Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 350 – 360
© TÜBİTAK
doi:10.3906/mat-1512-64

Positive solutions of first order boundary value problems with nonlinear nonlocal boundary conditions

Smita PATI[1], **Seshadev PADHI**[2,*]
[1]Department of Applied Mathematics, Cambridge Institute of Technology, Tatisilwai, Ranchi, India
[2]Department of Mathematics, Birla Institute of Technology, Mesra, Ranchi, India

Received: 15.12.2015 • **Accepted/Published Online:** 06.05.2016 • **Final Version:** 03.04.2017

Abstract: We consider the existence of positive solutions of the nonlinear first order problem with a nonlinear nonlocal boundary condition given by

$$\begin{aligned} x'(t) &= r(t)x(t) + \sum_{i=1}^{m} f_i(t, x(t)), \ \ t \in [0,1] \\ \lambda x(0) &= x(1) + \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)), \ \ \tau_j \in [0,1], \end{aligned}$$

where $r : [0,1] \to [0,\infty)$ is continuous, the nonlocal points satisfy $0 \leq \tau_1 < \tau_2 < ... < \tau_n \leq 1$, the nonlinear functions f_i and Λ_j are continuous mappings from $[0,1] \times [0,\infty) \to [0,\infty)$ for $i = 1,2,...,m$ and $j = 1,2,...,n$ respectively, and $\lambda > 1$ is a positive parameter. The Leray–Schauder theorem and Leggett–Williams fixed point theorem were used to prove our results.

Key words: Positive solutions, Leray–Schauder fixed point theorem, nonlinear boundary conditions

1. Introduction

Consider the first order boundary value problem (BVP) with nonlinear nonlocal boundary condition

$$x'(t) = r(t)x(t) + \sum_{i=1}^{m} f_i(t, x(t)), \ \ t \in [0,1] \tag{1.1}$$

$$\lambda x(0) = x(1) + \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)), \ \ \tau_j \in [0,1], \tag{1.2}$$

where $r : [0,1] \to [0,\infty)$ is continuous, $f_i : [0,1] \times [0,\infty) \to [0,\infty)$, $i = 1,2,\cdots,m$, and the nonlocal points $\Lambda_j : [0,1] \times [0,\infty) \to [0,\infty), j = 1,2,...,n$ are continuous with $0 \leq \tau_1 < \tau_2 < ... < \tau_n \leq 1$, the nonlinear functions Λ_j satisfy

$$0 \leq \Lambda_j(t,x) \leq x\Psi_j(t,x), \ \ t \in [0,1], \, x \in [0,\infty) \tag{1.3}$$

*Correspondence: ses_2312@yahoo.co.in

2010 *AMS Mathematics Subject Classification:* 34B08, 34B18, 34B15, 34B10.

for some positive continuous functions $\Psi_j : [0,1] \times [0,\infty) \to [0,\infty)$, and the scalar λ satisfies

$$\lambda > \left(1 + \sum_{j=1}^{n} \beta_j\right) \exp\left(\int_0^1 r(\eta)\, d\eta\right), \tag{1.4}$$

where

$$\beta_j = \max_{[0,1]\times[0,c]} \Psi_j(t,x) \tag{1.5}$$

for some real constant c.

The BVP (1.1)–(1.2) was first studied by Anderson [1], who used the Leggett–William multiple fixed point theorem to obtain three positive solutions of the BVP (1.1)–(1.2). The results obtained by Anderson [1] are motivated by the results obtained for time scales in [2, 5–9, 12, 16–19]. Recently, Padhi et al. [15] used the Leggett–Williams multiple fixed point theorem to establish a sufficient condition for the existence of at least three positive solutions of the BVP (1.1)–(1.2), and improved the results in [1].

Motivated by the work by Anderson [1], Çetin and Topal in [3] used monotone iteration and established an iterative scheme for the existence and approximation of two positive solutions of the nonlinear nonlocal first-order multipoint problem with sign changing nonlinearities

$$\begin{aligned} x'(t) + r(t)x(t) &= \sum_{i=1}^{m} f_i(t, x(t)), \;\; t \in [0,1] \\ x(0) = x(1) &+ \sum_{j=1}^{n} g_j(t_j, x(t_j)), \;\; \tau_j \in [0,1], \end{aligned}$$

where $r : [0,1] \to [0,\infty)$ is continuous, $f_i : [0,1] \times [0,\infty) \to (-\infty,\infty)$, $i = 1,2,\cdots,m$, and the nonlocal points $g_j : [0,1] \times [0,\infty) \to [0,\infty), j = 1,2,...,n$ are continuous, and satisfy $0 \le t_1 < t_2 < ... < t_n \le 1$.

