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A bivariate sampling series involving mixed partial derivatives

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Abstract: Recently Fang and Li established a sampling formula that involves only samples from the function and its first partial derivatives for functions from Bernstein space, $B^p_\sigma(\mathbb{R}^2)$. In this paper, we derive a general bivariate sampling series for the entire function of two variables that satisfy certain growth conditions. This general bivariate sampling formula involves samples from the function and its mixed and nonmixed partial derivatives. Some known sampling series will be special cases of our formula, like the sampling series of Parzen, Peterson and Middleton, and Gosselin. The truncated series of this formula are used to approximate functions from the Bernstein space so we establish a bound for the truncation error of this series based on localized sampling without decay assumption. Numerically, we compare our approximation results with the results of Fang and Li's sampling formula. Our formula gives us highly accurate approximations in comparison with the results of Fang and Li's formula.

Key words: Sampling series, contour integral, truncation error

1. Introduction

Denote by $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, $n \in \mathbb{N}$, the Banach space of all complex-valued Lebesgue measurable functions f of n-variables such that $|f|^p$ is integrable, with usual norm $\|.\|_p$. The Bernstein space, $B^p_\sigma(\mathbb{R}^n)$, $1 \le p < \infty$, is the class of all entire functions of exponential type σ, which belong to $L^p(\mathbb{R}^n)$ when restricted to $\mathbb{R}^n$. In other words, the Bernstein space, $B^p_\sigma(\mathbb{R}^n)$, is the class of all entire functions of n-variables that satisfy the growth condition

$$|f(\mathbf{z})| \le \sup_{\mathbf{x}\in\mathbb{R}^n} |f(\mathbf{x})| \exp \sigma \left(\sum_{k=1}^{n} |\Im z_k| \right), \quad \mathbf{z} := (z_1, \dots, z_n) \in \mathbb{C}^n, \tag{1.1}$$

and belong to $L^p(\mathbb{R}^n)$ when restricted to $\mathbb{R}^n$. Here $\Im z$ denotes the imaginary part of z. According to Schwartz's theorem cf. [21, p. 109],

$$B^p_\sigma(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : \text{supp}\, \widehat{f} \subset [-\sigma,\sigma]^n \right\},$$

where $\widehat{f}$ is the Fourier transform of f in the sense of generalized functions. For $f \in B^p_\sigma(\mathbb{R}^n)$, since the Fourier transform of f vanishes outside $[-\sigma,\sigma]^n$, we say that f is a bandlimited function with bandwidth σ. In particular, the space $B^2_\sigma(\mathbb{R}^n)$ is called Paley–Wiener space.

The sampling that uses samples from a function f of one variable and its derivatives up to r was first given by Linden and Abramson in 1960)[18]; for more detail, see [14, 16, 24]. The generalized sampling series

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for functions from the Bernstein space, $B_\sigma^p(\mathbb{R})$, states that if $f \in B_\sigma^p(\mathbb{R}^2)$, then it has the generalized sampling expansion involving derivatives, see, e.g., [25, 29, 31],

$$f(z) = \sum_{n=-\infty}^{\infty} \sum_{i+j+l=r} f^{(i)}(nh) \frac{\sin^{r+1}\left(\pi h^{-1} z\right)}{i!\,\ell!\,\,(z-nh)^{j+1}} \left[\frac{d^\ell}{dz^\ell}\left(\frac{z-nh}{\sin\left(\pi h^{-1} z\right)}\right)^{r+1}\right]_{z=nh}, \tag{1.2}$$

where $z \in \mathbb{C}$, $h := (r+1)\pi/\sigma$ and $r \in \mathbb{N}_\circ$. Series (1.2) converges uniformly on any compact subset of $\mathbb{C}$. The authors of [8, 29] studied the truncation error of the generalized sampling series (1.2) on a complex domain and a modification of this series with a Gaussian multiplier is given in [7, 9]. The special cases of (1.2) when $r = 0$ and $r = 1$ are useful in the approximation theory and its applications, cf., e.g., [2, 3, 6, 26, 27].

There are many results of multidimensional sampling series, cf., e.g., [15, 17, 20]. To the best of our knowledge, the first multidimensional sampling series using values from the function and its partial derivatives was introduced by Montgomery in 1965 [20]. A more general form of double sampling involving values of partial and mixed partial derivatives was given by Horng [15]. Recently, Fang and Li introduced a multidimensional version of the Hermite sampling theorem involving only samples from all the first partial derivatives for functions from Bernstein space $B_\sigma^p(\mathbb{R}^n)$, cf. [11, 17]. The bivariate sampling of Fang and Li states that if $f \in B_\sigma^p(\mathbb{R}^2)$, then we have the sampling series

$$\begin{aligned} f(x,y) \;=\; & \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left\{ f\left(\frac{2n\pi}{\sigma}, \frac{2m\pi}{\sigma}\right) + \left(x - \frac{2n\pi}{\sigma}\right) f'_x\left(\frac{2n\pi}{\sigma}, \frac{2m\pi}{\sigma}\right) \right. \\ & \left. + \left(y - \frac{2m\pi}{\sigma}\right) f'_y\left(\frac{2n\pi}{\sigma}, \frac{2m\pi}{\sigma}\right)\right\} \operatorname{sin}^2\left((\sigma/2)x - n\pi\right) \operatorname{sin}^2\left((\sigma/2)y - m\pi\right), \end{aligned} \tag{1.3}$$

where $(x, y) \in \mathbb{R}^2$ and the sinc function is defined as

$$\sin t = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Series (1.3) converges absolutely and uniformly on $\mathbb{R}^2$ [11]. In fact, we can verify that Fang and Li's formula, (1.3), is justified also on $\mathbb{C}^2$ and converges uniformly on any compact subset of $\mathbb{C}^2$. Recently, Asharabi and Prestin introduced a modification of series (1.3) with a bivariate Gaussian multiplier, cf. [10].

Motivated by formula (1.2) and the Fang–Li formula, (1.3), we derive a general bivariate sampling series for some classes of entire functions that satisfy some growth conditions. This new sampling expansion involving samples from the function and its mixed and nonmixed partial derivatives. The sampling series of Parzen, Peterson and Middleton, and Gosselin [12, 22, 23] will be special cases of our formula. To derive the desired formula, we develop the contour integrals technique [25] for functions of two variables.

We organize this paper as follows: the next section is devoted to some auxiliary results that will be used in the proofs of Sections 3 and 4. In Section 3, we present general bivariate sampling formulas with some useful special formulas. The truncation error bound of the general bivariate sampling is established in Section 4 for functions from the Bernstein space, $B_\sigma^p(\mathbb{R}^2)$. Numerical examples and comparisons are given in the last section.

Now we state some results that we will use later on. Assuming that $f(\zeta)$ has s-derivatives, then the

Leibniz formula for the product of several functions [28] is given by

$$\frac{d^s}{d\zeta^s} f^{r+1}(\zeta) = \sum_{k_1+k_2+\ldots+k_{r+1}=s} \frac{s!}{k_1!k_2!\ldots k_{r+1}!} \prod_{j=1}^{r+1} f^{(k_j)}(\zeta), \tag{1.4}$$

where the summation is taken over all partitions $(k_1, k_2, \ldots, k_n)$ of n into nonnegative integers k_j, $j = 1, 2, \ldots, r+1$ and $f^{(k)} := d^k f/d\zeta^k$. Assuming that $1/f$ has s-derivatives, then we have [13, p. 22]

$$\frac{d^s}{d\zeta^s}\left(\frac{1}{f(\zeta)}\right) = \sum \frac{(-1)^m\; s!m!}{k_1!k_2!\ldots k_\ell!\; f^{m+1}(\zeta)} \prod_{j=1}^{\ell} \left(\frac{f^{(j)}(\zeta)}{j!}\right)^{k_j}, \tag{1.5}$$

where the symbol Σ indicates summation over all solutions in nonnegative integers of the equation $\sum_{j=1}^{\ell} j\, k_j = s$ and $m = \sum_{j=1}^{\ell} k_j$.

