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Sufficient conditions on nonunitary operators that imply the unitary operators

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Abstract: In this paper, we give sufficient conditions on nonunitary operators on the Bergman space that imply the unitary operators.

Key words: Unitary operators, Toeplitz operators, composition operators, Berezin transform

1. Introduction

Let $dA(z)$ denote the Lebesgue area measure on the open unit disk $\mathbb{D}$, normalized so that the measure of the disk $\mathbb{D}$ equals 1. The Bergman space $L_a^2(\mathbb{D})$ is the Hilbert space consisting of analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$. For $z \in \mathbb{D}$, the Bergman reproducing kernel is the function $K_z \in L_a^2(\mathbb{D})$ such that $f(z) = \langle f, K_z \rangle$ for every $f \in L_a^2(\mathbb{D})$. The normalized reproducing kernel k_z is the function $\frac{K_z}{\|K_z\|_2}$. Here the norm $\|\cdot\|_2$ and the inner product $\langle , \rangle$ are taken in the space $L^2(\mathbb{D}, dA)$. For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$. Then $\{e_n\}$ forms an orthonormal basis for $L_a^2(\mathbb{D})$. Let $K(z, \bar{w}) = \overline{K_z(w)} = \frac{1}{(1-z\bar{w})^2} = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$. For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ with symbol ϕ is the operator on $L_a^2(\mathbb{D})$ defined by $T_\phi f = P(\phi f)$; here P is the orthogonal projection from $L^2(D, dA)$ onto $L_a^2(\mathbb{D})$.

Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$ an automorphism ϕ_a in $Aut(\mathbb{D})$ such that:
(i) $(\phi_a \ o \ \phi_a)(z) \equiv z$;
(ii) $\phi_a(0) = a, \phi_a(a) = 0$;
(iii) ϕ_a has a unique fixed point in $\mathbb{D}$.
In fact, $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$ for all a and z in $\mathbb{D}$. An easy calculation shows that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is $J_{\phi_a(z)} = |k_a(z)|^2 = \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}$. Given $z \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define a function $U_z f$ on $\mathbb{D}$ by $U_z f(w) = k_z(w) f(\phi_z(w))$. Notice that U_z is a bounded linear operator on $L^2(\mathbb{D}, dA)$ and $L_a^2(\mathbb{D})$ for all $z \in \mathbb{D}$. Furthermore, it can be verified that $U_z^2 = I$, the identity operator, $U_z^* = U_z, U_z(L_a^2(\mathbb{D})) \subset L_a^2(\mathbb{D})$ and $U_z((L_a^2(\mathbb{D}))^\perp) \subset (L_a^2(\mathbb{D}))^\perp$ for all $z \in \mathbb{D}$. Thus, $U_z P = P U_z$ for all $z \in \mathbb{D}$.

Let $\phi : \mathbb{D} \to \mathbb{D}$ be analytic. Define the composition operator C_ϕ from $L_a^2(\mathbb{D})$ into itself by $C_\phi f = f \circ \phi$.

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The operator C_ϕ is a bounded linear operator on $L_a^2(\mathbb{D})$ and $\|C_\phi\| \leq \frac{1+|\phi(0)|}{1-|\phi(0)|}$. Given $a \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define the function $C_a f = f o \phi_a$, where $\phi_a \in Aut(\mathbb{D})$. The map C_a is a composition operator on $L_a^2(\mathbb{D})$. Let $\mathcal{L}(H)$ denote the algebra of bounded, linear operators from a Hilbert space H into itself. Let $H(\mathbb{D})$ be the space of holomorphic functions from $\mathbb{D}$ into itself. Let us denote $E_{n,\phi} = \left\langle T_\phi \sqrt{n+1} z^n, \sqrt{n+1} z^n \right\rangle$.

If T is a compact operator on a separable Hilbert space H, then there exist orthonormal sets $\{u_n\}_{n=0}^\infty$ and $\{\sigma_n\}_{n=0}^\infty$ in H such that $Tx = \sum_{n=0}^\infty \lambda_n \langle x, u_n \rangle \sigma_n;\ x \in H$ where λ_n is the nth singular value of T. Given $0 < p < \infty$, we define the Schatten p-class of H, denoted by $S_p(H)$ or simply S_p, to be the space of all compact operators T on H with its singular value sequence $\{\lambda_n\}$ belonging to l^p (the p-summable sequence space). We will focus in the range $1 \leq p < \infty$. In this case, S_p is a Banach space with the norm $\|T\|_p = \left[\sum_n |\lambda_n|^p\right]^{\frac{1}{p}}$. The class S_1 is also called the trace class of H and S_2 is usually called the Hilbert–Schmidt class. One can easily verify that if T is a compact operator on H and $p \geq 1$, then $T \in S_p$ if and only if $|T|^p = (T^*T)^{\frac{p}{2}} \in S_1$ and $\|T\|_p^p = \||T|\|_p^p = \||T|^p\|_1$.