This work has been divided into three sections. Section 1 is the Introduction, which contains the basic notations and the fixed point theorems used in this paper. Statements of two theorems, namely the Leray–Schauder theorem and Leggett–Williams multiple fixed point theorem, are given in this section. In Section 3, we give an example to show that the use of the Leray–Schauder theorem is not sufficient to have a complete study on the existence of positive solutions. This forces us to use the Leggett–Williams multiple fixed point theorem in the example to find the number of positive solutions. Thus, we have applied the Leray–Schauder theorem and Leggett–Williams multiple fixed point theorem in Section 2 to obtain three theorems on the existence of positive solutions of the BVP (1.1)–(1.2). In Section 3, all these three theorems are applied to an example to get a complete study on the existence of positive solutions. The motivation for the study of this example for a particular case came from [4] and [14]. An extensive use the Leggett–Williams fixed point theorem can be found in [13] for the existence of positive periodic solutions of first order functional differential equations.

We shall use the following notations for our use in the sequel: Let X be a Banach space. For any cone K on X, we denote $K_a = \{x \in K : ||x|| < a\}$, $\overline{K}_a = \{x \in K : ||x|| \le a\}$ and

$$K(\psi, b, c) = \{x \in K; \psi(x) \ge b \,\text{and}\, ||x|| \le c\}$$

for any constants $a > 0, b > 0$, and $c > 0$. With the above notations, we now state the following fixed point theorems.

Theorem 1.1 *([10], Leray–Schauder) Let K be a convex subset of a Banach space X, $0 \in K$, $A : K \to K$ be a completely continuous operator. Then either (i) A has at least one fixed point in K, or (ii) the set $\{x \in K : x = \mu Ax, 0 < \mu < 1\}$ is unbounded.*

Theorem 1.2 ([11, Theorem 3.3]) *(Leggett–Williams fixed point theorem) Let $X = (X, \|.\|)$ be a Banach space and $K \subset X$ be a cone, and $c_4 > 0$ be a constant. Suppose there exists a concave nonnegative continuous function ψ on K with $\psi(x) \leq \|x\|$ for $x \in \overline{K}_{c_4}$ and let $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$ be a continuous compact map. Assume that there are numbers c_1, c_2, and c_3 with $0 < c_1 < c_2 < c_3 \leq c_4$ such that*
(i) $\{x \in K(\psi, c_2, c_3); \psi(x) > c_2\} \neq \phi$ and $\psi(Ax) > c_2$ for all $x \in K(\psi, c_2, c_3)$;
(ii) $\|Ax\| < c_1$ for all $x \in \overline{K}_{c_1}$;
(iii) $\psi(Ax) > c_2$ for all $x \in K(\psi, c_2, c_4)$ with $\|Ax\| > c_3$.
Then A has at least three fixed points x_1, x_2, and x_3 in $\overline{K}_{c_4}$. Furthermore, we have $x_1 \in \overline{K}_{c_1}$, $x_2 \in \{x \in K(\psi, c_2, c_4) : \psi(x) > c_2\}$, and $x_3 \in \overline{K}_{c_4} \setminus \{K(\psi, c_2, c_4) \cup \overline{K}_{c_1}\}$.

Remark 1 *According to the localization of the fixed points in Theorem 1.2, one of them is possibly a zero (namely $x_1 \in \overline{K}_{c_1}$). Thus, the operator A, stated in Theorem 1.2, has at least two positive fixed points and a zero fixed point as can be easily observed. Accordingly, the BVP (1.1)–(1.2) has two positive T-periodic solutions and a possible trivial solution (if the conditions of Theorem 1.2 are satisfied).*

2. Main results

Clearly, the BVP (1.1)–(1.2) is equivalent to the integral equation

$$x(t) = \sum_{i=1}^{m} \int_0^1 G(t,s) f_i(s, x(s))\, ds + \frac{\exp(\int_0^t r(\eta)\, d\eta) \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))}{\lambda - \exp(\int_0^1 r(\eta)\, d\eta)}, \tag{2.1}$$

where $G(t,s)$ is Green's kernel, given by

$$G(t,s) = \frac{\exp(\int_s^t r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \times \begin{cases} \lambda & ; \quad if\, 0 \leq s \leq t \leq 1 \\ \exp(\int_0^1 r(\eta)\, d\eta); & if\, 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Theorem 2.1 *Let*

$$f_{i0} = \limsup_{x \to 0+,\, 0 \leq t \leq 1} \frac{f_i(t,x)}{x} = 0, \; i = 1, 2, \cdots, m \tag{2.3}$$

hold. Then the BVP (1.1)–(1.2) has at least one positive solution.