2. Auxiliary results

In this section, we introduce some auxiliary results that we will use in proofs of main results in Section 3 and Section 4. Let S_1 be the set

$$S_1 := \{z \in \mathbb{C} : |z| \leq M \quad \text{and} \quad z \neq nh, \quad n \in \mathbb{Z}\}, \tag{2.1}$$

where $M > 0$, $r \in \mathbb{N}$ and $h = (r+1)\pi/\sigma$. Choose $z \in S_1$ and define a function g_r on the complex plane by

$$g_r(\zeta) = \frac{f(\zeta)}{(\zeta - z)\sin^{r+1}(\pi h^{-1}\zeta)}.$$

Lemma 2.1 *Let f be an entire function that satisfies one of the following growth conditions:*

$$|f(\zeta)| \leq \frac{C_f\, e^{\sigma|\Im\zeta|}}{1 + |\Re\zeta|}, \qquad |f(\zeta)| \leq \frac{C_f\, e^{\sigma|\Im\zeta|}}{1 + |\Im\zeta|}, \tag{2.2}$$

and let R_N be the rectangular path whose vertices are $\pm\tau_N \pm \mathrm{i}\tau_N$, $\tau_N = (N + 1/2)h$. Then the integral $\int_{R_N} g_r(\zeta)\, \mathrm{d}\zeta$ converges to zero uniformly on S_1 as $N \to \infty$.

Proof We prove this result only when f satisfies the first growth condition and the other proof is similar. Choose $N \in \mathbb{N}$ so large that $\tau_N \geq 2M$. Observing that for any $t \geq \delta > 0$

$$\sinh(t) \geq \alpha_\delta \mathrm{e}^t, \tag{2.3}$$

where $\alpha_\delta = \frac{\mathrm{e}^{2\delta}-1}{2\mathrm{e}^{2\delta}}$, we get, for $x \in \mathbb{R}$,

$$\left|\sin\left(\pi h^{-1}(x + \mathrm{i}\tau_N)\right)\right| \geq \left|\sinh(\pi h^{-1}\tau_N)\right| \geq \alpha_{\pi\delta/h}\, \mathrm{e}^{\pi h^{-1}\tau_N}. \tag{2.4}$$

Similarly, we have for $|y| > \delta$

$$\left|\sin\left(\pi h^{-1}(\tau_N + \mathrm{i}y)\right)\right| \geq \alpha_{\pi\delta/h}\, \mathrm{e}^{\pi h^{-1}|y|}. \tag{2.5}$$

For $z \in S_1$, $\tau_N \geq 2M$, and $x, y \in \mathbb{R}$ we have

$$|x + \mathrm{i}\tau_N - z| \geq \tau_N - M \qquad \text{and} \qquad |\tau_N + \mathrm{i}y - z| \geq \tau_N - M. \tag{2.6}$$

Let $R_{N,1}$ be the upper horizontal path, $R_{N,2}$ the right vertical path, $R_{N,3}$ the lower horizontal path, and $R_{N,4}$ the left vertical path of R_N. On $R_{N,1}$, ζ is written as

$$\zeta = x + \mathrm{i}\tau_N, \qquad -\tau_N \leq x \leq \tau_N,$$

and $g_r(\zeta)$ is bounded by

$$|g_r(x + \mathrm{i}\tau_N)| \leq \frac{C_f}{\alpha_{\pi\delta/h}(1+|x|)(\tau_N - M)},$$

where we have used the first growth of (2.2) and inequalities (2.4) and (2.6). Therefore,

$$\int_0^{\tau_N} |g_r(x + \mathrm{i}\tau_N)| \,\mathrm{d}x \leq \frac{C_f \ln(1+\tau_N)}{\alpha_{\pi\delta/h}(\tau_N - M)}. \tag{2.7}$$

On $R_{N,2}$, ζ is expressed as

$$\zeta = \tau_N + \mathrm{i}y, \qquad -\tau_N \leq y \leq \tau_N,$$

and $g_r(\zeta)$ is bounded by

$$|g_r(\tau_N + \mathrm{i}y)| \leq \frac{C_f}{\alpha_{\pi\delta/h}(1+\tau_N)(\tau_N - M)},$$

where we have used the first growth of (2.2), (2.5), and inequalities (2.5) and (2.6). Thus,

$$\int_0^{\tau_N} |g_r(\tau_N + \mathrm{i}y))| \,\mathrm{d}y \leq \frac{C_f}{\alpha_{\pi\delta/h}(\tau_N - M)}. \tag{2.8}$$

By the same computation it follows from (2.7) and (2.8) that

$$\int_{R_N} |g_r(\zeta)| \,\mathrm{d}\zeta \leq \frac{4C_f}{\alpha_{\pi\delta/h}} \left\{ \frac{\ln(1+\tau_N)}{\tau_N - M} + \frac{1}{\tau_N - M} \right\},$$

and $\int_{R_N} |g_r(\zeta)| \,\mathrm{d}\zeta$ converges to zero uniformly on S_1 as $N \to \infty$. □

For $z, w, \zeta, \eta \in \mathbb{C}$, we defined the following entire function:

$$\begin{aligned} \rho_r(z, w, \zeta, \eta) : \quad = \quad & \sin^{r+1}(\pi h^{-1} z) \sin^{r+1}(\pi h^{-1} \eta) + \sin^{r+1}(\pi h^{-1} w) \sin^{r+1}(\pi h^{-1} \zeta) \\ & - \sin^{r+1}(\pi h^{-1} z) \sin^{r+1}(\pi h^{-1} w). \end{aligned} \tag{2.9}$$

Lemma 2.2 *Let f be an entire function of two variables. For all $i, j \in \mathbb{N}_\circ$ such that $i, j \leq r$ and $n, m \in \mathbb{Z}$, we have*

$$\left[\frac{\partial^{i+j}}{\partial \eta^j \partial \zeta^i} f(\zeta, \eta) \rho_r(z, w, \zeta, \eta) \right]_{(\zeta,\eta)=(nh,mh)} = -\sin^{r+1}(\pi h^{-1} z) \sin^{r+1}(\pi h^{-1} w) f^{(i,j)}(nh, mh), \tag{2.10}$$

where $z, w \in \mathbb{C}$ and

$$f^{(i,j)} := \frac{\partial^{i+j} f}{\partial \eta^j \partial \zeta^i}, \qquad f^{(0,0)} := f.$$

Proof Using the Leibniz formula, we get for all $i \in \mathbb{N}_\circ$ and $i \le r$

$$\frac{\partial^i}{\partial \zeta^i}\Big\{f(\zeta,\eta)\rho_r(z,w,\zeta,\eta)\Big\}_{\zeta=nh} = \sum_{k=0}^{i}\binom{i}{k} f^{(k,0)}(nh,\eta)\rho_r^{(i-k)}(z,w,nh,\eta). \tag{2.11}$$

From the definition of ρ_r, (2.9), we can get

$$\rho_r^{(i-k)}(z,w,nh,\eta) = \begin{cases} \sin^{r+1}(\pi h^{-1}z)\left\{\sin^{r+1}(\pi h^{-1}\eta) - \sin^{r+1}(\pi h^{-1}w)\right\}, & k=i, \\ 0, & k<i. \end{cases}$$

Therefore,

$$\frac{\partial^i}{\partial \zeta^i}\Big\{f(\zeta,\eta)\rho_r(z,w,\zeta,\eta)\Big\}_{\zeta=nh} = f^{(i,0)}(nh,\eta)\sin^{r+1}(\pi h^{-1}z)\Big[\sin^{r+1}(\pi h^{-1}\eta) - \sin^{r+1}(\pi h^{-1}w)\Big]. \tag{2.12}$$

Using the same arguments for the variable η, we get (2.10). □

Lemma 2.3 *For $s \in \mathbb{Z}^+$, $\zeta \in \mathbb{C}$, and $n \in \mathbb{Z}$, we have*

$$\left[\frac{d^s}{d\zeta^s}\left(\frac{1}{\sin(\zeta)}\right)\right]_{\zeta=0} = \begin{cases} 0, & s \text{ is odd}, \\ \displaystyle\sum \frac{(-1)^m\, s!m!}{k_2!k_4!\ldots k_{2\ell}!}\prod_{j=1}^{\ell}\left(\frac{(-1)^j}{(2j+1)!}\right)^{k_{2j}}, & s \text{ is even}, \end{cases} \tag{2.13}$$

where the sum is taken over all solutions in nonnegative integers of the equation $\sum_{j=1}^{\ell} 2j\, k_{2j} = s$ and $m = \sum_{j=1}^{\ell} k_{2j}$.

