.15)–(2.17) of Theorem 2.1 and using relations (1.13) and (2.4), we get assertion (2.19). The proof of (2.20) is similar and (2.21) follows from identities (2.19) and (2.20).

Corollary 2.5. *If the integrals involved converge absolutely, then we have*

$$\mathcal{L}_{2n}\{y^{2n(\mu-v)} \mathcal{L}_{2n}\{f(x); \frac{1}{2^{1/n}y}\}; z\} = \frac{z^{n(v-\mu-3/2)}}{2^{\mu-v+1}} \mathcal{K}_{v-\mu-1,n}\{x^{n(\mu-v+3/2)} f(x); z\}, \tag{2.22}$$

$$\mathcal{L}_{2n}\{y^{2n(\mu-v)} \mathcal{L}_{2n}\{f(x); \frac{1}{2^{1/n}y}\}; z\} = \frac{2^{2v-\mu-2}}{z^{n(\mu+1/2)}} \Gamma(v) \mathcal{H}_{\mu,n}\{u^{n(\mu+1/2)} \mathcal{P}_{v,2n}\{f(x); u\}; z\}, \tag{2.23}$$

and

$$\mathcal{H}_{\mu,n}\{u^{n(\mu+1/2)} \mathcal{P}_{v,2n}\{f(x); u\}; z\} = \left(\frac{z^n}{2}\right)^{v-1} \frac{1}{n\Gamma(v)} \mathcal{K}_{v-\mu-1,n}\{x^{n(\mu-v+3/2)} f(x); z\}. \tag{2.24}$$

Proof We put

$$g(u) = u^\mu J_\mu(zu) \tag{2.25}$$

in (2.15) of Theorem 2.1. Using relation (1.5) and the formula [8, p. 185, Entry (30)], we have

$$\mathcal{L}_{2n}\{u^{n\mu} J_\mu(z^n u^n); y\} = \frac{z^{n\mu}}{n 2^{\mu+1}} y^{-2n(\mu+1)} \exp(-z^{2n}/4y^{2n}). \tag{2.26}$$

Utilizing (2.4) of Corollary 2.2, we have

$$\mathcal{P}_{v,2n}\{u^{n\mu} J_\mu(z^n u^n); x\} = \frac{z^{n(v-1)}}{n\Gamma(v) 2^{v-1}} x^{n(\mu-v+1)} K_{v-\mu-1}(z^n x^n). \tag{2.27}$$

Substituting the relations (2.25), (2.26), and (2.27) into (2.15) of Theorem 2.1, we get

$$\int_0^\infty y^{2n(v-\mu-1)-1} \exp(-z^{2n}/4y^{2n}) \mathcal{L}_{2n}\{f(x); y\} dy$$

$$= \frac{1}{n}\left(\frac{z^n}{2}\right)^{v-\mu-1} \int_0^\infty x^{n(\mu-v+3)-1} K_{v-\mu-1}(z^n x^n) f(x) dx. \tag{2.28}$$

Changing the variable of the integration to $y = \frac{1}{2u}$ and then using the definition (1.3) of the $\mathcal{L}_{2n}$-transform on the left-hand side of (2.28) and the definition (1.15) of the $\mathcal{K}_{v,n}$-transform on the right-hand side of (2.28), we obtain assertion (2.22).

To prove identity (2.23), we substitute the relations (2.25) and (2.26) into (2.16) of Theorem 2.1 and change the variable of the integration to $y = \frac{1}{2u}$ on the left-hand side. Then, using the definition (1.3) of the $\mathcal{L}_{2n}$-transform, we obtain

$$\left(\frac{z^n}{2}\right)^{\mu} \frac{1}{2^{2(v-\mu-1)}} \mathcal{L}_{2n}\{y^{2n(\mu-v)} \mathcal{L}_{2n}\{f(x); \frac{1}{2^{1/n}y}\}; z\}$$

$$= \Gamma(v) \int_0^\infty u^{n(\mu+2)-1} J_\mu(z^n u^n) \mathcal{P}_{v,2n}\{f(x); u\} du. \tag{2.29}$$

Using the definition (1.18) of the generalized Hankel transform on the right-hand side of (2.29), we arrive at assertion (2.23).

