Computation of conditional expectation based on the multidimensional J-process using Malliavin calculus related to pricing American options

Mohamed KHARRAT*

Department of Mathematics, Laboratory of Probability and Statistics, Faculty of Sciences of Sfax,
Sfax University, Tunisia

Abstract: In this work, we extend the uni-dimensional results, already found by Jerbi and Kharrat, for the multidimensional case: we compute the Malliavin weights related to the conditional expectation $\mathbb{E}(P_t(X_t)|(X_s))$ for $0 \leq s \leq t$, where the only state variable follows a multidimensional J-process.

Key words: Malliavin calculus, J-law, J-process, multidimensional J-process, conditional expectation, pricing American option

1. Introduction

Malliavin calculus is an especially promising tool to compute the value of the conditional expectation in order to resolve many problems in the field of financial mathematics. In 2005, Bally et al. [1] initiated the study of the evaluation of the American option pricing and they developed a theoretical expression of the following conditional expectation:

$$\mathbb{E}(P_t(X_t)|(X_s)) \tag{1}$$

where $0 \leq s \leq t$, P_t is the price of an American option at time t, and X_t is generated by the following process:

$$\begin{aligned} dX_t &= rX_tdt + \sigma X_t dW_t \\ X_0 &= x \ , \end{aligned}$$

where $x \in \mathbb{R}_+$, r represents the interest rate considered as constant, σ denotes the volatility, and W is a Brownian motion.

In 2011, Jerbi introduce the J-law and the J-process [3] as a generalization of the geometric Brownian motion. Then, in his paper [4], he elaborated a new closed form solution for pricing European options as an extension of the Black and Scholes model and showed that his model was equivalent to Heston's stochastic volatility model [2]. After that, Jerbi and Kharrat [5] showed the equivalence between the one-dimensional model (under J-process) and the stochastic volatility model. By using the J-process as one of the underlying assets, rather than the Brownian motion, the innovation consisted of extending Bally et al.'s work in order to consider both the skewness and kurtosis effects.

In their paper, Bally et al. [1] developed the previous problem for the multidimensional case where they assumed that X_t was generated by the multi-Brownian motion. However, their results did not take into

account the skewness and kurtosis effects. That is why their results were not in accordance with the reality of the financial market. In order to overcome this shortcoming, this model should be extended by considering the volatility as a stochastic state variable.

As an extension of the work by Jerbi and Kharrat [5] for the one-dimensional J-process, our aim is to give a formula of the conditional expectation given by (1) for the multidimensional J-process.

To do this, we first recall some concepts of the J-law and the J-process [2,5].

Definition 1 *Let V be a random variable. V is said to follow a standard J-law:*

$$V \sim J(\lambda, \theta)$$

if its distribution density is defined as follows:

$$f(v, \lambda, \theta) = \frac{1}{Jer(\lambda, \theta)\sqrt{2\pi}} e^{-\frac{1}{2}v^2} N(\lambda v + \theta) \ ,$$

where λ and θ are two reals and $N(.)$ represents the cumulative function of the standard normal distribution, and where the quantity $Jer(\lambda, \theta)$ is defined as follows: $Jer(\lambda, \theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}v^2} N(\lambda v + \theta) dv$.

In fact, for $\lambda = 0$, the J-law represents nothing else but the standard normal law.

Using the J-law, Jerbi proposed a new stochastic process as a generalization of the geometric Brownian motion [3]. Then Kharrat improved Jerbi's definition of the previous process (see [6], page 21).

