

Value distribution and normality of meromorphic functions involving partial sharing of small functions

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Abstract: In this paper, we generalize a value distribution result and use it to prove a normality criterion using partial sharing of small functions. Further in the sequel, various known normality criteria are improved and generalized on the domain $D := \{z : |z| < R, 0 < R \leq \infty\}$.

Key words: Normal families, meromorphic functions, differential monomials, sharing of values

1. Introduction and main results

We assume that the reader is familiar with the theory of normal families of meromorphic functions on a domain $D \subseteq \mathbb{C}$; one may refer to [5] for more information.

The idea of the sharing of values was introduced in the study of normality of families of meromorphic functions for the first time by Schwick [6] in 1989.

Two nonconstant meromorphic functions f and g are said to share a value $\omega \in \mathbb{C}$ IM (ignoring multiplicities) if f and g have the same ω -points counted with ignoring multiplicities. If multiplicities of ω -points of f and g are counted, then f and g are said to share the value ω CM. For deeper insight into the sharing of values by meromorphic functions, one may refer to [8].

In this paper all meromorphic functions are considered on $D := \{z : |z| < R, 0 < R \leq \infty\}$ excepting Theorem A and Theorem 1.1, where the domain is the whole complex plane. A meromorphic function $\omega(z)$ is said to be a *small function* of a meromorphic function $f(z)$ if $T(r, \omega) = o(T(r, f))$ as $r \rightarrow R$. Further, we say that a meromorphic function f shares a small function ω *partially* with a meromorphic function g if

$$\overline{E}(\omega, f) = \{z \in \mathbb{C} : f(z) - \omega(z) = 0\} \subseteq \overline{E}(\omega, g) = \{z \in \mathbb{C} : g(z) - \omega(z) = 0\},$$

where $\overline{E}(\omega, \phi)$ denotes the set of zeros of $\phi - \omega$ counted with ignoring multiplicities.

The function of the form $M[f] = f^{n_0}(f')^{n_1} \dots (f^{(k)})^{n_k}$ is called a *differential monomial* of f of degree $d = n_0 + n_1 + \dots + n_k$, where n_0, n_1, \dots, n_k are nonnegative integers.

In the present discussion, we have used the idea of partial sharing of small functions in the study of normality of families of meromorphic functions. One can verify that a good amount of results on normal families

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proved by using the sharing of values can be proved under a weaker hypothesis of partial sharing of values or small functions.

Lahiri and Dewan [4] proved the following result:

Theorem A *Let f be a transcendental meromorphic function and $F = (f)^{n_0}(f^{(k)})^{n_1}$, where $n_0(\geq 2)$, and n_1 and k are positive integers such that $n_0(n_0 - 1) + (1 + k)(n_0n_1 - n_0 - n_1) > 0$. Then*

$$\left[1 - \frac{1+k}{n_0+k} - \frac{n_0(1+k)}{(n_0+k)\{n_0+(1+k)n_1\}} \right] T(r, F) \leq \bar{N} \left(r, \frac{1}{F-\omega} \right) + S(r, F)$$

for any small function $\omega(\neq 0, \infty)$ of f .

It is natural to ask whether Theorem A remains valid for a general class of monomials. In this direction, we have proved that it does hold for a larger class of monomials. Precisely, we have:

Theorem 1.1 *Let f be a transcendental meromorphic function. Let*

$$F = f^{n_0}(f')^{n_1} \dots (f^{(k)})^{n_k}, \tag{1.1}$$

where k, n_0, n_1, \dots, n_k are nonnegative integers with $k \geq 1, n_0 \geq 2$, and $n_k \geq 1$ such that

$$n_0(n_0 - 1) + \sum_{j=1}^k (j+1)(n_0n_j - n_j - n_0) + (k-1)n_0 > 0. \tag{1.2}$$

Then

$$\left[1 - \frac{1 + \frac{k(k+1)}{2}}{n_0 + \frac{k(k+1)}{2}} - \frac{n_0 \left(1 + \frac{k(k+1)}{2} \right)}{\{n_0 + \frac{k(k+1)}{2}\} \{n_0 + \sum_{j=1}^k (j+1)n_j\}} + o(1) \right] T(r, F) \leq \bar{N} \left(r, \frac{1}{F-\omega} \right) + S(r, F) \tag{1.3}$$

for any small function $\omega(\neq 0, \infty)$ of f .