Proof Using the identity (1.5) for $f(\zeta) = \sin(\zeta)$, we obtain

$$\frac{d^s}{d\zeta^s}\left(\frac{1}{\sin(\zeta)}\right) = \sum \frac{(-1)^m\, s!m!}{k_1!k_2!\ldots k_\ell!\, \sin^{m+1}(\zeta)}\prod_{j=1}^{\ell}\left(\frac{\sin^{(j)}(\zeta)}{j!}\right)^{k_j}, \tag{2.14}$$

where the sum is taken over all solutions in nonnegative integers of the equation $\sum_{j=1}^{\ell} j\, k_j = s$ and $m = \sum_{j=1}^{\ell} k_j$. From [5, Eq. (22)], we have

$$\left[\frac{d^j}{d\zeta^j}\sin(\zeta)\right]_{\zeta=0} = \begin{cases} 0, & j \text{ is odd}, \\ \dfrac{(-1)^{j/2}}{j+1}, & j \text{ is even}, \end{cases} \tag{2.15}$$

When s is odd, we get at least one odd value of j for all solutions of $\sum_{j=1}^{\ell} j\, k_j = s$. Substituting from (2.15) into (2.14) and using the fact $\sin(0) = 1$, we get (2.13). □

For convenience, we set

$$\delta_s^r(\nu) := \left[\frac{d^s}{d\zeta^s}\left(\frac{\zeta-\nu h}{\sin\left(\pi h^{-1}\zeta\right)}\right)^{r+1}\right]_{\zeta=\nu h}. \tag{2.16}$$

In the following lemma, we prove that $|\delta_s^r(\nu)|$ is independent of ν.

Lemma 2.4 *Let $r,s\in\mathbb{N}_\circ$ such that $s\le r$. Then we have for all $\nu\in\mathbb{Z}$*

$$\delta_s^r(\nu) = (-1)^{(r+1)\nu}\left(\frac{h}{\pi}\right)^{r+1-s} \sum_{\substack{\ell_1+\ell_2+\ldots+\ell_{r+1}=s \\ \ell_i \text{ is even}}} \frac{s!}{\ell_1!\ell_2!\ldots\ell_{r+1}!}\prod_{i=1}^{r+1}\beta_i, \tag{2.17}$$

where

$$\beta_i = \sum \frac{(-1)^m\,(\ell_i)!(m_i)!}{K_{2,i}!K_{4,i}!\ldots K_{2\tau,i}!}\prod_{j=1}^{\tau}\left(\frac{(-1)^j}{(2j+1)!}\right)^{K_{2j,i}}, \tag{2.18}$$

such that the last sum is taken over all solutions in nonnegative integers of the equation $\sum_{j=1}^{\ell} 2j\,K_{2j,i}=\ell_i$ and $m_i=\sum_{j=1}^{\ell}K_{2j,i}$.

Proof From the definition of δ_s^r, we can see that

$$\delta_s^r(\nu) = \frac{d^s}{d\zeta^s}\left[\left(\frac{\zeta-\nu h}{\sin\left(\pi h^{-1}\zeta\right)}\right)^{r+1}\right]_{\zeta=\nu h} = (-1)^{(r+1)\nu}\left(\frac{h}{\pi}\right)^{r+1-s}\left[\frac{d^s}{d\zeta^s}\left(\frac{1}{\sin(\zeta)}\right)^{r+1}\right]_{\zeta=0}. \tag{2.19}$$

Applying the general Leibniz formula (1.4), we obtain

$$\left[\frac{d^s}{d\zeta^s}\left(\frac{1}{\sin(\zeta)}\right)^{r+1}\right]_{\zeta=0} = \sum_{\ell_1+\ell_2+\ldots+\ell_{r+1}=s}\frac{s!}{\ell_1!\ell_2!\ldots\ell_{r+1}!}\prod_{i=1}^{r+1}\left[\frac{d^{\ell_i}}{d\zeta^{\ell_i}}\left(\frac{1}{\sin(\zeta)}\right)\right]_{\zeta=0}, \tag{2.20}$$

where ℓ_j is a nonnegative number for all $j=1,2,\ldots,r+1$. Combining (2.20), (2.19), and (2.13) implies (2.17). □

3. General bivariate sampling

This section is devoted to establishing a generalized sampling formula for functions from different classes of bivariate entire functions that satisfy certain growth conditions. Some known results will be special cases of our formula. To derive this formula, we develop the contour integrals technique [25] for functions of two variables. In Section 2, we defined the set S_1 on the z-plane and here we define the set S_2 on the w-plane as follows:

$$S_2 := \{w\in\mathbb{C} : |w|\le M \quad \text{and} \quad w\ne mh, \quad m\in\mathbb{Z}\}. \tag{3.1}$$

Choose any $z\in S_1$, $w\in S_2$ and consider the kernel function

$$\mathcal{K}_r(z,w,\zeta,\eta) := \frac{f(\zeta,\eta)\rho_r(z,w,\zeta,\eta)}{(\zeta-z)(\eta-w)\sin^{r+1}(\pi h^{-1}\zeta)\sin^{r+1}(\pi h^{-1}\eta)}, \quad r\in\mathbb{N}_\circ, \tag{3.2}$$

where f is an entire function of two variables and ρ_r is defined in (2.9). The kernel $\mathcal{K}_r$ has a singularity of order one at all the points of the set $\{(z,\mathbb{C}),(\mathbb{C},w): z \text{ and } w \text{ are fixed in } \mathbb{C}\}$ and a singularity of order $r+1$ at all the points of the set $\{(nh,\mathbb{C}),(\mathbb{C},mh):\ n,m\in\mathbb{Z}\}$.

Theorem 3.1 *Let f be an entire function satisfying the following growth condition:*

$$|f(z,w)| \le \frac{C_f\, e^{\sigma(|\Im z|+|\Im w|)}}{(1+|\Re z|)\,(1+|\Re w|)}. \tag{3.3}$$

Then f can be expanded as the following bivariate sampling series

$$f(z,w)=\sum_{(n,m)\in\mathbb{Z}^2}\ \sum_{i+s+\ell=r}\ \sum_{j+\tau+k=r} f^{(i,j)}\,(nh,mh)\,\frac{\delta_\ell^r(n)\delta_k^r(m)\sin^{r+1}\left(\pi h^{-1}z\right)\sin^{r+1}\left(\pi h^{-1}w\right)}{i!\ell!j!k!(z-nh)^{s+1}(w-mh)^{\tau+1}}, \tag{3.4}$$

where $(z,w)\in\mathbb{C}^2$, $h=(r+1)\pi/\sigma$ and δ_ℓ^r is defined in (2.16). Series (3.4) converges uniformly on any compact subset of $\mathbb{C}^2$.