Proof Let $X = C[0,1]$; then X is a Banach space endowed with the sup. norm. From (2.3), there exist constants ϵ and $B > 0$ such that

$$f_i(t,x) < \epsilon x \; for \; 0 < x \leq B, \; 0 \leq t \leq 1 \text{ and } 1 \leq i \leq m$$

holds, where $\epsilon > 0$ is chosen such that it satisfies

$$0 < \epsilon < \frac{\lambda - \left(1 + \sum_{j=1}^{n} \beta_j\right) \exp\left(\int_0^1 r(\eta)\, d\eta\right)}{\lambda m \int_0^1 \exp(\int_s^1 (r(\eta)\, d\eta)\, ds)}. \tag{2.4}$$

On X, we define a convex set K by

$$K = \{x(t); x(t) \in X, x(t) \geq 0, x(t) \,\text{is nondecreasing}, x(t) \leq B, 0 \leq t \leq 1\}, \tag{2.5}$$

and an operator $A : K \to X$ by

$$(Ax)(t) = \sum_{i=1}^{m} \int_0^1 G(t,s) f_i(s, x(s))\, ds + \frac{\exp(\int_0^t r(\eta)\, d\eta) \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))}, \tag{2.6}$$

where $G(t,s)$ is Green's kernel given in (2.2). The operator A in (2.6) can be rewritten as

$$\begin{aligned} Ax(t) =& \sum_{i=1}^{m} \int_0^t \frac{\lambda \exp(\int_s^t r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} f_i(s, x(s))\, ds \\ &+ \sum_{i=1}^{m} \int_t^1 \frac{\exp(\int_0^1 r(\eta)\, d\eta) \exp(\int_s^t r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} f_i(s, x(s))\, ds \\ &+ \frac{\exp(\int_0^t r(\eta)\, d\eta) \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))}{\lambda - \exp(\int_0^1 r(\eta)\, d\eta)}. \end{aligned} \tag{2.7}$$

Let $x \in K$. Then the positivity of f_i, $i = 1, 2, ..., m$ and Λ_j, $j = 1, 2, ..., n$ shows that $(Ax)(t) \geq 0$ for all $t \in [0,1]$. Now we show that the fixed points of the operator A are the solutions of the BVP (1.1)–(1.2). In fact, if $x = Ax$, then from (2.7), we have

$$\begin{aligned} \lambda x(0) - x(1) =& \sum_{i=1}^{m} \int_0^1 \frac{\lambda \exp(\int_s^1 r(s)\, ds)}{(\lambda - \exp(\int_0^1 r(s)\, ds))} f_i(s, x(s))\, ds \\ &+ \lambda \frac{\sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \\ &- \sum_{i=1}^{m} \int_0^1 \frac{\lambda \exp(\int_s^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} f_i(s, x(s))\, ds \\ &- \frac{\exp(\int_0^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)), \\ &= \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)), \quad \tau_j \in [0,1]. \end{aligned}$$

Hence the boundary condition (1.2) is satisfied. Again differentiating (2.7) with respect to t, with $Ax = x$, we obtain

$$
\begin{aligned}
x'(t) =& \sum_{i=1}^{m} \int_0^t \frac{\lambda r(t) \exp(\int_s^t r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} f_i(s, x(s))\, ds + \sum_{i=1}^{m} \frac{\lambda}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} f_i(t, x(t)) \\
&+ \sum_{i=1}^{m} \int_t^1 \frac{r(t) \exp(\int_0^1 r(\eta)\, d\eta) \exp(\int_s^t r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} f_i(s, x(s))\, ds \\
&- \sum_{i=1}^{m} \frac{\exp(\int_0^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} f_i(t, x(t)) + \frac{r(t) \exp(\int_0^t r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)) \\
=& r(t) \left[\sum_{i=1}^{m} \int_0^t \lambda \frac{\exp(\int_s^t r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} f_i(s, x(s))\, ds \right] \\
&+ r(t) \left[\sum_{i=1}^{m} \int_t^1 \frac{\exp(\int_0^1 r(\eta)\, d\eta) \exp(\int_s^t r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} f_i(s, x(s))\, ds \right] \\
&+ r(t) \left[\frac{\exp(\int_0^t r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j)) \right] + \sum_{i=1}^{m} f_i(t, x(t)) \\
=& r(t)x(t) + \sum_{i=1}^{m} f_i(t, x(t)), \text{ (using (2.1))},
\end{aligned}
$$