The Berezin transform $\widetilde{\phi}$ of a function $\phi \in L^\infty(\mathbb{D})$ is defined to be the Berezin transform of the Toeplitz operator T_ϕ. In other words, $\widetilde{\phi} = \widetilde{T_\phi}$. Furthermore, $\widetilde{\phi}(z) = \widetilde{T_\phi}(z) = \langle T_\phi k_z, k_z \rangle = \langle P(\phi k_z), k_z \rangle = \langle \phi k_z, k_z \rangle$ for each $z \in \mathbb{D}$.

For $\phi \in L^2(\mathbb{D}, dA)$ and $\lambda \in \mathbb{D}$, let

$$\widetilde{\phi}(\lambda) = \langle \phi k_\lambda, k_\lambda \rangle = \int_{\mathbb{D}} \phi(z) \frac{(1-|\lambda|^2)^2}{|1-\bar{\lambda} z|^4} dA(z).$$

For more details, see [12]. A nice survey of earlier known results relating to the unitary operators on the Hilbert space can be found in [3, 4, 10, 11].

Theorem 1 **([4])** *Let $T, V, W \in \mathcal{L}(H)$, where T is a paranormal contraction operator, V is a coisometry, and W has a dense range. Assume that $TW = WV$. Then T is unitary. In particular, if W is injective and has a dense range, then V is also a unitary operator.*

Theorem 2 **([11])** *Let $A, V, X \in \mathcal{L}(H)$ be such that V, X are isometries and A^* is p-hyponormal. If $VX = XA$, then A is unitary.*

Theorem 3 **([4])** *Let $T, S, W \in \mathcal{L}(H)$ where W has a dense range. Assume that $TW = WS$ and $T^*W = WS^*$. Then T is unitary if S is unitary.*

Theorem 4 **([3])** *Let T be a k-paranormal contraction, and let*

$$M = \{x \in H : \|T^{*n} x\| \geq \varepsilon_x > 0 \ \ for \ \ n = 1, 2, \cdots\}.$$

Then $T|M$ is unitary.

Corollary 1 ([3]) *Let A be a k-paranormal contraction, let B be a right invertible operator with a power bounded right inverse B_1, and let X be an operator with dense range such that $AX = XB$. Then A is unitary.*

Theorem 5 ([10]) *If T is a k-paranormal contraction operator, V has a right inverse V_r, which is power bounded, and operator W has a dense range such that $TW = WV_r$, and then $T^*W = WV_r$. Moreover, T is unitary.*

Main results

Proposition 1 *Let $\phi \in L^\infty(\mathbb{D})$ be such that $\|\phi\|_\infty \leq 1$. Suppose that $\zeta = \inf_{z\in\mathbb{D}} |\widetilde{\phi}(z)| > 0$ and there exists a sequence $\mu = \{\psi_n\}_{n\geq 0} \subset \mathbb{D}$ such that*

$$\lambda_\phi^\mu = \left(\sum_{n=0}^{\infty}(1 - 2Re\ (\widetilde{\phi}(\bar{\psi_n})E_{n,\phi}) + |\widetilde{\phi}(\psi_n)|^2)\right)^{\frac{1}{2}} < \infty. \tag{1.1}$$

If $\zeta > \lambda_\phi^\mu$, and $T_\phi^{-1} = T_{\phi o \phi_z}$ for some $z \in \mathbb{D}$, then T_ϕ is unitary.