Proof We consider z, w, and η to be arbitrary fixed complex parameters and we consider the kernel $\mathcal{K}_r(z,w,\zeta,\eta)$ as a function of ζ. Let $\mathcal{R}_{1,N}$ be the rectangular path whose vertices are $\pm(N+\frac{1}{2})h\pm \mathrm{i}(N+\frac{1}{2})h$ on the ζ-plane. Applying the classical Cauchy integral formula on the ζ-plane, see, e.g., [1, p. 141], [19, Chapter 3], we obtain

$$\frac{1}{2\pi i}\oint_{\mathcal{R}_{1,N}}\mathcal{K}_r(z,w,\zeta,\eta)\,\mathrm{d}\zeta=\mathrm{Res}\left(\mathcal{K}_r;(z,\eta)\right)+\sum_{n=-N}^{N}\mathrm{Res}\left(\mathcal{K}_r;(nh,\eta)\right), \tag{3.5}$$

where $\mathrm{Res}\left(\mathcal{K}_r;(\cdot,\cdot)\right)$ is denoted the residue of the function $\mathcal{K}_r$ at the point $(\cdot,\cdot)$. Now we consider the right-hand side of (3.5) as a function of η and z,w are the arbitrary fixed complex parameters. Let $\mathcal{R}_{2,N}$ be the rectangular path whose vertices are $\pm(N+\frac{1}{2})h\pm \mathrm{i}(N+\frac{1}{2})h$ on the η-plane. Applying the classical Cauchy integral formula on the η-plane, we get

$$\frac{1}{(2\pi i)^2}\oint_{\mathcal{R}_{2,N}}\oint_{\mathcal{R}_{1,N}}\mathcal{K}_r(z,w,\zeta,\eta)\,\mathrm{d}\zeta\mathrm{d}\eta=\mathrm{Res}\left(\mathcal{K}_r;(z,w)\right)+\sum_{(n,m)=(-N,-N)}^{(N,N)}\mathrm{Res}\left(\mathcal{K}_r;(nh,mh)\right). \tag{3.6}$$

The residue at each point is

$$\mathrm{Res}\left(\mathcal{K}_r;(z,w)\right)=f(z,w), \tag{3.7}$$

and for $-N\le n,m\le N$

$$\mathrm{Res}\left(\mathcal{K}_r;(nh,mh)\right)=\frac{1}{(r!)^2}\lim_{\eta\to mh}\frac{\partial^r}{\partial\eta^r}\left\{\frac{D(z,w,\eta)}{(\eta-w)}\left(\frac{\eta-mh}{\sin(\pi h^{-1}\eta)}\right)^{r+1}\right\}, \tag{3.8}$$

where

$$D(z,w,\eta):=\lim_{\zeta\to nh}\frac{\partial^r}{\partial\zeta^r}\left\{\frac{f(\zeta,\eta)\rho_r(z,w,\zeta,\eta)}{(\zeta-z)}\left(\frac{\zeta-nh}{\sin(\pi h^{-1}\zeta)}\right)^{r+1}\right\}.$$

Using the general Leibniz formula and (2.16), we get

$$\begin{aligned} D(z,w,\eta) &= -\sum_{i+s+\ell=r} s!\begin{pmatrix} r \\ i,s,\ell \end{pmatrix} \frac{\left[\frac{\partial^i}{\partial\zeta^i} f(\zeta,\eta)\rho_r(z,w,\zeta,\eta)\right]_{\zeta=nh}}{(z-nh)^{s+1}} \left[\frac{d^\ell}{d\zeta^\ell}\left(\frac{\zeta-nh}{\sin\left(\pi h^{-1}\zeta\right)}\right)^{r+1}\right]_{\zeta=nh} \\ &= -r!\sum_{i+s+\ell=r}\left[\frac{\partial^i}{\partial\zeta^i} f(\zeta,\eta)\rho_r(z,w,\zeta,\eta)\right]_{\zeta=nh} \frac{\delta_\ell^r(n)}{i!\ell!(z-nh)^{s+1}}. \end{aligned} \tag{3.9}$$

Substituting from (3.9) in (3.8) and applying the general Leibniz formula and using the equalities (2.16) and (2.10) yields

$$\begin{aligned} \operatorname{Res}\left(\mathcal{K}_r;(nh,mh)\right) &= \sum_{i+s+\ell=r}\sum_{j+\tau+k=r} \frac{\delta_\ell^r(n)\delta_k^r(m)\left[\frac{\partial^{i+j}}{\partial\eta^j\partial\zeta^i} f(\zeta,\eta)\rho_r(z,w,\zeta,\eta)\right]_{(\zeta,\eta)=(nh,mh)}}{i!\ell!j!k!(z-nh)^{s+1}(w-mh)^{\tau+1}} \\ &= -\sum_{i+s+\ell=r}\sum_{j+\tau+k=r} f^{(i,j)}(nh,mh)\frac{\delta_\ell^r(n)\delta_k^r(m)\sin^{r+1}\left(\pi h^{-1}z\right)\sin^{r+1}\left(\pi h^{-1}w\right)}{i!\ell!j!k!(z-nh)^{s+1}(w-mh)^{\tau+1}}. \end{aligned} \tag{3.10}$$

Combining (3.10), (3.7), and (3.5), we get

$$\begin{aligned} f(z,w) &- \sum_{(n,m)=(-N,-N)}^{(N,N)}\sum_{i+s+\ell=r}\sum_{j+\tau+k=r} f^{(i,j)}(nh,mh)\frac{\delta_\ell^r(n)\delta_k^r(m)\sin^{r+1}\left(\pi h^{-1}z\right)\sin^{r+1}\left(\pi h^{-1}w\right)}{i!\ell!j!k!(z-nh)^{s+1}(w-mh)^{\tau+1}} \\ &= \frac{1}{(2\pi i)^2}\oint_{\mathcal{R}_{2,N}}\oint_{\mathcal{R}_{1,N}} \mathcal{K}_r(z,w,\zeta,\eta)\,\mathrm{d}\zeta\mathrm{d}\eta. \end{aligned} \tag{3.11}$$

With the use of the Cauchy integral formula in one dimension, the integral in the right-hand side of (3.11) may be expanded to obtain the following representation:

$$\begin{aligned} \oint_{\mathcal{R}_{2,N}}\oint_{\mathcal{R}_{1,N}} \mathcal{K}_r(z,w,\zeta,\eta)\,\mathrm{d}\zeta\mathrm{d}\eta &= 2\pi\mathrm{i}\sin^{r+1}(\pi h^{-1}z)\oint_{\mathcal{R}_{1,N}}\frac{f(\zeta,w)\,\mathrm{d}\zeta}{(\zeta-z)\sin^{r+1}(\pi h^{-1}\zeta)} \\ &= 2\pi\mathrm{i}\sin^{r+1}(\pi h^{-1}w)\oint_{\mathcal{R}_{2,N}}\frac{f(z,\eta)\,\mathrm{d}\eta}{(\eta-w)\sin^{r+1}(\pi h^{-1}\eta)} \\ &- \oint_{\mathcal{R}_{2,N}}\oint_{\mathcal{R}_{1,N}}\frac{\sin^{r+1}(\pi h^{-1}z)\sin^{r+1}(\pi h^{-1}w)\ f(\zeta,\eta)\,\mathrm{d}\zeta\mathrm{d}\eta}{(\zeta-z)\sin^{r+1}(\pi h^{-1}\zeta)(\eta-w)\sin^{r+1}(\pi h^{-1}\eta)}. \end{aligned}$$

Since f satisfies the growth condition (3.3), then all integrals in the right-hand side of (3.12) converge uniformly to zero on $S_1\times S_2$ as $N\to\infty$, see Lemma 2.1. Therefore, the integral in the right-hand side of (3.11) converges uniformly to zero on $S_1\times S_2$ as $N\to\infty$ and the sampling series (3.4) converges uniformly on $S_1\times S_2$. When $z=nh$, $n\in\mathbb{Z}$, it is easy to verify that the sampling expansion

$$f(nh,w)=\sum_{m=-\infty}^{\infty}\sum_{j+\tau+k=r} f^{(i,j)}(nh,mh)\frac{\delta_k^r(m)\sin^{r+1}\left(\pi h^{-1}w\right)}{j!k!(w-mh)^{\tau+1}}$$

holds and converges uniformly on any compact subset of the w-plane, see [25, Theorem 3.1]. Similarly, when $w = mh$, $m \in \mathbb{Z}$, we have the sampling representation

$$f(z, mh) = \sum_{n=-\infty}^{\infty} \sum_{i+s+\ell=r} f^{(i,j)}(nh, mh) \frac{\delta_\ell^r(n) \sin^{r+1}\left(\pi h^{-1} z\right)}{i!\ell!(z-nh)^{s+1}},$$

which converges uniformly on any compact subset of the z-plane. The equality (3.4) holds for each point $(z, w) = (nh, mh)$, $n, m \in \mathbb{Z}$. Hence, the sampling representation (3.4) holds for any $(z, w) \in \mathbb{C}^2$ such that $|z| \le M$ and $|w| \le M$. Since $M > 0$ is arbitrary, then series (3.4) is convergent uniformly on any compact subset of $\mathbb{C}^2$. □

Theorem 3.2 *Let f be an entire function satisfying the following growth condition:*

$$|f(z, w)| \le \frac{C_f\, e^{\sigma(|\Im z| + |\Im w|)}}{(1 + |\Im z|)\,(1 + |\Im w|)}. \tag{3.12}$$

Then f can be expanded as (3.4) and the series converges uniformly on any compact subset of $\mathbb{C}^2$.