which shows that $x(t)$ satisfies (1.1). Moreover, $(Ax)'(t) = r(t)x(t) + \sum_{i=1}^{m} f_i(t, x(t))$, $t \in [0, 1]$ shows that Ax is nondecreasing, $t \in [0, 1]$.

Next, for $0 < x \leq B$, we have

$$
\begin{aligned}
(Ax)(t) \leq& ||Ax|| = Ax(1) = \sum_{i=1}^{m} \int_0^1 G(1, s) f_i(s, x(s))\, ds + \frac{\exp(\int_0^1 r(\eta)\, d\eta) \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \\
\leq& \frac{\exp(\int_0^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \left[\lambda \sum_{i=1}^{m} \int_0^1 \exp(-\int_0^s r(\eta)\, d\eta) f_i(s, x(s))\, ds + x(t) \sum_{j=1}^{n} \beta_j \right] \qquad (2.8) \\
\leq& \frac{\exp(\int_0^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \left[\lambda\epsilon \sum_{i=1}^{m} \int_0^1 \exp(-\int_0^s r(\eta)\, d\eta) \|x\|\, ds + \|x\| \sum_{j=1}^{n} \beta_j \right] \\
\leq& \frac{B \exp(\int_0^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \left[\lambda\epsilon m \int_0^1 \exp(-\int_0^s r(\eta)\, d\eta)\, ds + \sum_{j=1}^{n} \beta_j \right] \\
\leq& B. \qquad (2.9)
\end{aligned}
$$

This proves that $A(K) \subset K$. One may verify that A is completely continuous.

In order to use Theorem 1.1, we consider $x \in K$ with $x(t) = \mu(Ax)(t)$, $0 < \mu < 1$. Then, using (2.9), we have

$$x(t) = \mu(Ax)(t) < (Ax)(t) \leq B,$$

which implies that the set

$$\{x \in K : x = \mu Ax, 0 < \mu < 1\}$$

is bounded. Hence, by Theorem 1.1, the operator A has a fixed point in X, which is a positive solution of the BVP (1.1)–(1.2). This completes the proof of the theorem. □

Theorem 2.2 *Let*

$$f_{i\infty} = \limsup_{x\to\infty,\, 0\le t\le 1} \frac{f_i(t,x)}{x} = 0, \;\; i = 1, 2, \cdots, m \tag{2.10}$$

hold. Then the BVP (1.1)–(1.2) has at least one positive solution.

Proof Let $X = C[0,1]$; then X is a Banach space endowed with the sup. norm. From (2.10), there exist positive constants ϵ and N such that

$$f_i(t,x) < \epsilon x \;\; for \;\; x \ge N, \;\; 0 \le t \le 1 \text{ and } 1 \le i \le m,$$

where ϵ is chosen such that (2.4) is satisfied. Let

$$\gamma = \max_{0\le t\le 1, 0\le x\le N, 1\le i\le m} f_i(t,x).$$

Then

$$f_i(t,x) < \epsilon x + \gamma \;\; for \;\; x \ge 0, \;\; 0 \le t \le 1 \text{ and } 1 \le i \le m.$$

For the above choice of ϵ and γ, we consider a constant B by

$$B \ge \frac{\lambda\gamma m \int_0^1 \exp(\int_s^1 r(\eta)\, d\eta)\, ds}{\lambda - \left(1 + \sum_{j=1}^n \beta_j\right) \exp\left(\int_0^1 r(\eta)\, d\eta\right) - \lambda\epsilon m \int_0^1 \exp(\int_s^1 r(\eta)\, d\eta)\, ds}. \tag{2.11}$$