Definition 2 *Let $(\Omega, \mathfrak{F}, P, (\mathfrak{F}_t))$ be a filtered probability space. A stochastic process $(X_t)_{t \geq 0}$ is said to follow a J-process if we have:*

- *X_t is an $\mathfrak{F}_t$-adapted and continuous stochastic process;*

- *for $s < t$, $X_t - X_s$ follows M_{t-s};*

- *$dX_t = \alpha(X_t, t)dt + \beta(X_t, t)dM_t$,*

where α and β are functions of both X_t and time t. If we put $dM_t = U\sqrt{dt}$, where U is a standard random variable based on the above defined J-law: $U = \frac{V - E(V)}{\sigma(V)}$, where $V \sim J(\lambda, \theta)$, $E(V) = \lambda Z(\lambda, \theta)$,

$\sigma_V^2 = 1 - \frac{\lambda^2 \theta}{1+\lambda^2} Z(\lambda, \theta) - \lambda^2 Z^2(\lambda, \theta)$, and $Z(\lambda, \theta) = \frac{e^{-\frac{\theta^2}{2(1-\lambda^2)}}}{Jer(\lambda, \theta)\sqrt{2\pi(1-\lambda^2)}}$.

Then the J-process can also be written as follows:

$$dX_t = \alpha(X_t, t)dt + \beta(X_t, t)U\sqrt{dt}.$$

The outline of this paper is as follows. In the second section, we present the hypothesis and the nature of the problem. In the third section, we present our theoretical framework followed by our results. The fourth section will be reserved for the conclusion.

2. Nature of the problem

In the following, we extend the definition of the J-process to the multidimensional case.

Definition 3 *Let $(\Omega, \mathfrak{F}, P, (\mathfrak{F}_t))$ be a filtered probability space. A multidimensional J-process is a stochastic process $X = (X_t)_{t \in [0, +\infty[}$ in $\mathbb{R}^d$ such that*

- *X is an $\mathfrak{F}_t$-adapted and continuous stochastic process;*

- *for $s < t$, $X_t^i - X_s^i$ has J-law distribution, i.e. follows M_{t-s}, and it is independent of $\mathfrak{F}_s$.*

Let X denote the underlying asset price process, following a multidimensional J-process defined by

$$
\begin{aligned}
dX_t &= rX_t dt + \sigma X_t dM_t \\
X_0 &= x \ ,
\end{aligned}
$$

where $x \in \mathbb{R}_+^d$, $r \in \mathbb{R}_+^d$, with $r_i = r_0$ for any $i = 1, \cdots, d$, and r_0 represents the interest rate at the initial time considered as constant, σ denotes the $d \times d$ volatility matrix assumed to be nondegenerate and a subtriangle matrix, and M is a d-dimensional J-process.

Hence, any component of X_t can be written as

$$
X_t^i = x_i exp\left(\left(r_i - \frac{1}{2} \sum_{j=1}^i \sigma_{ij}^2 \right) t + \sum_{j=1}^i \sigma_{ij} M_t^j \right) \ , \qquad \text{with } i = 1, \cdots, d. \tag{2}
$$

In order to price the American option, we must evaluate this conditional expectation

$$
\mathbb{E}(P_t(X_t)|X_s = \alpha),
$$

where $0 \le s \le t$, $\alpha \in \mathbb{R}_+^d$, and P_t represents the American option price at time t, which is an $\mathbb{R}^d$ measurable function.

In the following section, we will present our theoretical framework followed by our main results in order to solve the above problem.

3. Theoretical framework

Let $l_t = (l_t^1, \cdots, l_t^d)$ be a fixed C^1 function. Moreover, let us set the following:

$$
\widetilde{X}_t^i = x_i exp\left(\left(r_i - \frac{1}{2} \sum_{j=1}^i \sigma_{ij}^2 \right) t + l_t^i + \sigma_{ii} M_t^i \right) \ , \qquad \text{with } i = 1, \cdots, d. \tag{3}
$$

As a first result, we study a transformation allowing us to handle the new process $\widetilde{X}$ instead of X.