The proof of assertion (2.24) follows from identities (2.22) and (2.23).

Remark 2.4. *Setting $v = 1$ in Corollary 2.5, we have for $Re(\mu) > -1$,*

$$\mathcal{L}_{2n}\{y^{2n(\mu-1)} \mathcal{L}_{2n}\{f(x); \frac{1}{2^{1/n}y}\}; z\} = \frac{z^{-n(\mu+1/2)}}{n 2^\mu} \mathcal{K}_{\mu,n}\{x^{n(\mu+1/2)} f(x); z\}, \tag{2.30}$$

$$\mathcal{L}_{2n}\{y^{2n(\mu-1)} \mathcal{L}_{2n}\{f(x); \frac{1}{2^{1/n}y}\}; z\} = \frac{2^{-\mu}}{z^{n(\mu+1/2)}} \mathcal{H}_{\mu,n}\{u^{n(\mu+1/2)} \mathcal{P}_{2n}\{f(x); u\}; z\}, \tag{2.31}$$

and

$$\mathcal{H}_{\mu,n}\{u^{n(\mu+1/2)} \mathcal{P}_{2n}\{f(x); u\}; z\} = \frac{1}{n} \mathcal{K}_{\mu,n}\{x^{n(\mu+1/2)} f(x); z\}, \tag{2.32}$$

where we use the fact that $K_\mu(x)$ is an even function with respect to the index and relationship (1.13). If we set $\mu = -\frac{1}{2}$ in (2.32) and use the relations

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos(x), \tag{2.33}$$

$$K_{-1/2}(x) = \left(\frac{\pi}{2x}\right)^{1/2} \exp(-x), \tag{2.34}$$

we obtain

$$2n \mathcal{F}_{c,n}\{\mathcal{P}_{2n}\{f(x); u\}; z\} = \pi \mathcal{L}_n\{f(x); z\}. \tag{2.35}$$

Corollary 2.6. *If the integrals involved converge absolutely, then we have*

$$\int_0^\infty y^{2n(v-\mu)-1} \mathcal{L}_{2n}\{g(u); y\} dy = \frac{\Gamma(v)}{\Gamma(\mu)} \int_0^\infty x^{2n\mu-1} \mathcal{P}_{v,2n}\{g(u); x\} dx, \tag{2.36}$$

$$\int_0^\infty y^{2n(v-\mu)-1} \mathcal{L}_{2n}\{g(u); y\} dy = \frac{\Gamma(v-\mu)}{2n} \int_0^\infty u^{2n(\mu-v+1)-1} g(u) du, \tag{2.37}$$

and

$$\int_0^\infty x^{2n\mu-1} \mathcal{P}_{v,2n}\{g(u); x\} dx = \frac{B(v-\mu, \mu)}{2n} \int_0^\infty u^{2n(\mu-v+1)-1} g(u) du, \tag{2.38}$$

where $0 < Re(\mu) < Re(v)$ and $B(x,y)$ [15, p. 18] denotes the beta function.

Proof If we put

$$f(x) = x^{2n(\mu-1)} \tag{2.39}$$

into (2.15) of Theorem 2.1, we get

$$\int_0^\infty y^{2nv-1}\mathcal{L}_{2n}\{x^{2n(\mu-1)}; y\}\mathcal{L}_{2n}\{g(u); x\}dx = \frac{\Gamma(v)}{2n}\int_0^\infty \frac{\mathcal{P}_{v,2n}\{g(u); x\}}{x^{-2n\mu+1}}dx. \tag{2.40}$$

Using relation (1.5) and the known formula [8, p. 137, Entry (1)] on the left-hand side of (2.40), we have assertion (2.36).

