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Existence and global attractivity of periodic solutions in a max-type system of difference equations

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Abstract: We consider in this paper the following system of difference equations with maximum

$$\begin{cases} x(n+1) = & \max\{f_1(n,x(n)), g_1(n,y(n))\} \\ y(n+1) = & \max\{f_2(n,x(n)), g_2(n,y(n))\} \end{cases}, \quad n = 0, 1, 2, \ldots,$$

where f_i, g_i, $i = 1, 2$, are real-valued functions with periodic coefficients. We use the Banach fixed point theorem to get a sufficient condition under which this system admits a unique periodic solution. Moreover, we show that this periodic solution attracts all the solutions of the current system. Some examples are also given to illustrate our results.

Key words: Max-type difference equations, nonautonomous difference equations, periodic solutions, Banach fixed point theorem, global attractivity

1. Introduction and preliminaries

It is a famous problem that focuses on the treatment of periodic solutions of nonautonomous difference equations with periodic coefficients. This type of equation has arisen by the observance of periodic phenomena in discrete models, especially in mathematical ecology and population dynamics. Studying the existence of periodic solutions of such equations is recently of great interest. Many mathematicians have done research on this topic and the majority of them have used in their analysis fixed point theorems such as Krasnoselskii's theorem and the theorem of Schauder (see, e.g., [1, 2, 6, 7, 9, 10, 15, 16]). However, for the question of the global attractivity of these periodic solutions, in fact, there are no general or basic results to use and there are only a few articles published on this problem. One can see for example [5, 8].

Max-type difference equations and their systems have also been around for many years and their studies have attracted the attention of several researchers; we refer the reader to [3, 11–14] and the references cited therein.

The present work is a combination of these two areas of research in the domain of difference equations. We consider the following functional nonautonomous max-type system:

$$\begin{cases} x(n+1) = & \max\{f_1(n,x(n)), g_1(n,y(n))\} \\ y(n+1) = & \max\{f_2(n,x(n)), g_2(n,y(n))\} \end{cases}, n = 0, 1, 2, \ldots, \tag{1}$$

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where the initial values $x(0), y(0)$ are real numbers, the functions

$$f_i, g_i : \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \quad \text{for} \quad i = 1, 2,$$

are ω-periodic in n and $\omega \geq 1$ is an integer, $\mathbb{R}$ is the set of real numbers and $\mathbb{N}$ is the set of nonnegative integers.

Throughout this paper we denote $\{0, 1, 2, \ldots, \omega - 1\}$ by $[0, \omega - 1]$, and for $(x, y) = \{(x(n), y(n))\}_{n \in \mathbb{N}}$ a sequence in $\mathbb{R}^2$, we define the maximum norm

$$\|(x, y)\| = \max_{n \in [0, \omega - 1]} |(x(n), y(n))|_0,$$

where $|.|_0$ denotes the infinity norm in $\mathbb{R}^2$, i.e. $|(x(n), y(n))|_0 = \max\{|x(n)|, |y(n)|\}$.

In addition, we assume that the functions f_i, g_i, $i = 1, 2$, are contraction mappings, that is, there exist constants $L_i, K_i \in [0, 1)$, $i = 1, 2$, such that for all $u, v \in \mathbb{R}$, $n \in [0, \omega - 1]$,

$$|f_i(n, u) - f_i(n, v)| \leq L_i |u - v| \text{ and } |g_i(n, u) - g_i(n, v)| \leq K_i |u - v|. \tag{2}$$

We will show that under this condition System (1) has a unique periodic solution with period ω that attracts all the solutions of this system.
Recall that a solution $\{(x(n), y(n))\}_{n \geq 0}$ of System (1) is called periodic with period ω (or ω-periodic) if there exists an integer $\omega \geq 1$ such that

$$x(n + \omega) = x(n) \text{ and } y(n + \omega) = y(n), \text{ for all } n \geq 0.$$

We say that a solution $\{(x(n), y(n))\}_{n \geq 0}$ of System (1) is eventually periodic with period ω if there exists an integer $N \geq 0$ such that $\{(x(n), y(n))\}_{n \geq N}$ is periodic with period ω, that is,

$$x(n + \omega) = x(n) \text{ and } y(n + \omega) = y(n), \text{ for all } n \geq N.$$

The following lemma describes when a solution of System (1) converges to a periodic solution. One can consult [4].