Now we define a convex set K on X by (2.5) and an operator $A : K \to X$ by (2.6), where $G(t,s)$ is Green's kernel given in (2.2). One may verify that A is completely continuous. Proceeding as in Theorem 2.1, we can prove that a fixed point of the operator A in the cone K is equivalent to the existence of a positive solution of the BVP (1.1)–(1.2), $(Ax)(t) \ge 0$ and Ax is nondecreasing for $0 \le t \le 1$. Now we show that $Ax \le B$ for $0 \le t \le 1$, where B is defined in (2.11). For $0 < x \le B$, from (2.8), we have

$$\begin{aligned}
(Ax)(t) \le & \frac{\exp(\int_0^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \left[\lambda \sum_{i=1}^m \int_0^1 \exp(-\int_0^s r(\eta)\, d\eta)(\epsilon\|x\| + \gamma)\, ds + \|x\| \sum_{j=1}^n \beta_j \right] \\
\le & \frac{\exp(\int_0^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \left[\lambda m(\epsilon B + \gamma) \int_0^1 \exp(-\int_0^s r(\eta)\, d\eta)\, ds + B \sum_{j=1}^n \beta_j \right] \\
\le & B.
\end{aligned} \tag{2.12}$$

This proves that $A(K) \subset K$.

Next suppose that $x \in K$ with $x(t) = \mu(Ax)(t)$, $0 < \mu < 1$. Then, using (2.12), we have

$$x(t) = \mu(Ax)(t) < (Ax)(t) \le B,$$

which, in turn, implies that the set

$$\{x \in K : x = \mu Ax, 0 < \mu < 1\}$$

is bounded. Hence, by Theorem 1.1, the operator A has a fixed point in X, which is a positive solution of the BVP (1.1)–(1.2). This completes the proof of the theorem. □

Remark 2 *Theorems 2.1 and 2.2 require at least one of the conditions (2.3) or (2.10) to guarantee the existence of a positive solution of the BVP (1.1)–(1.2). In Section 3, we give an example, that is the BVP (3.1)–(3.2), where both Theorems 2.1 and 2.2 are applicable, guaranteeing the existence of a positive solution independently. Therefore, now we are confronted with a question regarding the number of positive solutions admitted by the BVP (3.1)–(3.2). In this case, we shall use the following theorem regarding the number of positive solutions admitted by the BVP (3.1)–(3.2).*

Theorem 2.3 *Suppose that the conditions (2.3) and (2.10) are satisfied, and there exists a positive constant $c_2 > 0$ such that*

$$\sum_{i=1}^{m} \int_0^1 \exp\Big(\int_s^1 r(\eta)\,d\eta\Big) f_i(s, x(s))\,ds + \frac{\sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta)\,d\eta))} > c_2\Big[\lambda - \exp\Big(\int_0^1 r(\eta)\,d\eta\Big)\Big],\ c_2 \le x \le \lambda c_2 \quad (2.13)$$

for $0 \le t \le 1$ holds. Then the BVP (1.1)–(1.2) has at least two positive solutions.

Proof Let $X = C[0,1]$ be a Banach space endowed with the sup. norm. On the space X, we define a cone K by

$$K = \{x \in X; x(t) \ge 0, x(t)\, nondecreasing, t \in [0,1]\}$$

and an operator $A : K \to X$ by (2.6), where $G(t,s)$ is Green's kernel given in (2.2). Proceeding as in the lines of Theorem 2.1, we can show that $A(K) \subset K$ and $: K \to K$ is completely continuous. Further, the existence of a positive solution of the BVP (1.1)–(1.2) is equivalent to the existence of a fixed point of the operator A in K.

First we consider (2.10). Then there exist constants $\epsilon > 0$ and $N > 0$ such that

$$f_i(t,x) < \epsilon x \ for\ x \ge N,\ 0 \le t \le 1 \text{ and } 1 \le i \le m,$$

where $\epsilon > 0$ is chosen so that it satisfies the property (2.4). Let

$$\gamma = \max_{0 \le t \le 1, 0 \le x \le N, 1 \le i \le m} f_i(t,x).$$

Then

$$f_i(t,x) < \epsilon x + \gamma \ for\ x \ge 0,\ 0 \le t \le 1 \text{ and } 1 \le i \le m.$$

Choose a constant $c_4 > 0$ such that

$$c_4 \ge \left\{ \lambda c_2, \frac{\lambda \gamma m \int_0^1 \exp(\int_s^1 r(\eta)\,d\eta)\,ds}{\lambda - \left(1 + \sum_{j=1}^{n} \beta_j\right) \exp\left(\int_0^1 r(\eta)\,d\eta\right) - \lambda \epsilon m \int_0^1 \exp(\int_s^1 r(\eta)\,d\eta)\,ds} \right\}.$$