Proposition 1 *For any $t \ge 0$, there exists a function $F_t(.) : \mathbb{R}_+^d \to \mathbb{R}_+^d$ such that F_t is invertible, and*

$$
\begin{aligned}
X_t &= F_t(\widetilde{X}_t) \\
\widetilde{X}_t &= F_t^{-1}(X_t).
\end{aligned}
$$

Proof *Let* t, l, *and* x *be fixed. From Equation (3), we may deduce that*

$$M_t^i = \frac{1}{\sigma_{ii}} \left(ln\frac{\widetilde{X}_t^i}{x_i} - \left(r_i - \frac{1}{2} \sum_{j=1}^i \sigma_{ij}^2 \right) t - l_t^i \right)$$

and, by inserting this in Equation (2), we get

$$X_t^i = x_i exp \left(\left(\left(r_i - \frac{1}{2} \sum_{j=1}^i \sigma_{ij}^2 \right) t - \sum_{j=1}^i \frac{\sigma_{ij}}{\sigma_{jj}} \left(\left(r_j - \frac{1}{2} \sum_{j=1}^i \sigma_{jj}^2 \right) t + l_t^j \right) \right) \prod_{j=1}^i (\frac{\widetilde{X}_t^i}{x_i})^{\frac{\sigma_{ij}}{\sigma_{jj}}} .$$

Hence, after setting $\widetilde{\sigma}_{ij} = \frac{\sigma_{ij}}{\sigma_{jj}}$ *with* $i, j = 1, \cdots, d$, *let* $F_t = (F_t^1, \cdots, F_t^d)$ *satisfy*

$$ln(F_t(y)) = -\widetilde{\sigma} l_t + \widetilde{\sigma} lny + (I - \widetilde{\sigma}) \left(lnx + (r - \frac{1}{2}\sigma^2)t \right),$$

where its inverse function is given by

$$ln(F_t^{-1}(z)) = l_t + \widetilde{\sigma}^{-1} lnz + (I - \widetilde{\sigma}^{-1}) \left(lnx + (r - \frac{1}{2}\sigma^2)t \right).$$

Let us note that $\widetilde{\sigma}^{-1}$ is easy to compute because $\widetilde{\sigma}$ is a triangular matrix. Moreover, $\widetilde{\sigma}^{-1}$ is also triangular and $(\widetilde{\sigma}^{-1})_{ii} = 1$ for any i. Hence, the function F_t and its inverse $G_t = F_t^{-1}$ (such that $X_t = F_t(\widetilde{X}_t)$ and $\widetilde{X}_t = G_t(X_t)$) are respectively given by

$$F_t^i(y) = y_i \left(exp(-\sum_{j=1}^i \widetilde{\sigma}_{ij} l_t^j) \right) \prod_{j=1}^{i-1} \left(\frac{y_i e^{-(r_j - \frac{1}{2}\sum_{j=1}^i \sigma_{jj}^2)t}}{x_j} \right)^{\widetilde{\sigma}_{ij}} , \qquad with \ i = 1, \cdots, d \ and \ y \in \mathbb{R}_+^d \qquad (4)$$

and

$$G_t^i(z) = z_i exp(l_t^i) \prod_{j=1}^{i-1} \left(\frac{z_i e^{-(r_j - \frac{1}{2}\sum_{j=1}^i \sigma_{jj}^2)t}}{x_j} \right)^{\widetilde{\sigma}_{ij}^{-1}} , \qquad with \ i = 1, \cdots, d \ and \ z \in \mathbb{R}_+^d . \qquad (5)$$

By using the process $\widetilde{X}$, mainly the fact that all its components are independent, we can easily get a first formula for the conditional expectation starting from the one-dimensional case, by using the results already found by Jerbi and Kharrat [5].

Now we are ready to present and demonstrate the following theorem.

Theorem 1 *Let* $0 \le s \le t$ *be fixed and let* $X_t = X_s W$, *with* $W = e^{(r - \frac{1}{2}\sigma^2)(t-s) + \sigma(M_t - M_s)}$. *Let us assume that* W *is independent of* X_s *and that its density is* $\Gamma(\gamma(W))$.