Corollary 2.7. *If the integrals involved converge absolutely, then we have*

$$\mathcal{P}_{\mu,2n}\{\mathcal{P}_{v,2n}\{g(u); x\}; t\}$$

$$= \frac{1}{\Gamma(v)}\int_0^\infty y^{2n(\mu+v-1)-1}\exp(t^{2n}y^{2n})\Gamma(-\mu+1, t^{2n}y^{2n})\mathcal{L}_{2n}\{g(u); y\}dy, \tag{2.41}$$

$$\mathcal{G}_n\{x^{-2n+1}\mathcal{P}_{v,2n}\{g(u); x\}; t\}$$

$$= \frac{\sqrt{\pi}}{\Gamma(v)}\int_0^\infty y^{n(2v-1)-1}\exp(t^{2n}y^{2n})erfc(t^n y^n)\mathcal{L}_{2n}\{g(u); y\}dy, \tag{2.42}$$

where $0 < Re(\mu) < Re(v)$.

Proof We put

$$f(x) = (x^{2n} + t^{2n})^{-\mu} \tag{2.43}$$

in relation (2.15) of Theorem 2.1. Using the known formula [8, p. 137, Entry (4)] and the definition (1.11) of the $\mathcal{P}_{v,2n}$-transform, we obtain

$$\int_0^\infty y^{2nv-1}\mathcal{L}_{2n}\{(x^{2n} + t^{2n})^{-\mu}; y\}\mathcal{L}_{2n}\{g(u); y\}dy$$

$$= \int_0^\infty y^{2n(v+\mu-1)-1}\exp(t^{2n}y^{2n})\Gamma(-\mu+1, t^{2n}y^{2n})\mathcal{L}_{2n}\{g(u); y\}dy$$

$$= \Gamma(v)\int_0^\infty \frac{x^{2n-1}}{(x^{2n}+t^{2n})^\mu}\mathcal{P}_{v,2n}\{g(u); x\}\, dx = \Gamma(v)\mathcal{P}_{\mu,2n}\{P_{v,2n}\{g(u); x\}; t\}. \tag{2.44}$$

In order to prove (2.42), we substitute $f(x) = (x^{2n} + t^{2n})^{-1/2}$ and the formula [8, p. 135, Entry (18)] into (2.15) of Theorem 2.1. We thus obtain

$$\int_0^\infty y^{2nv-1}\mathcal{L}_{2n}\{(x^{2n} + t^{2n})^{-1/2}; y\}\mathcal{L}_{2n}\{g(u); y\}dy$$

$$= \sqrt{\pi}\int_0^\infty y^{n(2v-1)-1}\exp(t^{2n}y^{2n})erfc(t^n y^n)\mathcal{L}_{2n}\{g(u);y\}dy$$

$$= \Gamma(v)\int_0^\infty x^{2n-1}\frac{\mathcal{P}_{v,2n}\{g(u);x\}}{(x^{2n}+t^{2n})^{1/2}}dx. \tag{2.45}$$

If we use the definition (1.12) of the $\mathcal{G}_n$-transform on the right-hand side of (2.45), we arrive at assertion (2.42).

3. Illustrative examples

Example 3.1. *We show*

$$\mathcal{P}_{v,2n}\{x^{2n\mu}\exp(-a^{2n}x^{2n});y\} = \frac{\Gamma(\mu+1)}{2}a^{2n(v-\mu-1)}\Psi(v,v-\mu;a^{2n}y^{2n}) \tag{3.1}$$

where $Re(\mu) > -1$, $Re(v) > 0$, $v-\mu \notin \mathbb{Z}$ and $\Psi(a,b;z)$ is the Tricomi hypergeometric function [14, p. 517].