Note that the sum in (1.2) runs over the orders of derivatives present in F .

Note: When f has no poles then Theorem 1.1 holds without condition (1.2).

As an application of Theorem 1.1, we prove a normality criterion using the idea of partial sharing of small functions.

Theorem 1.2 *Let \mathcal{F} be a family of meromorphic functions such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least $k \geq 2$. Let n_0, n_1, \dots, n_k be nonnegative integers with $n_0 \geq 2, n_k \geq 1$ such that*

$$n_0(n_0 - 1) + \sum_{j=1}^k (j+1)(n_0n_j - n_0 - n_j) + (k-1)n_0 > 0.$$

Let $\omega(z)$ be a small function of each $f \in \mathcal{F}$ having no zeros and poles at the origin. If there exists $\tilde{f} \in \mathcal{F}$ such that $M[f]$ share ω partially with $M[\tilde{f}]$, for every $f \in \mathcal{F}$, then \mathcal{F} is a normal family.

Consequently, we can prove:

Theorem 1.3 *Let \mathcal{F} be a family of meromorphic functions such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least $k \geq 2$. Let n_0, n_1, \dots, n_k be nonnegative integers with $n_0 \geq 2, n_k \geq 1$ such that*

$$n_0(n_0 - 1) + \sum_{j=1}^k (j + 1)(n_0 n_j - n_0 - n_j) + (k - 1)n_0 > 0.$$

Let $\omega(z)$ be a small function of each $f \in \mathcal{F}$ having no zeros and poles at the origin. If $M[f]$ and $M[g]$ share ω , for each pair $f, g \in \mathcal{F}$, then \mathcal{F} is a normal family.

Corollary 1.4 *Let \mathcal{F} be a family of meromorphic functions such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least $k \geq 2$. Let n_0, n_1, \dots, n_k be nonnegative integers with $n_0 \geq 2, n_k \geq 1$ such that*

$$n_0(n_0 - 1) + \sum_{j=1}^k (j + 1)(n_0 n_j - n_0 - n_j) + (k - 1)n_0 > 0.$$

Let $\omega(z)$ be a small function of each $f \in \mathcal{F}$ having no zeros and poles at the origin. If $M[f] - \omega$ has no zero, for every $f \in \mathcal{F}$, then \mathcal{F} is a normal family.

Further, one can see that Theorem 4.1 of Hu and Meng [3] may be generalized to a class of monomials as follows:

Theorem 1.5 *Let $k \in \mathbb{N}$ and \mathcal{F} be a family of nonconstant meromorphic functions such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k . Let n_0, n_1, \dots, n_k be nonnegative integers with $n_0 \geq 2, n_k \geq 1$ such that*

$$n_0(n_0 - 1) + \sum_{j=1}^k (j + 1)(n_0 n_j - n_0 - n_j) + (k - 1)n_0 > 0.$$

Let $\omega(z)$ be a small function of each $f \in \mathcal{F}$ having no zeros and poles at the origin. If, for each $f \in \mathcal{F}$, $(M[f] - \omega)(z) = 0$ implies $|f^{(k)}(z)| \leq A$, for some $A > 0$, then \mathcal{F} is a normal family.

2. Proof of main results

Proof of Theorem 1.1: Since n_0 is positive, by [7] (Theorem 1, p. 792), f and F have the same order of growth and hence $T(r, \omega) = S(r, F)$ as $r \rightarrow \infty$. Precisely, ω is a small function of f iff ω is a small function of F .