For any $\mathbb{R}^d$ *-measurable function* P_t, *and for* $\alpha \in \mathbb{R}_+^d$, *we have*

$$\mathbb{E}(P_t(X_t)|X_s = \alpha) = \frac{\mathbb{E}\left(H(X_s - \alpha)\Upsilon_s[P_t](X_t)\right)}{\mathbb{E}\left(H(X_s - \alpha)\Upsilon_s[1](X_t)\right)} ,$$

where

$$\Upsilon_s[f](X_t) = \prod_{i=1}^{d} \frac{(\sigma_{M_s})^2 f(X_t^i)}{\sigma X_s \sqrt{t-s}} \left(M_s + 2\frac{\sigma\sqrt{t-s}}{\sigma_{M_s}} - \lambda\frac{N'(\lambda M_s + \theta)}{N(\lambda M_s + \theta)} + \frac{\sigma}{\sigma_{M_s}} W \frac{\gamma'(W)\Gamma'(\gamma(W))}{\Gamma(\gamma(W))} \right) .$$

Let us note that $N(\ )$ represents the cumulative function of the standard normal distribution and H is the Heaviside function, with $\widetilde{X}_s = G_s(X_s)$ and $\widetilde{\alpha} = G_s(\alpha)$.

Proof Let us set $\widetilde{P}_t(y) = P_t \circ F_t(y)$; $y \in \mathbb{R}_+^d$, F_t being defined in (4).

Since $X_t = F_t(\widetilde{X}_t)$ for any t, we can easily deduce that

$$\mathbb{E}(P_t(X_t)|X_s = \alpha) = \mathbb{E}(\widetilde{P}_t(\widetilde{X}_t)|\widetilde{X}_s = G_s(\alpha)) .$$

Then, by setting $\widetilde{\alpha} = G_s(\alpha)$, it is sufficient to prove that

$$\mathbb{E}(\widetilde{P}_t(\widetilde{X}_t)|\widetilde{X}_s = \widetilde{\alpha}) = \frac{\mathbb{E}\left(H(\widetilde{X}_s - \widetilde{\alpha})\Upsilon_s[\widetilde{P}_t](\widetilde{X}_t)\right)}{\mathbb{E}\left(H(\widetilde{X}_s - \widetilde{\alpha})\Upsilon_s[1](\widetilde{X}_t)\right)} .$$

Let us assume that $\widetilde{P}_t(\widetilde{X}_t) = \widetilde{P}_t^1(\widetilde{X}_t^1).\widetilde{P}_t^2(\widetilde{X}_t^2)\ldots\widetilde{P}_t^d(\widetilde{X}_t^d)$, i.e. $\widetilde{P}_t$ can be separated in the product of d-measurable functions, where each one depends only on a single variable.

In such a case, we clearly have

$$\mathbb{E}(\widetilde{P}_t(\widetilde{X}_t)|\widetilde{X}_s = \widetilde{\alpha}) = \prod_{i=1}^{d} \mathbb{E}(\widetilde{P}_t^i(\widetilde{X}_t^i)|\widetilde{X}_s^i = \widetilde{\alpha}_i) .$$

$\square$

Now it is easy to show that, for each $\widetilde{X}_t^i$, we can apply the result already found by Jerbi and Kharrat in [5]. Then we get the following result:

$$\begin{aligned}
\mathbb{E}(\widetilde{P}_t(\widetilde{X}_t)|\widetilde{X}_s = \widetilde{\alpha}) &= \prod_{i=1}^{d} \mathbb{E}(\widetilde{P}_t^i(\widetilde{X}_t^i)|\widetilde{X}_s^i = \widetilde{\alpha}_i) \\
&= \prod_{i=1}^{d} \frac{\mathbb{E}\left(H(\widetilde{X}_s^i - \widetilde{\alpha}^i)\Upsilon_s^i[\widetilde{P}_t^i](\widetilde{X}_t^i)\right)}{\mathbb{E}\left(H(\widetilde{X}_s^i - \widetilde{\alpha}^i)\Upsilon_s^i[1](\widetilde{X}_t^i)\right)} ,
\end{aligned}$$