Proof Following the proof of the above theorem, we can replace the growth condition (3.3) by (3.12) and obtain the expansion (3.4). □

Theorem 3.3 *Let $f \in B_\sigma^p(\mathbb{R}^2)$, $1 \le p < \infty$, and then we have the following bivariate sampling expansion:*

$$f(z, w) = \sum_{(n,m)\in\mathbb{Z}^2} \sum_{i+s+\ell=r} \sum_{j+\tau+k=r} f^{(i,j)}(n\mathfrak{h}, m\mathfrak{h}) \frac{\delta_\ell^r(n)\delta_k^r(m) \sin^{r+1}\left(\pi\mathfrak{h}^{-1} z\right) \sin^{r+1}\left(\pi\mathfrak{h}^{-1} w\right)}{i!\ell!j!k!(z-n\mathfrak{h})^{s+1}(w-m\mathfrak{h})^{\tau+1}}, \tag{3.13}$$

where $\mathfrak{h} = (r+1)\pi/\sigma'$ and $\sigma' \ge \sigma$. Series (3.13) converges uniformly on any compact subset of $\mathbb{C}^2$.

Proof Since $f \in B_\sigma^p(\mathbb{R}^2)$, then function f has the following growth:

$$|f(z, w)| \le A_f\, \mathrm{e}^{\sigma(|\Im z| + |\Im w|)}, \tag{3.14}$$

where A_f is a positive number. Let $\sigma' > \sigma$. Since f satisfies (3.14), it is easy to verify that f satisfies the following growth:

$$|f(z, w)| \le \frac{A'_f\, e^{\sigma'(|\Im z| + |\Im w|)}}{(1 + |\Im z|)\,(1 + |\Im w|)}.$$

By Theorem 3.2, the sampling expansion (3.13) holds and the series converges uniformly on any compact subsets of $\mathbb{C}^2$. Finally, it is not hard to check that the series (3.13) is valid when $\sigma = \sigma'$. □

The following known sampling series is a special case of our series (3.13):

Corollary 3.4 *Let $f \in B_\sigma^p(\mathbb{R}^2)$, $1 \le p < \infty$, and then we have*

$$f(z, w) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f\left(\frac{n\pi}{\sigma}, \frac{m\pi}{\sigma}\right) \sin(\sigma z - n\pi) \sin(\sigma w - m\pi), \quad (z, w) \in \mathbb{C}^2. \tag{3.15}$$

Series (3.15) converges uniformly on any compact subset of $\mathbb{C}^2$.

Proof Letting $r=0$ in (3.13) implies $i=s=\ell=j=\tau=k=0$, and from (2.17), we get

$$\frac{\delta_0^0(\nu)\sin\left(\pi\mathfrak{h}^{-1}\zeta\right)}{\zeta-\nu\mathfrak{h}}=\sin\left(\sigma\zeta-\nu\pi\right),\quad\forall\,\zeta\in\mathbb{C},\nu\in\mathbb{Z},$$

since $\mathfrak{h}=\pi/\sigma$. Therefore, (3.15) is proved. □

Remark 3.5 *Formula (3.15) goes back to Parzen (1956), Peterson and Middleton (1962), and Gosselin (1963) [12, 22, 23]. They proved it for real-valued functions from Paley–Wiener space, $B_\sigma^2(\mathbb{R}^2)$. Since $B_\sigma^2(\mathbb{R}^2)\subseteq B_\sigma^p(\mathbb{R}^2)$ for $p\geq 2$, then (3.15) extends the result of Parzen, Peterson and Middleton, and Gosselin for complex-valued functions from Bernstein space, $B_\sigma^p(\mathbb{R}^2)$, $1\leq p<\infty$.*

The generalized sampling series (3.13) uses samples from the function itself and its mixed and nonmixed derivatives up to order $2r$. In the special case when $r=1$, we get the following series:

Corollary 3.6 *Let $f\in B_\sigma^p(\mathbb{R}^2)$, $1\leq p<\infty$, and then we have the sampling series*

$$\begin{aligned} f(z,w) &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left\{f\left(\frac{2n\pi}{\sigma},\frac{2m\pi}{\sigma}\right)+\left(z-\frac{2n\pi}{\sigma}\right)f'_z\left(\frac{2n\pi}{\sigma},\frac{2m\pi}{\sigma}\right)+\left(w-\frac{2m\pi}{\sigma}\right)f'_w\left(\frac{2n\pi}{\sigma},\frac{2m\pi}{\sigma}\right)\right.\\ &+ \left.\left(z-\frac{2n\pi}{\sigma}\right)\left(w-\frac{2m\pi}{\sigma}\right)f''_{zw}\left(\frac{2n\pi}{\sigma},\frac{2m\pi}{\sigma}\right)\right\}\sin^2\left((\sigma/2)z-n\pi\right)\sin^2\left((\sigma/2)w-m\pi\right), \end{aligned}\tag{3.16}$$

which converges uniformly on any compact subset of $\mathbb{C}^2$.

Proof From (2.16), we have $\delta_0^1(\nu)=(\mathfrak{h}/\pi)^2$ and $\delta_1^1(\nu)=0$ for all $\nu\in\mathbb{Z}$. Since $f\in B_\sigma^p(\mathbb{R}^2)$, then the bandwidth of functions is equal to σ and hence $\mathfrak{h}=2\pi/\sigma$. Letting $r=1$ in (3.13) implies (3.16). □

Remark 3.7 *The sampling series (3.16) and Fang and Li's series (1.3) are valid for the same class of functions, $B_\sigma^p(\mathbb{R}^2)$, $1\leq p<\infty$, but our formula gives us high accuracy of approximations, see Section 5. Series (3.16) is a version of the Hermite series, which involves sample values from the function itself and its partial and mixed partial derivatives of functions $f\in B_\sigma^p(\mathbb{R}^2)$, while series (1.3) does not involve samples from mixed partial derivatives of the function.*

4. Truncation error bound

The bounds of the truncation error of sampling series (1.2), when $r=0$ and $r=1$, have been studied widely under the assumption that f satisfies a decay condition, cf., e.g., [4, 5] and their references. In 2012, Ye and Song studied the truncation error of Whittaker–Kotelnikov–Shannon sampling, which is a special case of (1.2) when $r=0$, for real-valued functions from $B_\sigma^p(\mathbb{R})$ based on localized sampling without decay assumption [30]. In this section, we extend the technique of [30] to find a bound for the truncation error of the series (3.4). For any positive number N, we truncate the series (3.4) as follows:

$$T_N^r[f](z,w)=\sum_{(n,m)\in\mathbb{Z}_N^2(z,w)}\sum_{i+s+\ell=r}\sum_{j+\tau+k=r}f^{(i,j)}\left(n\mathfrak{h},m\mathfrak{h}\right)\frac{\delta_\ell^r(n)\delta_k^r(m)\sin^{r+1}\left(\pi\mathfrak{h}^{-1}z\right)\sin^{r+1}\left(\pi\mathfrak{h}^{-1}w\right)}{i!l!j!k!(z-n\mathfrak{h})^{s+1}(w-m\mathfrak{h})^{\tau+1}},\tag{4.1}$$

where $(z, w) \in \mathbb{C}^2$ and

$$\mathbb{Z}_N^2(z,w) := \Big\{(n,m) \in \mathbb{Z}^2 : \ |\lfloor \mathfrak{h}^{-1}\Re z\rfloor - n| \le N, \ |\lfloor \mathfrak{h}^{-1}\Re w\rfloor - m| \le N\Big\}. \tag{4.2}$$

Here $\lfloor x\rfloor$ is the integer part of x. That is, if we want to estimate f, we only sum over values of f on a part of $\mathfrak{h}\mathbb{Z}$ near $\Re z$ and $\Re w$. Now we introduce two introductory lemmas that we will use in the proof of the main result of this section.