Now, by the second fundamental theorem of Nevanlinna, for three small functions (see [2], p. 47), we have:

$$[1 + o(1)]T(r, F) \leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \omega}\right) + S(r, F). \tag{2.1}$$

Next, we have

$$\begin{aligned} \bar{N}\left(r, \frac{1}{F}\right) &\leq \bar{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^k \bar{N}_0\left(r, \frac{1}{f^{(j)}}\right) \\ &\leq \bar{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^k j \left[\bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f)\right] + S(r, f) \\ &= \bar{N}\left(r, \frac{1}{f}\right) + \frac{k(k+1)}{2} \left[\bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f)\right] + S(r, f), \end{aligned}$$

where $\bar{N}_0\left(r, \frac{1}{f^{(j)}}\right)$ is the number of those zeros of $f^{(j)}$ in $|z| \leq r$ that are not the zeros of f .

That is,

$$\bar{N}\left(r, \frac{1}{F}\right) \leq \left[1 + \frac{k(k+1)}{2}\right] \bar{N}\left(r, \frac{1}{f}\right) + \frac{k(k+1)}{2} \bar{N}(r, f) + S(r, f). \tag{2.2}$$

Also, we can see that

$$N\left(r, \frac{1}{F}\right) - \bar{N}\left(r, \frac{1}{F}\right) \geq \left[(k+1)n_0 + \sum_{j=1}^k n_j - 1\right] \bar{N}_{(k+1)}\left(r, \frac{1}{f}\right) + (n_0 - 1) \bar{N}_k\left(r, \frac{1}{f}\right), \tag{2.3}$$

where $\bar{N}_{(k+1)}\left(r, \frac{1}{f}\right)$ and $\bar{N}_k\left(r, \frac{1}{f}\right)$ are the counting functions ignoring multiplicities of those zeros of f whose multiplicity is $\geq k+1$ and $\leq k$, respectively.

Now from (2.2) and (2.3), we get

$$\begin{aligned} \bar{N}\left(r, \frac{1}{F}\right) &\leq \left[1 + \frac{k(k+1)}{2}\right] \bar{N}_{(k+1)}\left(r, \frac{1}{f}\right) \\ &+ \frac{\left[1 + \frac{k(k+1)}{2}\right]}{n_0 - 1} \left[N\left(r, \frac{1}{F}\right) - \bar{N}\left(r, \frac{1}{F}\right) - \left((k+1)n_0 + \sum_{j=1}^k n_j - 1\right) \bar{N}_{(k+1)}\left(r, \frac{1}{f}\right)\right] \\ &+ \frac{k(k+1)}{2} \bar{N}(r, f) + S(r, f). \end{aligned}$$

That is,

$$\begin{aligned} \left[1 + \frac{\left(1 + \frac{k(k+1)}{2}\right)}{n_0 - 1}\right] \bar{N}\left(r, \frac{1}{F}\right) &\leq \left(1 + \frac{k(k+1)}{2}\right) \left(1 - \frac{(k+1)n_0 + \sum_{j=1}^k n_j - 1}{n_0 - 1}\right) \bar{N}_{(k+1)}\left(r, \frac{1}{f}\right) \\ &+ \frac{1 + \frac{k(k+1)}{2}}{n_0 - 1} N\left(r, \frac{1}{F}\right) + \frac{k(k+1)}{2} \bar{N}(r, f) + S(r, f). \end{aligned}$$

Since $\overline{N}(r, f) = \overline{N}(r, F)$ and $S(r, f) = S(r, F)$, we have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{F}\right) &\leq \frac{1 + \frac{k(k+1)}{2}}{n_0 + \frac{k(k+1)}{2}} N\left(r, \frac{1}{F}\right) + \frac{\left(\frac{k(k+1)}{2}\right)(n_0 - 1)}{n_0 + \frac{k(k+1)}{2}} \overline{N}(r, f) + S(r, f) \\ &= \frac{1 + \frac{k(k+1)}{2}}{n_0 + \frac{k(k+1)}{2}} N\left(r, \frac{1}{F}\right) + \frac{\left(\frac{k(k+1)}{2}\right)(n_0 - 1)}{n_0 + \frac{k(k+1)}{2}} \overline{N}(r, F) + S(r, F). \end{aligned}$$