Demonstration. If we set

$$g(x) = x^{2n\mu}\exp(-a^{2n}x^{2n}) \tag{3.2}$$

in assertion (2.1) of Lemma 2.1, and use relationship (1.5) and the known formula [8, p. 144, Entry(3)], we get

$$\mathcal{L}_{2n}\{x^{2n\mu}\exp(-a^{2n}x^{2n});u\} = \frac{1}{2n}\Gamma(\mu+1)(u^{2n}+a^{2n})^{-\mu-1}. \tag{3.3}$$

Multiplying both sides of (3.2) by $u^{2n(v-1)}$ and applying the $\mathcal{L}_{2n}$-transform, we obtain

$$\mathcal{L}_{2n}\{u^{2n(v-1)}\mathcal{L}_{2n}\{x^{2n\mu}\exp(-a^{2n}x^{2n});u\};y\}$$

$$= \frac{\Gamma(\mu+1)}{2n}\mathcal{L}_{2n}\{u^{2n(v-1)}(u^{2n}+a^{2n})^{-\mu-1};y\}. \tag{3.4}$$

Using relationship (1.5) once again and the formula [14, p. 18, Entry 2.1.3-(1)], we have

$$\mathcal{L}\{u^{v-1}(u+a^{2n})^{-\mu-1};y^{2n}\} = \Gamma(v)a^{2n(v-\mu-1)}\Psi(v,v-\mu;a^{2n}y^{2n}). \tag{3.5}$$

Substituting (3.2), (3.4), and (3.5) into (2.1) of Lemma 2.1, we obtain assertion (3.1).

Example 3.2. *We show*

$$\mathcal{P}_{v,2n}\{Ei(-\frac{a^{2n}}{x^{2n}});y\} = \frac{-\Gamma(v-1)}{4n^2a^n(v-1)}y^{-2nv+3n}\exp(a^{2n}/2y^{2n})W_{-v+3/2,0}(\frac{a^{2n}}{y^{2n}}), \tag{3.6}$$

where $Re(v) > 1$, $W_{\lambda,\mu}(x)$ denotes Whittaker's function [11].

Demonstration. We set

$$g(x) = Ei(-\frac{a^{2n}}{x^{2n}}) \tag{3.7}$$

in identity (2.1). $Ei(x)$ is the exponential integral function. The exponential integral $Ei(x) = -E_1(-x)$ is defined by

$$E_1(x) = \int_x^\infty \frac{\exp(-u)}{u} du. \tag{3.8}$$

Using (1.5) and the known identity [14, p. 136, Entry 3.4.1-(13)], we get

$$\mathcal{L}_{2n}\{Ei(-\frac{a^{2n}}{x^{2n}}); u\} = -\frac{1}{2n^2u^{2n}} K_0(2a^n u^n). \tag{3.9}$$

Multiplying both sides of (3.9) by $u^{2n(v-1)}$, applying the $\mathcal{L}_{2n}$-transform, and then using relation (1.5) once more and the known formula [14, p. 353, Entry 3.16.2-(3)], we obtain

$$\mathcal{L}_{2n}\{u^{2n(v-2)}\mathcal{L}_{2n}\{Ei(-\frac{a^{2n}}{x^{2n}}); u\}; y\}$$

$$= \mathcal{L}_{2n}\{-\frac{1}{n}u^{2n(v-2)}K_0(2a^n u^n); y\}$$

$$= -\frac{1}{4n^2a^n}(y^{2n})^{-v+3/2}(\Gamma(v-1))^2 \exp(\frac{a^{2n}}{y^{2n}})W_{-v+3/2,0}(\frac{a^{2n}}{y^{2n}}). \tag{3.10}$$

Substituting (3.7) and (3.10) in (2.1) of Lemma 2.1, we arrive at assertion (3.6).

Example 3.3. *We show*

$$\mathcal{P}_{v,2n}\{erf(a^n x^n); y\} = \frac{a^{n(v-\frac{3}{2})}}{2n(v-1)}(y^n)^{-v+\frac{1}{2}} \exp(-\frac{a^{2n}y^{2n}}{2})W_{\frac{3-2v}{4},\frac{2v-3}{4}}(a^{2n}y^{2n}), \tag{3.11}$$

where $y > 0$, $Re(v) > 1$.