Lemma 4.1 *Let $p, q > 1$ such that $1/p + 1/q = 1$. Then for any $\zeta \in \mathbb{C}$ and $j \in \mathbb{N}_\circ$, we have*

$$\sum_{|\lfloor \mathfrak{h}^{-1}\Re\zeta\rfloor - k| > N} \left|\sin\left(\pi\mathfrak{h}^{-1}\zeta - k\pi\right)\right|^{(j+1)q} \le C_j\, e^{j\pi\mathfrak{h}^{-1}|\Im\zeta|}\, N^{-j-1/p}, \tag{4.3}$$

where $\mathfrak{h} = (r+1)\pi/\sigma'$, $\sigma' \ge \sigma > 0$ and $C_{j,p} = 2\pi^{-(j+1)}\left(\frac{pj+1}{p-1}\right)^{1/p-1}$.

Proof The sum in (4.3) is periodic with period $\mathfrak{h}$, as a function of ζ. Indeed, for all $\zeta \in \mathbb{C}$, we have

$$\sum_{\left|\lfloor \mathfrak{h}^{-1}\Re(\zeta+\mathfrak{h})\rfloor - k\right| > N} \left|\sin\left(\pi\mathfrak{h}^{-1}(\zeta+\mathfrak{h}) - k\pi\right)\right|^{(j+1)q} = \sum_{\left|\lfloor \mathfrak{h}^{-1}\Re\zeta\rfloor - (k-1)\right| > N} \left|\sin\left(\pi\mathfrak{h}^{-1}\zeta - (k-1)\pi\right)\right|^{(j+1)q}.$$

Replacing $k-1$ by k, the $\mathfrak{h}$-periodicity of the sum of (4.3) is proved. From the periodicity of the sum in (4.3), it is sufficient to prove the lemma only on the strip $S_\mathfrak{h} = \{\zeta \in \mathbb{C} : 0 \le \Re\zeta < \mathfrak{h}\}$. Using the definition of the sinc function and the facts that $|\sin(\zeta)| \le \exp(|\Im\zeta|)$ and $|\Re\zeta| \le |\zeta|$, we obtain

$$\left|\sin\left(\pi\mathfrak{h}^{-1}\zeta - k\pi\right)\right| \le \frac{\exp(\pi\mathfrak{h}^{-1}|\Im\zeta|)}{|\pi\mathfrak{h}^{-1}\Re\zeta - k\pi|}. \tag{4.4}$$

It can be verified that for any $\zeta \in S_\mathfrak{h}$

$$\begin{aligned}
\left(\sum_{\left|\lfloor \mathfrak{h}^{-1}\Re\zeta\rfloor - k\right| > N} \left|\sin\left(\pi\mathfrak{h}^{-1}\zeta - k\pi\right)\right|^{(j+1)q}\right)^{1/q} &\le \frac{e^{(j+1)\pi\mathfrak{h}^{-1}|\Im\zeta|}}{\pi^{(j+1)}}\left(\sum_{\left|\lfloor \mathfrak{h}^{-1}\Re\zeta\rfloor - k\right| > N} \frac{1}{|\mathfrak{h}^{-1}\Re\zeta - k|^{(j+1)q}}\right)^{1/q}\\
&\le \frac{e^{(j+1)\pi\mathfrak{h}^{-1}|\Im\zeta|}}{\pi^{(j+1)}}\left(\sum_{|k|>N} \frac{1}{|k|^{(j+1)q}}\right)^{1/q}\\
&\le \frac{2e^{(j+1)\pi\mathfrak{h}^{-1}|\Im\zeta|}}{\pi^{(j+1)}}\left(\int_N^\infty \frac{1}{t^{(j+1)q}}dt\right)^{1/q}\\
&= C_{j,p}\, e^{(j+1)\pi\mathfrak{h}^{-1}|\Im\zeta|}\, N^{-j-1/p},
\end{aligned} \tag{4.5}$$

where we have used that the sequence of function $1/|\mathfrak{h}^{-1}\Re\zeta - k|^{(j+1)q}$ attains its maximum in the strip $S_\mathfrak{h}$ at $\zeta = 0$ and $\lfloor \mathfrak{h}^{-1}\Re\zeta\rfloor = 0$ in $S_\mathfrak{h}$. □

Remark 4.2 *The special case of (4.3) when $\zeta \in \mathbb{R}$ and $j = 1$ was considered by Ye and Song in [30, p. 415].*

For convenience, we let

$$\mathcal{S}_{N,q}^{s,\tau}(z,w) := \left(\sum_{(n,m)\in\mathbb{Z}^2\setminus\mathbf{Z}_N^2(z,w)} \left|\sin^{s+1}\left(\pi\mathfrak{h}^{-1}z - n\pi\right)\sin^{\tau+1}\left(\pi\mathfrak{h}^{-1}w - m\pi\right)\right|^q \right)^{1/q}. \tag{4.6}$$

Lemma 4.3 *Let $p,q>1$ and $1/p+1/q=1$. For any $(z,w)\in\mathbb{C}^2$ and $s,\tau\in\mathbb{N}_\circ$, we have*

$$\mathcal{S}_{N,q}^{s,\tau}(z,w) \le p\left(C_{s,p}N^{-s} + C_{\tau,p}N^{-\tau}\right) \mathrm{e}^{(s+1)\pi\mathfrak{h}^{-1}|\Im z|}\mathrm{e}^{(\tau+1)\pi\mathfrak{h}^{-1}|\Im w|}N^{-1/p}. \tag{4.7}$$

Proof From the definitions of $\mathcal{S}_{N,q}^{s,\tau}(z,w)$ and $\mathbb{Z}_N^2(z,w)$, we obtain

$$\begin{aligned} \left(\mathcal{S}_{N,q}^{s,\tau}(z,w)\right)^q \le & \sum_{|\lfloor\mathfrak{h}^{-1}\Re z\rfloor - n|>N} \left|\sin\left(\pi\mathfrak{h}^{-1}z-n\pi\right)\right|^{(s+1)q} \sum_{m=-\infty}^{\infty}\left|\sin\left(\pi\mathfrak{h}^{-1}w-m\pi\right)\right|^{(\tau+1)q} \\ & + \sum_{n=-\infty}^{\infty}\left|\sin\left(\pi\mathfrak{h}^{-1}z-n\pi\right)\right|^{(s+1)q} \sum_{|\lfloor\mathfrak{h}^{-1}\Re w\rfloor - m|>N}\left|\sin\left(\pi\mathfrak{h}^{-1}w-m\pi\right)\right|^{(\tau+1)q}. \end{aligned} \tag{4.8}$$