Lemma 1 *Let $\{(x(n), y(n))\}_{n \geq 0}$ be a solution of System (1). Suppose that there exist real numbers $l_0, l_1, \ldots, l_{\omega-1}$ and $k_0, k_1, \ldots, k_{\omega-1}$, such that*

$$\lim_{n \to \infty} x(\omega n + j) = l_j, \ \lim_{n \to \infty} y(\omega n + j) = k_j, \quad \text{for all} \quad j = 0, 1, \ldots, \omega - 1$$

and let $\{(\widetilde{x}(n), \widetilde{y}(n))\}_{n \geq 0}$ be the ω-periodic sequence of real numbers such that for every integer j with $0 \leq j \leq \omega - 1$, we have

$$\widetilde{x}(\omega n + j) = l_j, \ \widetilde{y}(\omega n + j) = k_j, \quad \text{for all} \quad n = 0, 1, \ldots.$$

Then the following statements are true:

- $\{(\widetilde{x}(n), \widetilde{y}(n))\}_{n \geq 0}$ *is a ω-periodic solution of System (1),*
- $\lim_{n \to \infty} (x(\omega n + j), y(\omega n + j)) = (\widetilde{x}(j), \widetilde{y}(j)), \quad$ *for* $j = 0, 1, \ldots, \omega - 1$.

Now we state Banach fixed-point theorem (also known as the contraction mapping principle), which enables us to prove the existence of a periodic solution of System (1).

Theorem 1 *Let $\mathbb{X}$ be a nonempty complete metric space with a contraction mapping $T : \mathbb{X} \to \mathbb{X}$. Then T has a unique fixed point z in $\mathbb{X}$.*

2. Main results

We start by studying the existence of periodic solutions of System (1) by using Banach fixed-point theorem. To this end, let $\mathbb{X}$ be the set of all ω-periodic sequences in $\mathbb{R}^2$, which is a Banach space with the maximum norm defined in the previous section. Thus, we need to construct a contraction mapping T on $\mathbb{X}$, and to do this let us use c, a real number, such that

$$0 < c \leq \frac{1 - \max_{1 \leq i \leq 2}\{L_i, K_i\}}{2},$$

where L_i, K_i, $i = 1, 2$, are given by (2). Rewrite the first equation of System (1) as

$$\begin{aligned} x(n+1) &= cx(n) + \max\{f_1(n, x(n)), g_1(n, y(n))\} - cx(n) \\ &= cx(n) + \max\{f_1(n, x(n)) - cx(n), g_1(n, y(n)) - cx(n)\} \end{aligned}$$

and similarly

$$y(n+1) = cy(n) + \max\{f_2(n, x(n)) - cy(n), g_2(n, y(n)) - cy(n)\}.$$

We define two mappings, T_1 and T_2, as follows:

$$(T_1(x, y))(n) = \max\{f_1(n, x(n)) - cx(n), g_1(n, y(n)) - cx(n)\}$$

and

$$(T_2(x, y))(n) = \max\{f_2(n, x(n)) - cy(n), g_2(n, y(n)) - cy(n)\},$$

for all $(x, y) \in \mathbb{X}$, $n \in \mathbb{N}$.

Arguing as in [15], we get the following lemma, which is important for our study.

Lemma 2 *Let $(x, y) = \{(x(n), y(n))\}_{n \geq 0} \in \mathbb{X}$. Then (x, y) is a solution of System (1) if and only if*

$$x(n) = (c^{-w} - 1)^{-1} \sum_{i=1}^{w} c^{-i} (T_1(x, y))(n + i - 1)$$

and

$$y(n) = (c^{-w} - 1)^{-1} \sum_{i=1}^{w} c^{-i} (T_2(x, y))(n + i - 1),$$

for all $n \in [0, \omega - 1]$.