For $x \in \overline{K}_{c_4}$, we have

$$\begin{aligned}(Ax)(t) \leq & ||Ax|| = Ax(1) = \sum_{i=1}^{m} \int_0^1 G(1,s) f_i(s,x(s))\, ds + \frac{\exp(\int_0^1 r(\eta)\, d\eta) \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \\ \leq & \frac{\exp(\int_0^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \left[\lambda \sum_{i=1}^{m} \int_0^1 \exp(-\int_0^s r(\eta)\, d\eta) f_i(s,x(s))\, ds + x(t) \sum_{j=1}^{n} \beta_j \right] \\ \leq & \frac{\exp(\int_0^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \left[\lambda \sum_{i=1}^{m} \int_0^1 \exp(-\int_0^s r(\eta)\, d\eta)(\epsilon \|x\| + \gamma)\, ds + \|x\| \sum_{j=1}^{n} \beta_j \right] \\ \leq & \frac{\exp(\int_0^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \left[\lambda m(\epsilon c_4 + \gamma) \int_0^1 \exp(-\int_0^s r(\eta)\, d\eta)\, ds + c_4 \sum_{j=1}^{n} \beta_j \right] \\ \leq & c_4,\end{aligned}$$

that is, $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$.

Next we consider (2.3). Then there exist constants ϵ and $c_1 \in (0, c_2)$ such that

$$f_i(t,x) < \epsilon x \text{ for } 0 < x \leq c_1,\ 0 \leq t \leq 1 \text{ and } 1 \leq i \leq m,$$

where ϵ satisfies the property (2.4).

For $x \in \overline{K}_{c_1}$, we have

$$\begin{aligned}(Ax)(t) \leq & ||Ax|| = Ax(1) = \sum_{i=1}^{m} \int_0^1 G(1,s) f_i(s,x(s))\, ds + \frac{\exp(\int_0^1 r(\eta)\, d\eta) \sum_{j=1}^{n} \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \\ \leq & \frac{\exp(\int_0^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \left[\lambda \sum_{i=1}^{m} \int_0^1 \exp(-\int_0^s r(\eta)\, d\eta) f_i(s,x(s))\, ds + x(t) \sum_{j=1}^{n} \beta_j \right] \\ \leq & \frac{\exp(\int_0^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \left[\lambda \epsilon \sum_{i=1}^{m} \int_0^1 \exp(-\int_0^s r(\eta)\, d\eta) \|x\|\, ds + \|x\| \sum_{j=1}^{n} \beta_j \right] \\ \leq & \frac{c_1 \exp(\int_0^1 r(\eta)\, d\eta)}{(\lambda - \exp(\int_0^1 r(\eta)\, d\eta))} \left[\lambda \epsilon m \int_0^1 \exp(-\int_0^s r(\eta)\, d\eta)\, ds + \sum_{j=1}^{n} \beta_j \right] \\ < & c_1.\end{aligned}$$

This proves the condition (ii) of Theorem 1.2.

Set $c_3 = \lambda c_2$. In order to verify the condition (i) of Theorem 1.2, we set $\theta(t) = \lambda c_2$ for $t \in [0,1]$. Let $\psi(t) = \min_{t \in [0,1]} x(t)$ be a nonnegative concave functional on K. Since $\psi(\theta(t)) = \min_{t \in [0,1]} \theta(t) = \lambda c_2 > c_2; c_2 \leq \psi(x), \|x\| = \lambda c_2$, then the set $\{x \in K; c_2 \leq \psi(x), \|x\| \leq \lambda c_2\}$ is nonempty. Let $x \in (K, \psi, c_2, c_3)$; then

$c_2 \le \psi(x) \le x \le \|x\| = x(1) = \lambda c_2 = c_3$, and hence

$$\begin{aligned}\psi(Ax)(t) = Ax(0) &= \sum_{i=1}^{m}\int_0^1 G(0,s)f_i(s,x(s))\,ds + \frac{\sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j))}{(\lambda-\exp(\int_0^1 r(\eta)\,d\eta))}\\ &= \frac{1}{(\lambda-\exp(\int_0^1 r(\eta)\,d\eta))}\left[\sum_{i=1}^{m}\int_0^1 \exp(\int_s^1 r(\eta)\,d\eta)f_i(s,x(s))\,ds + \sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j))\right]\\ &\ge \frac{1}{(\lambda-\exp(\int_0^1 r(\eta)\,d\eta))}\sum_{i=1}^{m}\int_0^1 \exp(\int_s^1 r(\eta)\,d\eta)f_i(s,x(s))\,ds + \frac{\sum_{j=1}^{n}\Lambda_j(\tau_j,x(\tau_j))}{(\lambda-\exp(\int_0^1 r(\eta)\,d\eta))}\\ &> c_2 \ (\text{using}(2.13))\end{aligned}$$

holds. Thus, the condition (i) of Theorem 1.2 is satisfied.