Demonstration. We put

$$g(x) = erf(a^n x^n) \tag{3.12}$$

in identity (2.1) of Lemma 2.1. Using relation (1.5) and the identity [8, p. 176, Entry (4)], we find

$$\mathcal{L}_{2n}\{erf(a^n x^n); u\} = \frac{1}{2n}a^n u^{-2n}(u^{2n} + a^{2n})^{-1/2}. \tag{3.13}$$

Multiplying both sides of (3.13) by $u^{2n(v-1)}$, applying the $\mathcal{L}_{2n}$-transform, and then using (1.3) once more and the known identity [8, p. 139, Entry (22)], we obtain assertion (3.11).

Remark 3.5. *If we set* $v = 2\mu + \frac{3}{2}$, *we get for* $y > 0$, $Re(\mu) > -\frac{3}{4}$,

$$\mathcal{P}_{2\mu+\frac{3}{2},2n}\{erf(a^n x^n); y\} = \frac{a^{2n\mu}}{n(4\mu+1)(y^n)^{2\mu+1}} \exp(-\frac{a^{2n}y^{2n}}{2})W_{-\mu,\mu}(a^{2n}y^{2n}). \tag{3.14}$$

Example 3.4. *We show for* $Re(\mu - v) > -4$,

$$\mathcal{K}_{v-\mu-1,n}\{x^{n(\mu-v+\frac{7}{2})}J_2(a^n x^n); z\} = \frac{a^{2n}}{n^2}\frac{\Gamma(\mu-v+4)}{(z^{2n}+a^{2n})^{\mu-v+4}}\frac{2^{\mu-v+3}}{z^{n(v-\mu-3/2)}} \tag{3.15}$$

and

$$\mathcal{H}_{\mu,n}\{u^{n(\mu+1/2)}\mathcal{P}_{v,2n}\{x^{2n}J_2(a^n x^n); u\}; z\} = \frac{z^{n(\mu+\frac{1}{2})}\Gamma(\mu-v+4)}{2^{2v-\mu-4}n^3\Gamma(v)}\frac{a^{2n}}{(z^{2n}+a^{2n})^{\mu-v+4}}, \tag{3.16}$$

where J_2 *is the Bessel function of the first kind of order* 2*.*

Demonstration. If we put

$$f(x) = x^{2n}J_2(a^n x^n) \tag{3.17}$$

in relation (2.22) of Corollary 2.5, then use (1.5) and the formula [14, p. 264, Entry 3.12.2.(25)], we have

$$\mathcal{L}_{2n}\{x^{2n}J_2(a^n x^n); \frac{1}{2^{1/n}y}\} = \frac{2^3 a^{2n}}{n}y^{6n}\exp(-a^{2n}y^{2n}) \tag{3.18}$$

where $J_2(a^n x^n)$ is the Bessel function of the first kind of order 2 and $Re(a) > 0$. Multiplying by $y^{2n(\mu-v)}$ and applying the $\mathcal{L}_{2n}$-transform for both sides of (3.18), using the relation (1.5) and the known formula [14, p. 28, Entry 2.2.1.(2)], we find that

$$\frac{8a^{2n}}{n}\mathcal{L}_{2n}\{y^{2n(\mu-v+3)}\exp(-a^{2n}y^{2n}); z\} = (\frac{2a^n}{n})^2\frac{\Gamma(\mu-v+4)}{(z^{2n}+a^{2n})^{\mu-v+4}}, \tag{3.19}$$

where $Re(\mu - v) > -4$. Substituting relations (3.18) and (3.19) into identities (2.22) and (2.23), we arrive at assertions (3.15) and (3.16), respectively.

4. Conclusion

Generalized integral transforms could be used in many areas of applied mathematics. Different types of generalized integral transforms were investigated similarly before and many related articles could be found in literature.

For example, using a new integral transform, Aghili and Ansari gave a Cauchy type fractional diffusion equation on fractals and expressed its solution in terms of the Laplace type integral in [4]. In addition, generalized integral transforms were used to solve singular integral equations and partial fractional differential equations in [1, 2]. Furthermore, the fundamental solutions of the single-order and distributed-order Cauchy type fractional diffusion equations were given using generalized integral transforms in [3].

In conclusion, many other infinite integrals, as in [8, 9], could be evaluated in this manner applying the lemma, theorem, and corollaries considered in this paper.

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