Since $|\sin(\zeta)| \le \exp(|\Im\zeta|)$, then we have

$$\sum_{k=-\infty}^{\infty}\left|\sin\left(\pi\mathfrak{h}^{-1}\zeta-k\pi\right)\right|^{(j+1)q} \le \mathrm{e}^{jq\pi\mathfrak{h}^{-1}|\Im\zeta|}\sum_{k=-\infty}^{\infty}\left|\sin\left(\pi\mathfrak{h}^{-1}\zeta-k\pi\right)\right|^{q}, \tag{4.9}$$

for all $j\in\mathbb{N}_\circ$ and $\zeta\in\mathbb{C}$. Combining the inequality of [2, Lemma 2.5]

$$\sum_{k=-\infty}^{\infty}\left|\sin\left(\pi\mathfrak{h}^{-1}\zeta-k\pi\right)\right|^{q} \le p^q\ \mathrm{e}^{q\pi\mathfrak{h}^{-1}|\Im\zeta|},$$

and (4.9) with (4.8) implies (4.7). □

Theorem 4.4 *Let $f\in B_\sigma^p(\mathbb{R}^2)$, $1\le p<\infty$. Then for all $(z,w)\in\mathbb{C}^2$, we have*

$$|(f-T_N^r[f])(z,w)| \le \mathcal{B}\ \mathfrak{h}^{-2/p}\|f\|_p\ D_N^r\ \mathrm{e}^{(r+1)\pi\mathfrak{h}^{-1}(|\Im z|+|\Im w|)}\ N^{-1/p}, \tag{4.10}$$

where $\mathcal{B}$ is positive constant, $\mathfrak{h}=(r+1)\pi/\sigma'$, $\sigma'\ge\sigma$, and

$$D_N^r := \sum_{i+s+\ell=r}\frac{\sigma^i\,|\delta_\ell^r(n)|}{i!\ell!}\left(\frac{\pi}{\mathfrak{h}}\right)^{s+1}N^{-s} + \sum_{j+\tau+k=r}\frac{\sigma^j\,|\delta_k^r(n)|}{j!k!}\left(\frac{\pi}{\mathfrak{h}}\right)^{\tau+1}N^{-\tau}. \tag{4.11}$$

Proof Since $f \in B^p_\sigma(\mathbb{R}^2)$, we can apply the expansion (3.4). Together with (4.1) and the triangle inequality, we obtain

$$\begin{aligned} |(f - T^r_N[f])(z,w)| &\leq \sum_{(n,m)\notin \mathbb{Z}^2_N(z,w)} \sum_{i+s+\ell=r} \sum_{j+\tau+k=r} \left| f^{(i,j)}(n\mathfrak{h}, m\mathfrak{h}) \frac{\delta^r_\ell(n)\delta^r_k(m)\sin^{r+1}\left(\pi\mathfrak{h}^{-1}z\right)\sin^{r+1}\left(\pi\mathfrak{h}^{-1}w\right)}{i!\ell!j!k!(z-n\mathfrak{h})^{s+1}(w-m\mathfrak{h})^{\tau+1}} \right| \\ &= \sum_{i+s+\ell=r} \sum_{j+\tau+k=r} \sum_{(n,m)\notin \mathbb{Z}^2_N(z,w)} \left| f^{(i,j)}(n\mathfrak{h}, m\mathfrak{h}) \frac{\delta^r_\ell(n)\delta^r_k(m)\sin^{r+1}\left(\pi\mathfrak{h}^{-1}z\right)\sin^{r+1}\left(\pi\mathfrak{h}^{-1}w\right)}{i!\ell!j!k!(z-n\mathfrak{h})^{s+1}(w-m\mathfrak{h})^{\tau+1}} \right|, \end{aligned} \tag{4.12}$$

where the interchange of the sums is justified by the absolute convergence of the series (3.4). It is easy to see that

$$\frac{\sin^{r+1}\left(\pi\mathfrak{h}^{-1}\zeta\right)}{(\zeta-\nu\mathfrak{h})^{s+1}} = (-1)^{(s+1)\nu}(\pi\mathfrak{h}^{-1})^{s+1}\sin^{r-s}\left(\pi\mathfrak{h}^{-1}\zeta\right)\sin^{s+1}\left(\pi\mathfrak{h}^{-1}\zeta-\nu\pi\right). \tag{4.13}$$

Combining (4.13) and (4.12) with $|\sin(\zeta)| \leq \exp(|\Im\zeta|)$, we obtain

$$\begin{aligned} |(f - T^r_N[f])(z,w)| &\leq \sum_{i+s+\ell=r} \frac{A^r_\ell}{i!\ell!}\left(\frac{\pi}{\mathfrak{h}}\right)^{s+1} \mathrm{e}^{(r-s)\pi\mathfrak{h}^{-1}|\Im z|} \sum_{j+\tau+k=r} \frac{A^r_k}{j!k!}\left(\frac{\pi}{\mathfrak{h}}\right)^{\tau+1} \mathrm{e}^{(r-\tau)\pi\mathfrak{h}^{-1}|\Im w|} \\ &\quad \times \sum_{(n,m)\notin \mathbb{Z}^2_N(z,w)} \left| f^{(i,j)}(n\mathfrak{h}, m\mathfrak{h}) \sin^{s+1}\left(\pi\mathfrak{h}^{-1}z-n\pi\right)\sin^{\tau+1}\left(\pi\mathfrak{h}^{-1}w-m\pi\right)\right|, \end{aligned} \tag{4.14}$$

where we have used that the sequences $A^r_\ell := |\delta^r_\ell(n)|$ and $A^r_k := |\delta^r_k(m)|$ are independent of $n, m \in \mathbb{Z}$, see Lemma 2.4. Let $p, q > 1$ such that $1/p + 1/q = 1$. Applying the general Hölder inequality yields

$$\begin{aligned} &\sum_{(n,m)\notin \mathbb{Z}^2_N(z,w)} \left| f^{(i,j)}(n\mathfrak{h}, m\mathfrak{h}) \sin^{s+1}\left(\pi\mathfrak{h}^{-1}z-n\pi\right)\sin^{\tau+1}\left(\pi\mathfrak{h}^{-1}w-m\pi\right)\right| \\ &\qquad \leq \left(\sum_{(n,m)\in\mathbb{Z}^2} \left| f^{(i,j)}(n\mathfrak{h}, m\mathfrak{h})\right|^p \right)^{1/p} \mathcal{S}^{s,\tau}_{N,q}(z,w), \end{aligned} \tag{4.15}$$

where $\mathcal{S}^{s,\tau}_{N,q}(z,w)$ is defined in (4.6). Because of $f \in B^p_\sigma(\mathbb{R}^2) \subset B^p_{\sigma'}(\mathbb{R}^2)$, we have $f^{(i,j)} \in B^p_{\sigma'}(\mathbb{R}^2)$ for all $i, j \in \mathbb{N}$ and (see [21, pp. 123–124])

$$\left(\sum_{(n,m)\in\mathbb{Z}^2} \left| f^{(i,j)}(n\mathfrak{h}, m\mathfrak{h})\right|^p \right)^{1/p} \leq \mathcal{B}\,\mathfrak{h}^{-2/p}\|f^{(i,j)}\|_p \leq \mathcal{B}\,\mathfrak{h}^{-2/p}(\sigma')^{i+j}\|f\|_p, \tag{4.16}$$

where we used the Bernstein inequality [21, p. 116] in the last step of (4.16). □

Corollary 4.5 *Let $f \in B^p_\sigma(\mathbb{R}^2)$, $1 \leq p < \infty$. For all $(x,y) \in \mathbb{R}^2$, we have the following uniform bound:*

$$|(f - T^r_N[f])(z,w)| \leq \mathcal{B}\,\mathfrak{h}^{-2/p}\|f\|_p D^r_N N^{-1/p}, \tag{4.17}$$

where $\mathfrak{h} = (r+1)\pi/\sigma'$, $\sigma' \geq \sigma$ and D^r_N is defined in (4.11).