Therefore, (2.1) yields

$$[1 + o(1)]T(r, F) \leq \overline{N}\left(r, \frac{1}{F - \omega}\right) + \frac{1 + \frac{k(k+1)}{2}}{n_0 + \frac{k(k+1)}{2}} N\left(r, \frac{1}{F}\right) + \frac{n_0(1 + \frac{k(k+1)}{2})}{n_0 + \frac{k(k+1)}{2}} \overline{N}(r, F) + S(r, F). \tag{2.4}$$

Also, if f has a pole of multiplicity p , then F has a pole of multiplicity

$$n_0 p + n_1(p + 1) + \dots + n_k(p + k) \geq n_0 + 2n_1 + \dots + (k + 1)n_k = n_0 + \sum_{j=1}^k (j + 1)n_j,$$

and, therefore,

$$N(r, F) \geq \left[n_0 + \sum_{j=1}^k (j + 1)n_j \right] \overline{N}(r, F). \tag{2.5}$$

Finally, from (2.4) and (2.5), we find that $[1 + o(1)]T(r, F) \leq \overline{N}\left(r, \frac{1}{F - \omega}\right) + \frac{1 + \frac{k(k+1)}{2}}{n_0 + \frac{k(k+1)}{2}} N\left(r, \frac{1}{F}\right)$

$$+ \frac{n_0 \left(1 + \frac{k(k+1)}{2}\right)}{\left(n_0 + \frac{k(k+1)}{2}\right) \left(n_0 + \sum_{j=1}^k (j + 1)n_j\right)} N(r, F) + S(r, F).$$

That is,

$$\begin{aligned} &\left[1 - \frac{1 + \frac{k(k+1)}{2}}{n_0 + \frac{k(k+1)}{2}} - \frac{n_0 \left(1 + \frac{k(k+1)}{2}\right)}{\left(n_0 + \frac{k(k+1)}{2}\right) \left(n_0 + \sum_{j=1}^k (j + 1)n_j\right)} + o(1) \right] T(r, F) \\ &\leq \overline{N}\left(r, \frac{1}{F - \omega}\right) + S(r, F). \end{aligned}$$

□

For the proof of Theorem 1.2, besides Theorem 1.1, we also need the following lemma, which is a straightforward generalization of Lemma 3 in [1].

Lemma 2.1 *Let f be a nonconstant rational function with only zeros of multiplicity at least k , where $k \geq 2$. Let $n_0, n_1, n_2, \dots, n_k$ be nonnegative integers with $n_0 \geq 2$ and $n_k \geq 1$. Let $\omega \neq 0$ be a finite complex number. Then $M[f] - \omega$ has at least two distinct zeros.*

Proof of Theorem 1.2: Since normality is a local property, we may assume that $D = \mathbb{D}$. Suppose \mathcal{F} is not normal in \mathbb{D} . In particular, suppose that \mathcal{F} is not normal at $z = 0$. Then, by Zalcman’s lemma (see [9]), there exist a sequence $\{f_n\}$ of functions in \mathcal{F} , a sequence $\{z_n\}$ of complex numbers in \mathbb{D} with $z_n \rightarrow 0$ as $n \rightarrow \infty$, and a sequence $\{\rho_n\}$ of positive real numbers with $\rho_n \rightarrow 0$ as $n \rightarrow \infty$ such that the sequence $\{g_n\}$ defined by

$$g_n(z) = \rho^{-\alpha} f_n(z_n + \rho_n z); 0 \leq \alpha < k,$$

converges locally uniformly to a nonconstant meromorphic function $g(z)$ in \mathbb{C} with respect to the spherical metric. Moreover, $g(z)$ is of order at most 2. By Hurwitz’s theorem, the zeros of $g(z)$ have multiplicity at least k .