Finally, suppose that $x \in K(\psi, c_2, c_3)$ with $\|Ax\| > \lambda c_2 = c_3$. Then

$$\psi(Ax) = (Ax)(0) \ge \frac{1}{\lambda}(Ax)(1) = \frac{\|Ax\|}{\lambda} > \frac{\lambda c_2}{\lambda} = c_2$$

implies that the condition (iii) of Theorem 1.2 is satisfied. Hence the BVP (1.1)–(1.2) has at least three solutions. Consequently, the BVP (1.1)–(1.2) has at least two positive solutions. This completes the proof of the theorem. □

Remark 3 *Anderson in [1] and Padhi et al. in [15] applied the Leggett–Williams multiple fixed point theorem [11] to obtain sufficient conditions for the existence of three positive solutions of the BVP (1.1)–(1.2). Although the conditions of Theorem 2.3 imply the conditions obtained in [15] and [1], the conditions of Theorem 2.3 are easy to use. We have used Theorem 2.3 to complete the example given in Section 3.*

3. A complete example

Consider the first order BVP

$$x'(t) = x(t) + \frac{x^n(t)}{1+x^m(t)}, \quad t \in [0,1] \tag{3.1}$$

$$\lambda x(0) = x(1) + \Lambda(\tau, x(\tau)), \quad \tau \in [0,1], \tag{3.2}$$

where $\lambda > \frac{3e}{2}$,

$$\Lambda(t,x) = \begin{cases} \frac{x}{4}\left(1 + e^{-\frac{1}{x(t)-1}}\right), & x > 1\\ \frac{x}{2}, & x \le 1,\end{cases}$$

and m and n are nonnegative integers satisfying the property

$$1 > \frac{(\lambda-e)}{(e-1)}\lambda^{n-1}\frac{m}{m-n+1}\left(\frac{m-n+1}{n-1}\right)^{\frac{n-1}{m}} \quad \text{for } 1 < n < m. \tag{3.3}$$

Here $r(t) \equiv 1$ and $f(t,x) = \frac{x^n}{1+x^m}$, $t \in [0,1]$. Clearly $\Lambda(t,x) \le \frac{x}{2}$ with $\beta = \frac{1}{2}$. Consequently, $\lambda > (1+\beta)e = \frac{3e}{2}$.

First we consider the case $0 \leq n \leq 1$. Then $\lim_{x\to\infty} \frac{f(t,x)}{x} = 0$ implies that Theorem 2.2 can be applied to the BVP (3.1)–(3.2). On the other hand, $\liminf_{x\to 0+} \frac{f(t,x)}{x} \neq 0$ implies that Theorem 2.1 cannot be applied to the BVP (3.1)–(3.2). Hence, by Theorem 2.2, the BVP (3.1)–(3.2) has a positive solution.

Next we consider the case when $n \geq m+1$. Then

$$\lim_{x\to\infty} \frac{f(t,x)}{x} = \lim_{x\to\infty} \frac{x^{n-1}}{1+x^m} = \lim_{x\to\infty} \frac{x^{n-m-1}}{\frac{1}{x^m}+1} \neq 0$$

implies that Theorem 2.2 cannot be applied to this example. On the other hand,

$$\lim_{x\to 0} \frac{f(t,x)}{x} = \lim_{x\to 0} \frac{x^{n-1}}{1+x^m} = 0 \tag{3.4}$$

implies, by Theorem 2.1, that the BVP (3.1)–(3.2) has a positive solution. Note that (3.4) holds for any $n > 1$. Thus, for any $n > 1$, the BVP (3.1)–(3.2) has a positive solution.