Proof From [5] it follows that the Toeplitz operator T_ϕ is invertible on $L_a^2(\mathbb{D})$, since $T_\phi^{-1} = T_{\phi o\phi_z} = U_z T_\phi U_z$ for some $z \in \mathbb{D}$. This implies T_ϕ^{-1} is unitarily equivalent to T_ϕ. Therefore, $\|T_\phi^{-1}\| = \|T_\phi\| \leq \|\phi\|_\infty \leq 1$. Thus, for any $f \in L_a^2(\mathbb{D}), \|f\| = \|T_\phi^{-1}T_\phi f\| \leq \|T_\phi f\| \leq \|f\|$. Hence, $\|T_\phi f\| = \|f\|$, which implies $T_{\overline{\phi}}T_\phi = I$. Furthermore, since $\|T_{\overline{\phi}}\| = \|T_\phi\| \leq \|\phi\|_\infty \leq 1$ and $\|(T_{\overline{\phi}})^{-1}\| = \|(T_\phi^{-1})^*\| = \|T_\phi^{-1}\| = \|T_\phi\| \leq \|\phi\|_\infty \leq 1$, we get for any $g \in L_a^2(\mathbb{D}), \|g\| = \|(T_{\overline{\phi}})^{-1}T_{\overline{\phi}}g\| \leq \|T_{\overline{\phi}}g\| \leq \|g\|$. Thus, $\|T_{\overline{\phi}}g\| = \|g\|$, which implies that $T_\phi T_{\overline{\phi}} = I$. Hence, T_ϕ is unitary. □

Theorem 6 *Let $\phi \geq 0$. If $V \in \mathcal{L}(L_a^2(\mathbb{D}))$ be an isometry such that $T_\phi - V \in S_p, 1 \leq p < \infty$. Then V is unitary.*

Proof The Schatten ideal $S_p, 1 \leq p < \infty$ is a two-sided ideal. Given that $T_\phi - V \in S_p, 1 \leq p < \infty$. Hence, $T_\phi V - V^*T_\phi = V^*(V - T_\phi) - (V^* - T_\phi)V \in S_p$. Hence, $T_\phi^2 - I = (V^* + T_\phi)(T_\phi - V) + T_\phi V - V^*T_\phi \in S_p$. As T_ϕ is positive, $(T_\phi + I)$ is invertible and so $T_\phi - I = (T_\phi^2 - I)(T_\phi + I)^{-1} \in S_p, 1 \leq p < \infty$. So $V - I = (T_\phi - I) - (T_\phi - V) \in S_p$. Hence, $V - I = A$, say, is compact. Now $V = I + A$ is isometric and hence one-one, so $\ker(I + A) = \{0\}$ and hence -1 is not an eigenvalue of the compact operator A; otherwise, $\ker(I + A)$ would contain a nonzero eigenvector of A with corresponding eigenvalue -1. Therefore, by the Fredholm alternative [6], $A - (-1)I(= V)$ is invertible and hence unitary. □

Theorem 7 *Let $\phi \in H(\mathbb{D})$ and $\psi \in L^\infty(\mathbb{D})$ such that $\psi \geq 0$. If $T_\psi \leq Re(C_\phi^* T_\psi)$,*

$\lim_{|z|\to 1^-} \dfrac{1 - |z|^2}{1 - |\phi(z)|^2} = 0$, *and* $\frac{1+|\phi(0)|}{1-|\phi(0)|} \leq 1$*; then C_ϕ is unitary.*

Proof For $f \in L_a^2(\mathbb{D})$, by Heinz inequality [7], we obtain

$$\begin{aligned}\langle T_\psi f, f\rangle &\leq \langle Re(C_\phi^* T_\psi) f, f\rangle \\ &= Re\langle C_\phi^* T_\psi f, f\rangle \\ &\leq |\langle C_\phi^* T_\psi f, f\rangle| \\ &= |\langle T_\psi f, C_\phi f\rangle| \\ &\leq \langle T_\psi f, f\rangle^{\frac{1}{2}} \langle T_\psi C_\phi f, C_\phi f\rangle^{\frac{1}{2}}.\end{aligned}$$

Hence, $\langle T_\psi f, f\rangle \leq \langle C_\phi^* T_\psi C_\phi f, f\rangle$ for all $f \in L_a^2(\mathbb{D})$, so $T_\psi \leq C_\phi^* T_\psi C_\phi$. The operator $T_\psi^{\frac{1}{2}} C_\phi$ is compact [12] since $\lim\limits_{|z|\to 1^-} \dfrac{1-|z|^2}{1-|\phi(z)|^2} = 0$. Let $M = T_\psi^{\frac{1}{2}} C_\phi$. Then

$$MM^* = T_\psi^{\frac{1}{2}} C_\phi C_\phi^* T_\psi^{\frac{1}{2}} \leq T_\psi.$$