Let $\alpha = \frac{\sum_{j=1}^k j n_j}{\sum_{j=0}^k n_j} < k$. Then

$$\begin{aligned} M[g_n](z) &= (g_n(z))^{n_0} (g'_n(z))^{n_1} \cdots (g_n^{(k)}(z))^{n_k} \\ &= \rho_n^{-\alpha n_0} (f_n(z_n + \rho_n z))^{n_0} \rho_n^{-\alpha n_1 + n_1} (f'_n(z_n + \rho_n z))^{n_1} \cdots \rho_n^{-\alpha n_k + k n_k} (f_n^{(k)}(z_n + \rho_n z))^{n_k} \\ &= \rho^{-\alpha \sum_{j=0}^k n_j + \sum_{j=1}^k j n_j} (f_n(z_n + \rho_n z))^{n_0} (f'_n(z_n + \rho_n z))^{n_1} \cdots (f_n^{(k)}(z_n + \rho_n z))^{n_k} \\ &= M[f_n](z_n + \rho_n z). \end{aligned}$$

On every compact subset of \mathbb{C} that contains no poles of g , we have

$$M[f_n](z_n + \rho_n z) - \omega(z_n + \rho_n z) = M[g_n](z) - \omega(z_n + \rho_n z) \rightarrow M[g](z) - \omega_0$$

spherically uniformly, where $\omega_0 = \omega(0)$.

Since g is a nonconstant meromorphic function of order at most 2 and $\omega_0 \neq 0, \infty$, it immediately follows that $M[g] \neq \omega_0$. Using Theorem 1.1 and Lemma 2.1, $M[g] - \omega_0$ has at least two distinct zeros, say w_0 and v_0 . Choose $r > 0$ such that the open disks $D(w_0, r) = \{z : |z - w_0| < r\}$ and $D(v_0, r) = \{z : |z - v_0| < r\}$ are disjoint and their union contains no zeros of $M[g] - \omega_0$ different from w_0 and v_0 respectively. Then, by Hurwitz’s theorem, we see that for sufficiently large n , there exist points $w_n \in D(w_0, r)$ and $v_n \in D(v_0, r)$ such that

$$(M[f_n] - \omega)(z_n + \rho_n w_n) = 0,$$

and

$$(M[f_n] - \omega)(z_n + \rho_n v_n) = 0.$$

Since, by hypothesis, $M[f_n]$ share ω partially with $M[\tilde{f}]$, for every n , it follows that

$$(M[\tilde{f}] - \omega)(z_n + \rho_n w_n) = 0,$$

and

$$(M[\tilde{f}] - \omega)(z_n + \rho_n v_n) = 0.$$

By letting $n \rightarrow \infty$, and noting that $z_n + \rho_n w_n \rightarrow 0, z_n + \rho_n v_n \rightarrow 0$, we find that

$$(M[\tilde{f}] - \omega)(0) = 0.$$

Since the zeros of $M[\tilde{f}] - \omega$ have no accumulation point, $z_n + \rho_n w_n = 0$ and $z_n + \rho_n v_n = 0$ for sufficiently large n . That is, $D(w_0, r) \cap D(v_0, r) \neq \emptyset$, a contradiction. \square

Proof of Theorem 1.5: As established in the proof of Theorem 1.2, we similarly find that $M[g] \neq \omega_0$. By Theorem 1.1 and Lemma 2.6 in [10], $M[g] - \omega_0$ has at least one zero, w_0 , say. By Hurwitz's theorem, there is a sequence of complex numbers $\{w_n\}$ such that $w_n \rightarrow w_0$ as $n \rightarrow \infty$, and

$$(M[f_n] - \omega)(z_n + \rho_n w_n) = 0.$$

Again, since $k > \alpha$,

$$\begin{aligned} |g_n^{(k)}(w_n)| &= \rho_n^{k-\alpha} |f_n^{(k)}(z_n + \rho_n w_n)| \\ &\leq \rho_n^{(k-\alpha)} A \\ &= A \rho_n^{k - \frac{\sum_{j=1}^k j n_j}{\sum_{j=0}^k n_j}} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore, $g^{(k)}(w_0) = \lim_{n \rightarrow \infty} g_n^{(k)}(w_n) = 0$
 $\Rightarrow M[g](w_0) = 0 \neq \omega_0$, which is a contradiction. \square