Finally, we consider the case $1 < n < m$. Since both the conditions (2.3) or (2.10) are satisfied, we shall use Theorem 1.2 to find the number of positive solutions of the BVP (3.1)–(3.2). For this, we need to find a constant $c_2 > 0$ such that

$$\int_0^1 e^{1-s} \frac{x^n(s)}{1+x^m(s)}\, ds > c_2(\lambda - e) \text{ for } c_2 \leq \|x\| \leq \lambda c_2 \tag{3.5}$$

holds. For $x \in K$ and $c_2 \leq \|x\| \leq \lambda c_2$, we have

$$\int_0^1 e^{1-s} \frac{x^n(s)}{1+x^m(s)}\, ds > \frac{c_2^n}{1+\lambda^m c_2^m}(e-1).$$

This, in turn, implies that (3.5) holds if

$$1 > \frac{(\lambda - e)}{(e-1)} \frac{(1+\lambda^m c_2^m)}{c_2^{n-1}} \tag{3.6}$$

holds. Set $c_2 = \frac{1}{\lambda}\left(\frac{n-1}{m-n+1}\right)^{1/m}$, which is the minimizer of $\frac{(\lambda-e)}{(e-1)}\frac{(1+\lambda^m c_2^m)}{c_2^{n-1}}$. Then the inequality (3.6) follows from (3.3). Thus for the case $1 < n < m$, the BVP (3.1)–(3.2) has at least two positive solutions.

Acknowledgment

The authors are thankful to the referee for his/her comments in revising the manuscript to the present form.

References

[1] Anderson DR. Existence of three solutions for a first order problem with nonlinear nonlocal boundary conditions. J Math Anal Appl 2013; 408: 318-323.

[2] Anderson DR. Existence of solutions for first order multipoint problems with changing-sign nonlinearities. J Diff Equ Appl 2008; 14: 657-666.

[3] Çetin E, Topal FS. Existence of solutions for a first-order nonlocal boundary value problem with changing-sign nonlinearity. Turk J Math 2015; 39: 556-563.

[4] Dix JG, Padhi S, Pati S. Existence of three nonnegative periodic solutions for functional differential equations and applications to Hematopoiesis. PanAmerican Math J 2009; 19: 27-36.

[5] Gao CH, Luo H. Positive solutions to nonlinear nonlocal BVPs with parameter on time scales. Bound Value Probl 2011; 15: 198598.

[6] Gilbert H. Existence theorem for first order equations on time scales with Δ-Caratheodory functions. Adv Diff Equ 2010; 20: 650827.

[7] Goodrich CS. Existence of a positive solutions to a first order p-Laplacian bvp on a time scale. Nonl Anal 2011; 74: 1926-1936.

[8] Goodrich CS. Positive solutions to boundary value problems with nonlinear boundary conditions. Nonl Anal 2011; 75: 417-432.

[9] Graef JR, Kong L. First order singular boundary value problems with p-Laplacian on time scales. J Diff Equ Appl 2011; 17: 831-839.

[10] Granas A, Dugundji J. Fixed Point Theory. New York, NY, USA: Springer-Verlag, 2003.

[11] Leggett RW, Williams LR. Multiple positive fixed points of nonlinear operators on ordered Banach Spaces. Indiana Univ Math J 1979; 28: 673-688.

[12] Otero-Espinar V, Vivero DR. The existence and approximation of extremal solutions to several first order discontinuous dynamic equations with nonlinear boundary value conditions. Nonl Anal 2008; 68: 2027-2037.

[13] Padhi S, Graef JR, Srinivasu PDN. Periodic Solutions of First order Functional Differential Equations in Population Dynamics. New Delhi, India: Springer India, 2014.

[14] Padhi S, Pati S. Multiple periodic solutions for a nonlinear first order functional difference equation with application to hematopoiesis model. Comm Appl Anal 2014; 18: 1-10.

[15] Padhi S, Pati S, Hota DK. Positive solutions of boundary value problems with nonlinear nonlocal boundary conditions. Opus Math 2016; 36: 69-79.

[16] Shu W, Deng C. Three positive solutions of nonlinear first order boundary value problems on time scales. Int J Pure Appl Math 2010; 63: 129-136.

[17] Tian Y, Ge WG. Existence and uniqueness results for nonlinear first-order three-point boundary value problems on time scales. Nonl Anal 2008; 69: 2833-2842.

[18] Zhao YH. First-order boundary value problem with nonlinear boundary condition on time scales. Discrete Dyn Nat Soc 2011; 8: 845107.

[19] Zhao YH, Sun JP. Monotone iterative technique for first-order nonlinear periodic boundary value problems on time scales. Adv Diff Equ 2010; 10: 620459.