Hence, $0 \leq C_\phi^* T_\psi C_\phi - T_\psi \leq C_\phi^* T_\psi C_\phi - T_\psi^{\frac{1}{2}} C_\phi C_\phi^* T_\psi^{\frac{1}{2}} = M^*M - MM^*$. That is, the operator M is hyponormal. Hence, M is normal [2] as M is compact. Therefore, $T_\psi = C_\phi^* T_\psi C_\phi = T_\psi^{\frac{1}{2}} C_\phi C_\phi^* T_\psi^{\frac{1}{2}}$ and hence C_ϕ^* is an isometry on $\overline{Ran\ (T_\psi)}$. Furthermore, T_ψ commutes with C_ϕ and also with C_ϕ^*, so

$$C_\phi^* C_\phi T_\psi = C_\phi^* T_\psi C_\phi = T_\psi = T_\psi C_\phi C_\phi^*.$$

Hence, C_ϕ is unitary. □

Theorem 8 *Let $\phi \in L^\infty(\mathbb{D})$ be such that $\phi \geq 0$ with $\|\phi\|_\infty \leq 1$ and $\|T_{1+\phi}\| < 1$. Then T_ϕ can expressed as the mean of two unitary operators.*

Proof Since $\phi \geq 0$, T_ϕ is positive on $L_a^2(\mathbb{D})$. Then, by ([1], Theorem 3.1), for every unitary operator U on $L_a^2(\mathbb{D})$, we obtain, $\|U - T_\phi\| \leq \|I + T_\phi\| = \|T_{1+\phi}\| < 1$. Since $\|U - T_\phi\| < 1$, that implies $\|I - U^* T_\phi\| < 1$ so that $U^* T_\phi$ and T_ϕ are invertible. Let $T_\phi = VQ$ be the polar decomposition of T_ϕ with V as partial isometry and Q as positive operator on $L_a^2(\mathbb{D})$. Since T_ϕ is invertible, V is unitary and Q is a positive invertible operator on the Bergman space $L_a^2(\mathbb{D})$.

Since $\|T_\phi\| \leq 1$, that implies $\|Q\| \leq 1$. Therefore, $I - Q^2$ is a positive operator and $\|I - Q^2\| \leq 1$. Let us define $W_1 = Q + i(I - Q^2)^{\frac{1}{2}}$ and $W_2 = Q - i(I - Q^2)^{\frac{1}{2}}$. One can easily observe that $W_1^* = W_2$ and $W_1 W_1^* = Q^2 + I - Q^2 = I$. Similarly, $W_1^* W_1 = I$. Hence, $W_1 W_1^* = W_1^* W_1 = I$ and also $W_2 W_2^* = W_2^* W_2 = I$. That implies that W_1 and W_2 are two unitary operators on the Bergman space $L_a^2(\mathbb{D})$. Therefore, $T_\phi = VQ = V(\frac{W_1 + W_2}{2}) = \frac{1}{2}(VW_1 + VW_2) = \frac{V_1 + V_2}{2}$ where $V_1 = VW_1$ and $V_2 = VW_2$ are two unitary operators on $L_a^2(\mathbb{D})$. The result follows. □

Definition 1 *An operator $T \in \mathcal{L}(H)$ is a **Fredholm** operator if and only if range of T is closed, dim $\ker T$ is finite, and dim $\ker T^*$ is finite.*

Let $\mathcal{F}(H)$ denote the collection of Fredholm operators on H. Recall that the index of an operator $T \in \mathcal{L}(H)$ denoted as $i(T)$ is a function from $\mathcal{F}(H)$ to $\mathbb{Z}$ defined by $i(T) = \dim\ \ker T$ - $\dim\ \ker T^*$. For more details, see [9].

Corollary 2 *Let $\phi \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$. If $T_\phi \in \mathcal{L}(L^2_a(\mathbb{D}))$ has index zero then the Toeplitz operator T_ϕ can be expressed as the mean of two unitary operators.*

Proof Since $\phi \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$, so $\|T_\phi\| \leq \|\phi\|_\infty \leq 1$. Hence, $\|T_\phi\| \leq 1$. Let $T_\phi = UQ$ be the polar decomposition of T_ϕ where U is a partial isometry and Q is a positive operator on $L^2_a(\mathbb{D})$. If a Toeplitz operator T_ϕ with symbol ϕ has index zero then $\dim(\ker(T_\phi)) = \dim(\ker(T^*_\phi))$. Thus, the partial isometry U of an operator T_ϕ can be extended to a unitary operator. Therefore, the corollary is evident from the above Theorem 8. □