3. Conclusions

Though our results do generalize and improve the results of Hu and Meng [3] and Ding et al. [1] when the domain D is $\{z : |z| < R, 0, R \leq \infty\}$, there seems no way of proving our results on an arbitrary domain since the idea of a small function on an arbitrary domain is not available, as far as we know. However, by making certain modifications in the proofs of results of Hu and Meng [3] and Ding et al. [1], one can easily extend and improve these results on an arbitrary domain with a shared value being a nonzero complex value. Precisely, one obtains:

Theorem 3.1 *Let \mathcal{F} be a family of nonconstant meromorphic functions on a domain D with all zeros of each $f \in \mathcal{F}$ having multiplicity at least k , where $k \geq 2$. Let $\omega \neq 0$ be a finite complex number and n_0, n_1, \dots, n_k be nonnegative integers with $n_0 \geq 2$ and $n_1 + n_2 + \dots + n_k \geq 1$. If there exists $\tilde{f} \in \mathcal{F}$ such that $M[f]$ share ω partially with $M[\tilde{f}]$ for every $f \in \mathcal{F}$, then \mathcal{F} is normal on D .*

The condition that f has only zeros of multiplicity at least k in Theorem 3.1 is sharp. For example, consider the open unit disk \mathbb{D} , an integer $k \geq 2$, a nonzero complex number ω , and the family

$$\mathcal{F} = \{f_m(z) = mz^{k-1}; m = 1, 2, 3, \dots\}.$$

Obviously, each $f_m \in \mathcal{F}$ has only a zero of multiplicity $k - 1$, and for distinct positive integers m and l we find that $f_m^2 f_m^{(k)}$ and $f_l^2 f_l^{(k)}$ share ω IM and \mathcal{F} is not normal at $z = 0$.

Also, $\omega \neq 0$ in Theorem 3.1 is essential. For example, let $\mathcal{F} = \{f_m\}$, where $f_m(z) = \frac{1}{e^{mz} + 1}; m = 1, 2, \dots$ and $z \in \mathbb{D}$. Choosing $k = 2, n = 2, n_1 = 1$, and $n_2 = 0$, we have

$$M[f_m] = f_m^2 f'_m = -\frac{me^{mz}}{(e^{mz} + 1)^4} \neq 0.$$

Thus, for any $f, g \in \mathcal{F}$, $M[f]$ and $M[g]$ share 0 IM, but we see that \mathcal{F} is not normal in \mathbb{D} .

Theorem 3.2 Let \mathcal{F} be a family of nonconstant holomorphic functions on a domain D with all zeros of each $f \in \mathcal{F}$ having multiplicity at least k , where $k \geq 2$. Let $\omega \neq 0$ be a finite complex number and n_0, n_1, \dots, n_k be nonnegative integers with $n_0 \geq 1$ and $n_1 + n_2 \dots + n_k \geq 1$. If there exists $\tilde{f} \in \mathcal{F}$ such that $M[f]$ share ω partially with $M[\tilde{f}]$ for every $f \in \mathcal{F}$, then \mathcal{F} is normal on D .

As an illustration of Theorem 3.2, we have the following example:

Example 3.3 Consider $\mathcal{F} = \{f_m(z) = me^{\frac{z}{m}} : m \in \mathbb{N}\}$, defined on \mathbb{C} . Take $k = 2, n = 1, n_1 = 0$, and $n_2 = 1$. Then

$$M[f_m] = f_m f_m'' = e^{\frac{2z}{m}},$$

and $M[f_m] = 1$ iff $\frac{2z}{m} = 2k\pi i, k \in \mathbb{Z}$ iff $z = mk\pi i$

for

$$m = 1; z = 0, \pm\pi i, \pm 2\pi i, \pm 3\pi i, \dots$$

$$m = 2; z = 0, \pm 2\pi i, \pm 4\pi i, \pm 6\pi i, \dots$$

$$m = 3; z = 0, \pm 3\pi i, \pm 6\pi i, \pm 9\pi i, \dots$$

and so on.

Thus, for each $m \geq 2$, $M[f_m]$ share 1 partially with $M[f_1]$. Since $f_m \rightarrow \infty$ uniformly on each compact subset of \mathbb{C} , it follows that \mathcal{F} is normal in \mathbb{C} .

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