Remark 4.6 *Letting $r = 0$ in (4.17), we exactly get Asharabi and Prestin's bound in [10, Lemma 2.2].*

5. Numerical examples and comparisons

In this section, we discuss three examples that are devoted to a numerical comparison between the new sampling formula (3.13) and Fang and Li's sampling formula (1.3). We restrict ourselves in Example 1 to the cases $r = 1, 2$ and the other examples to the case $r = 1$. We approximate the function f at the points $(x_i, y_j) = (i - 1/2, j - 1/2)$ where $i, j \in \mathbb{Z}^+$ and we summarize the results in some tables and illustrate the absolute errors by figures. In all examples, we use $\sigma' = \sigma + 10^{-1}$ and we find that the sampling formula (3.13) gives us highly accurate approximations compared with the results of Fang and Li's formula (1.3). We truncate the series (1.3) as follows:

$$\begin{aligned} \mathcal{T}_N[f](x,y) &= \sum_{(n,m)\in\mathbb{Z}_N(x,y)} \left\{ f\left(\frac{2n\pi}{\sigma}, \frac{2m\pi}{\sigma}\right) + \left(x - \frac{2n\pi}{\sigma}\right) f'_x\left(\frac{2n\pi}{\sigma}, \frac{2m\pi}{\sigma}\right) \right. \\ &+ \left. \left(y - \frac{2m\pi}{\sigma}\right) f'_y\left(\frac{2n\pi}{\sigma}, \frac{2m\pi}{\sigma}\right) \right\} \sin^2\left((\sigma/2)x - n\pi\right) \sin^2\left((\sigma/2)y - m\pi\right), \end{aligned} \tag{5.1}$$

where $(x, y) \in \mathbb{R}^2$ and $\mathbb{Z}_N(x, y)$ is defined above.

Example 5.1 *Consider the following function from $B_1^2(\mathbb{R}^2)$:*

$$f(x,y) = \sin\left(\sqrt{x^2+1}\right) \sin\left(\sqrt{y^2+1}\right).$$

In Table 1, we approximate f using sampling formula (3.13) and Fang and Li's formula (1.3). Figures 1 and 2 show the graphs of the absolute error of formulas (1.3) and (3.13) respectively on the region $[0, \pi] \times [0, \pi]$ for $N = 12$.

Table 1. Error approximating f when $N = 12$.

$(x, x) \in \mathbb{R}^2$	Fang and Li's formula (1.3)	Sampling formula (3.13)	
	$\lvert f(x,y) - \mathcal{T}_N[f](x,y)\rvert$	$\lvert f(x,y) - T_N^1[f](x,y)\rvert$	$\lvert f(x,y) - T_N^2[f](x,y)\rvert$
$(0.5, 0.5)$	1.23678×10^{-3}	1.39765×10^{-4}	3.70169×10^{-7}
$(0.5, 1.5)$	3.76975×10^{-3}	5.58342×10^{-4}	4.73431×10^{-6}
$(1.5, 0.5)$	3.76975×10^{-3}	5.58342×10^{-4}	4.73431×10^{-6}
$(1.5, 1.5)$	1.40714×10^{-2}	6.86209×10^{-4}	6.18844×10^{-6}

Example 5.2 *In this example, we approximate the function*

$$f(x,y) = \frac{\sin(\pi x)\sin(\pi y)}{\pi^2(x^2-1)(y^2-1)} \in B_\pi^2(\mathbb{R}^2), \qquad (x,y) \in \mathbb{R}^2.$$

In Table 2, we show the numerical results with the absolute errors, and the graphs of the absolute errors are given in Figures 3 and 4.

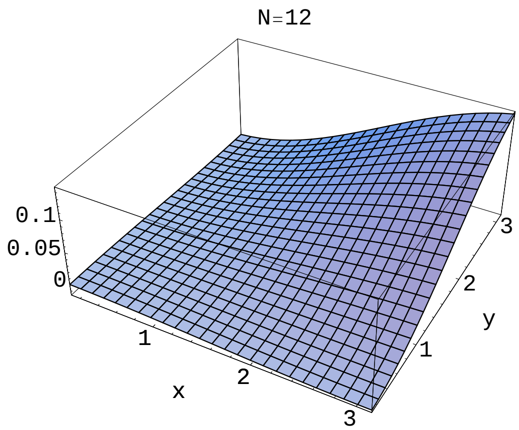

Figure 1. $f(x,y)-\mathcal{T}_{12}[f](x,y)$.

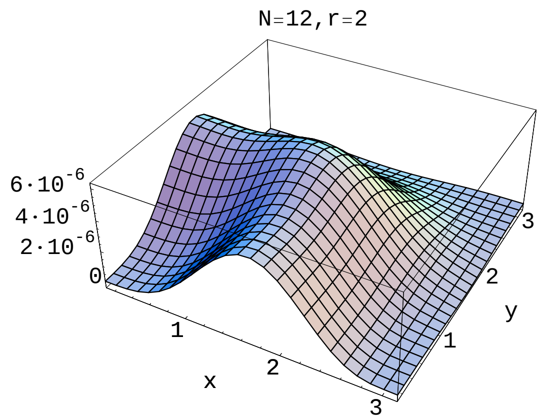

Figure 2. $f(x,y)-T_{12}^{2}[f](x,y)$.

Table 2. Error approximating f when $N=10$.

$(x,y)\in\mathbb{R}^2$	Fang and Li's formula (1.3)	Sampling formula (3.13)
	$f(x,y)-\mathcal{T}_N[f](x,y)$	$f(x,y)-T_N^1[f](x,y)$
$(0.5,0.5)$	0.180127	2.15377×10^{-6}
$(0.5,1.5)$	0.108076	3.27096×10^{-6}
$(1.5,0.5)$	0.108076	3.27096×10^{-6}
$(1.5,1.5)$	0.064846	3.14982×10^{-6}

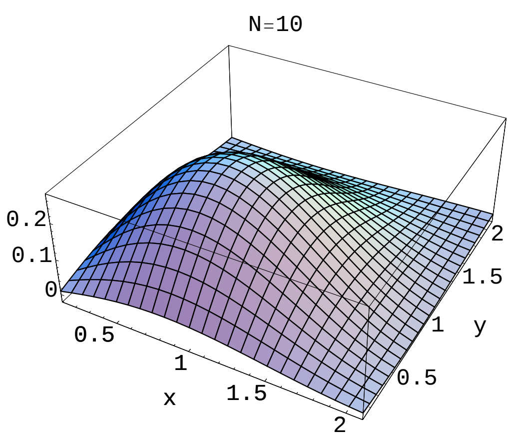

Figure 3. $f(x,y)-\mathcal{T}_{10}[f](x,y)$.

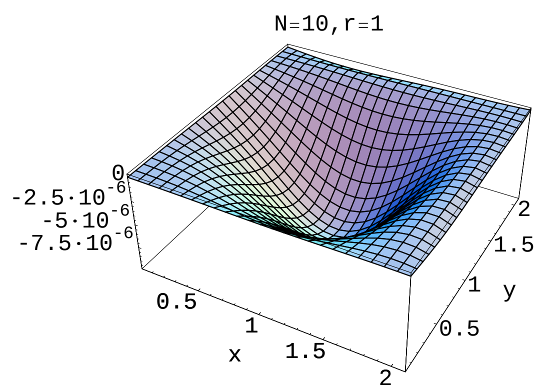

Figure 4. $f(x,y)-T_{10}^{1}[f](x,y)$.

Example 5.3 *The function*

$$f(z,w)=\sin^{(1)}(z)\ \sin^{(1)}(w),\qquad (z,w)\in\mathbb{C}^2,$$

belongs to $B_1^2(\mathbb{R}^2)$. *In this case, we apply Fang and Li's formula (1.3) because it is also justified on* $\mathbb{C}^2$ *as we mentioned in Section 1. Therefore, we approximate* f *using (3.13) and Fang and Li's formula (1.3).*

Table 3. Error approximating when $N = 15$.

$(z,w) \in \mathbb{C}^2$	Fang and Li's formula (1.3)	Sampling formula (3.13)
	$\left\|f(z,w) - \mathcal{T}_N[f](z,w)\right\|$	$\left\|f(z,w) - T_N^1[f](z,w)\right\|$
$(1+\mathrm{i}, 1+\mathrm{i})$	0.223874	1.23309×10^{-6}
$(1+\mathrm{i}, 2+\mathrm{i})$	0.270242	2.54352×10^{-6}
$(2+\mathrm{i}, 1+\mathrm{i})$	0.326213	2.54352×10^{-6}
$(2+\mathrm{i}, 2+\mathrm{i})$	0.326213	4.50832×10^{-6}

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