Corollary 3 *Let $\phi \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$. If $\|U_z - T_\phi\| < 1$, then the Toeplitz operator T_ϕ can be expressed as $\frac{1}{4}$ times the alternating finite series of four unitary operators. That is, $T_\phi = \sum_{k=1}^{4} \frac{(-1)^{k+1}}{4} U_k$ where U_k are unitary operators.*

Proof Since $\|\phi\|_\infty \leq 1$, so $\|T_\phi\| \leq \|\phi\|_\infty \leq 1$. Given that $\|U_z - T_\phi\| < 1$, then by ([8], Corollary-1) T_ϕ is invertible. Let $T_\phi = VQ$ be the polar decomposition of T_ϕ with V as partial isometry and Q as positive operator on $L^2_a(\mathbb{D})$. Since T_ϕ is invertible, so V is unitary and Q is a positive invertible operator on the Bergman space $L^2_a(\mathbb{D})$.

Since $\|T_\phi\| \leq 1$, that implies $\|Q\| \leq 1$. Therefore, $I - Q^2$ is a positive operator and $\|I - Q^2\| \leq 1$. Let us define $W_1 = Q + i(I - Q^2)^{\frac{1}{2}}$, $W_2 = -Q + i(I - Q^2)^{\frac{1}{2}}$, $W_3 = Q - i(I - Q^2)^{\frac{1}{2}}$, and $W_4 = -Q - i(I - Q^2)^{\frac{1}{2}}$. One may observe that $W_1^* = W_3, W_2^* = W_4$ and $W_1W_1^* = I, W_1^*W_1 = I$. Similarly, $W_2W_2^* = I, W_2^*W_2 = I$, $W_3W_3^* = I, W_3^*W_3 = I$, and $W_4W_4^* = I, W_4^*W_4 = I$. Hence, W_1, W_2, W_3 and W_4 are unitary operators on the Bergman space $L^2_a(\mathbb{D})$. Therefore, $T_\phi = VQ = V(\frac{W_1 - W_2 + W_3 - W_4}{4}) = \frac{1}{4}(VW_1 - VW_2 + VW_3 - VW_4) = \frac{V_1 - V_2 + V_3 - V_4}{4}$ where $V_1 = VW_1, V_2 = VW_2, V_3 = VW_3$, and $V_4 = VW_4$ are four unitary operators on $L^2_a(\mathbb{D})$. Hence, the result follows. □

Corollary 4 *If $W \in \mathcal{L}(L^2_a(\mathbb{D}))$ with $\|W\| \leq 1$ is of finite rank then WW^* and W^*W are unitarily equivalent.*

Proof Assume that $W \in \mathcal{L}(L^2_a(\mathbb{D}))$ and $\|W\| \leq 1$. Let $W = VQ$ be the polar decomposition of W with V as a partial isometry and Q is a positive operator on the Bergman space. Since the operator W is of finite rank, so $\dim(\ker W) = \dim(\ker W^*)$. Therefore, by using Corollary 2, we can conclude that the partial isometry V of the polar decomposition W extends to the unitary operator. Now

$$\begin{aligned} V^*WW^*V &= V^*VQQ^*V^*V \\ &= Q^2 \\ &= Q^*IQ \\ &= Q^*V^*VQ \\ &= W^*W. \end{aligned}$$

□

Theorem 9 *For a Toplitz operator* $T_\phi \in \mathcal{L}(L^2_a(\mathbb{D}))$, *let* $T^*_\phi T_\phi = S \oplus 0$ *defined on* $L^2_a(\mathbb{D}) = \overline{Range\ T^*_\phi} \oplus \ker T_\phi$ *and* $T_\phi T^*_\phi = T \oplus 0$ *defined on* $L^2_a(\mathbb{D}) = \overline{Range\ T_\phi} \oplus \ker T^*_\phi$. *Then* S *and* T *are unitarily equivalent.*

Proof Since $\overline{Range\ T^*_\phi} = \overline{Range\ (T^*_\phi T_\phi)^{\frac{1}{2}}}$ and $\overline{Range\ T_\phi} = \overline{Range\ (T_\phi T^*_\phi)^{\frac{1}{2}}}$ we may define $V : \overline{Range\ T^*_\phi} \to \overline{Range\ T_\phi}$ by $V((T^*_\phi T_\phi)^{\frac{1}{2}} f) = T_\phi f$ for $f \in L^2_a(\mathbb{D})$ and $W : \overline{Range\ T_\phi} \to \overline{Range\ T^*_\phi})$ by $W((T_\phi T^*_\phi)^{\frac{1}{2}} g) = T^*_\phi g$ for $g \in L^2_a(\mathbb{D})$. Then V and W are surjective isometries satisfying