where

$$\Upsilon_s^i[\widetilde{P}_t^i](\widetilde{X}_t^i) = \frac{(\sigma_{M_s})^2 \widetilde{P}_t^i(\widetilde{X}_t^i)}{\sigma X_s \sqrt{t-s}} \left(M_s + 2\frac{\sigma\sqrt{t-s}}{\sigma_{M_s}} - \lambda\frac{N'(\lambda M_s + \theta)}{N(\lambda M_s + \theta)} + \frac{\sigma}{\sigma_{M_s}} W \frac{\gamma'(W)\Gamma'(\gamma(W))}{\Gamma(\gamma(W))} \right) ,$$

and

$$\Upsilon_s^i[1](\widetilde{X}_t^i) = \frac{(\sigma_{M_s})^2}{\sigma X_s \sqrt{t-s}} \left(M_s + 2\frac{\sigma\sqrt{t-s}}{\sigma_{M_s}} - \lambda\frac{N'(\lambda M_s + \theta)}{N(\lambda M_s + \theta)} + \frac{\sigma}{\sigma_{M_s}} W \frac{\gamma'(W)\Gamma'(\gamma(W))}{\Gamma(\gamma(W))} \right) .$$

By using the independence of the components of $\widetilde{X}$, we can easily infer that

$$\widetilde{\Upsilon}_s[\widetilde{P}_t](\widetilde{X}_t) = \prod_{i=1}^{d} \Upsilon_s^i[\widetilde{P}_t^i](\widetilde{X}_t^i) \ ,$$

and

$$\widetilde{\Upsilon}_s[1](\widetilde{X}_t) = \prod_{i=1}^{d} \Upsilon_s^i[1](\widetilde{X}_t^i).$$

Important remark

We should note that

- When we take $\lambda = 1$ and $\theta = 0$, we return to Bally et al.'s results [1].

- When $d = 1$, we return to Jerbi and Kharrat's results [5].

4. Conclusion

In this paper, we have performed two extensions. The first one is the extension of the work performed by Bally et al. [1] in order to take into account both the skewness and kurtosis effects, allowing us to compute the conditional expectation related to the pricing problem of the American put option. The second one consists of the extension of Jerbi and Kharrat's work [5] from the one-dimensional case to the multidimensional one. By using Malliavin calculus, the conditional expectation can also be written as a suitable ratio of ordinal expectations. With this new formula, the previous conditional expectation becomes much easier to compute, by using the Monte Carlo method. Moreover, the calculation of the conditional expectation, which represents the aim of our paper, can be applied in any problems including the skewness and kurtosis effects.

Acknowledgments

References

[1] Bally V, Caramellino L, Zanette A. Pricing and hedging American options by Monte Carlo methods using a Malliavin calculus approach. Monte Carlo Methods and Applications 2005; 11: 97-133.

[2] Heston SL. Closed form solution for options with stochastic volatility with application to bonds and currency options. Rev Financ Stud 1993; 6: 327-343.

[3] Jerbi Y. New statistical law named J-law and its features. Far East J Appl Math 2011; 50: 41-56.

[4] Jerbi Y. A new closed-form solution as an extension of the Black-Scholes formula allowing smile curve plotting. Quant Financ 2013; 15: 2041-2052.

[5] Jerbi Y, Kharrat M. Conditional expectation determination based on the J-process using Malliavin calculus applied to pricing American options. J Stat Comput Sim 2014; 84: 2465-2473.

[6] Kharrat M. American Option Pricing Using Malliavin Calculus. Saarbrcken, Germany: Lambert Academic Publishing, 2014.

Appendix

The proof of Equation (2) is similar to the case of the Black and Scholes equation.

An underlying asset following a multidimensional J-process is generated by this dynamic:

$$dX_t^i = X_t^i \left(r_i dt + \sum_{j=1}^{i} \sigma_{ij} dM_t^j \right) , \qquad with \ i = 1, \cdots, d.$$

Based on the It formula related to the J-process [4] and knowing the fact that this process is Martingale we get

$$X_t^i = x_i exp \left(\left(r_i - \frac{1}{2} \sum_{j=1}^{i} \sigma_{ij}^2 \right) t + \sum_{j=1}^{i} \sigma_{ij} M_t^j \right) , \qquad with \ i = 1, \cdots, d.$$