$$\begin{aligned}
\langle V(T^*_\phi T_\phi)^{\frac{1}{2}} f, (T^*_\phi T_\phi)^{\frac{1}{2}} g\rangle &= \langle T_\phi f, (T_\phi T^*_\phi)^{\frac{1}{2}} g\rangle \\
&= \langle f, T^*_\phi (T_\phi T^*_\phi)^{\frac{1}{2}} g\rangle \\
&= \langle f, (T^*_\phi T_\phi)^{\frac{1}{2}} T^*_\phi g\rangle \\
&= \langle (T^*_\phi T_\phi)^{\frac{1}{2}} f, W(T_\phi T^*_\phi)^{\frac{1}{2}} g\rangle \quad \text{for} \quad \text{all } f, g \in L^2_a(\mathbb{D}).
\end{aligned}$$

Thus, $V = W^*$. We have

$$\begin{aligned}
(V^*TV)(T^*_\phi T_\phi)^{\frac{1}{2}} f &= WTT_\phi f \\
&= W(T_\phi T^*_\phi) T_\phi f \\
&= W(T_\phi T^*_\phi)^{\frac{1}{2}} (T_\phi T^*_\phi)^{\frac{1}{2}} T_\phi f \\
&= T^*_\phi (T_\phi T^*_\phi)^{\frac{1}{2}} T_\phi f \\
&= (T^*_\phi T_\phi)(T^*_\phi T_\phi)^{\frac{1}{2}} f \\
&= S(T^*_\phi T_\phi)^{\frac{1}{2}} f,
\end{aligned}$$

which shows that $V^*TV = S$, completing the proof. □

Corollary 5 *Let* $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$. *If* $\langle TU_z k_z, k_z\rangle = \langle Sk_z, U_z k_z\rangle$ *for all* $z \in \mathbb{D}$ *then* $|PS|^2$ *is unitarily equivalent to* $|QT|^2$ *for any isometries* P *and* Q *in* $\mathcal{L}(L^2_a(\mathbb{D}))$.

Proof Suppose $\langle TU_z k_z, k_z\rangle = \langle Sk_z, U_z k_z\rangle$ for all $z \in \mathbb{D}$, and then $\langle U_z S k_z, k_z\rangle = \langle TU_z k_z, k_z\rangle$ for all $z \in \mathbb{D}$. That is, $TU_z = U_z S$. Thus, $S = U_z T U_z$ for all $z \in \mathbb{D}$. Therefore, $S^*S = U_z T^* T U_z$. Now $U_z |QT|^2 U_z = U_z T^* Q^* Q T U_z = U_z T^* T U_z = S^* S = S^* P^* P S = |PS|^2$ for any isometries P and Q in $\mathcal{L}(L^2_a(\mathbb{D}))$. □

References

[1] Aiken JG, Erdos JA, Goldstein JA. Unitary approximation of positive operators. Illinois J Math 1980; 24: 61-72.

[2] Douglas RG. Banach Algebra Techniques in Operator Theory. New York, NY, USA: Academic Press, 1972.

[3] Duggal BP. Contractions with a unitary part. J London Math Soc 1985; 31: 131-136.

[4] Goya E, Saito T. On intertwining by an operator having a dense range. Tohoku Math J 1981; 33: 127-131.

[5] Gurdal M, Sohret F. Some results for Toeplitz operators on the Bergman space. Appl Math Comput 2011; 218: 789-793.

[6] Halmos PR. Hilbert Space Problem Book. 2nd ed. New York, NY, USA: Springer-Verlag, 1982.

[7] Heinz E. On an Inequality for Linear Operators in Hilbert Space. Report on Operator Theory and Group Representations. Washington, DC, USA: National Academy of Sciences-National Research Council, 1995.

[8] Lin CS. The unilateral shift and a norm equality for bounded linear operators. P Am Math Soc 1999; 127: 1693-1696.

[9] Olsen CL. Unitary approximation. J Funct Anal 1989; 85: 392-419.

[10] Patel JM, Sheth IH. On intertwining by an operator having a dense range. Indian J Pure Ap Mat 1983; 14: 1077-1082.

[11] Rashid MHM. Some conditions on non-normal operators which imply normality. Thai J Math 2010; 8: 185-192.

[12] Zhu K. Operator Theory in Function Spaces. New York, NY, USA: Marcel Dekker, 1990.