Proof Assume that $(x, y) = \{(x(n), y(n))\}_{n \geq 0}$ is a ω-periodic solution of System (1). We will only do the proof for the first equality. The second one can be shown in a similar way and it will be omitted. We have from above

$$x(n+1) = cx(n) + (T_1(x, y))(n),$$

thus,

$$c^{-1}x(n+1)-x(n)=c^{-1}(T_1(x,y))(n),$$

$$c^{-2}x(n+2)-c^{-1}x(n+1)=c^{-2}(T_1(x,y))(n+1),$$

$$\dots$$

$$c^{-\omega}x(n+\omega)-c^{1-\omega}x(n+\omega-1)=c^{-\omega}(T_1(x,y))(n+\omega-1).$$

By summing these equations, we obtain

$$c^{-\omega}x(n+\omega)-x(n)=\sum_{i=1}^{w}c^{-i}(T_1(x,y))(n+i-1).$$

Since $x(n+\omega)=x(n)$, we get the result.
The converse implication is easily obtained and the proof is completed. □

Now we consider the map T defined on $\mathbb{X}$ as

$$(T(x,y))(n)=(c^{-w}-1)^{-1}\sum_{i=1}^{w}c^{-i}((T_1(x,y))(n+i-1),(T_2(x,y))(n+i-1)),$$

for all $n\in[0,\omega-1]$. It is clear from the previous lemma that a fixed point of T is a periodic solution of System (1).

Theorem 2 *The mapping $T:\mathbb{X}\to\mathbb{X}$ is a contraction.*

Proof First, the periodicity properties of the functions $f_i,g_i,i=1,2$, guarantee that

$$(T_i(x,y))(n+\omega)=(T_i(x,y))(n),\text{ for } i=1,2,$$

and thus

$$(T(x,y))(n+\omega)=(T(x,y))(n).$$

To show that T is a contraction, let $(x,y),(z,t)\in\mathbb{X}$. We have

$$\begin{aligned}|(T_1(x,y))(n)-(T_1(z,t))(n)| &= |\max\{f_1(n,x(n))-cx(n),g_1(n,y(n))-cx(n)\}-\\ &\quad \max\{f_1(n,z(n))-cz(n),g_1(n,t(n))-cz(n)\}|\end{aligned}$$

We distinguish four cases:
Case 1. When

$$\max\{f_1(n,x(n)),g_1(n,y(n))\}=f_1(n,x(n))$$

and

$$\max\{f_1(n,z(n)),g_1(n,t(n))\}=f_1(n,z(n)).$$

This yields

$$\begin{aligned}|(T_1(x,y))(n)-(T_1(z,t))(n)| &= |f_1(n,x(n))-cx(n)-f_1(n,z(n))+cz(n)| \\ &\leq |f_1(n,x(n))-f_1(n,z(n))|+c|x(n)-z(n)| \\ &\leq (L_1+c)|x(n)-z(n)| \\ &\leq \left(\max_{1\leq i\leq 2}\{L_i,K_i\}+c\right)\max_{n\in[0,\omega-1]}\{|x(n)-z(n)|,|y(n)-t(n)|\} \\ &\leq \left(\max_{1\leq i\leq 2}\{L_i,K_i\}+c\right)\max_{n\in[0,\omega-1]}|(x(n),y(n))-(z(n),t(n))|_0 \\ &= \left(\max_{1\leq i\leq 2}\{L_i,K_i\}+c\right)\|(x,y)-(z,t)\|.\end{aligned}$$

Case 2. When

$$\max\{f_1(n,x(n)),g_1(n,y(n))\}=f_1(n,x(n))$$

and

$$\max\{f_1(n,z(n)),g_1(n,t(n))\}=g_1(n,t(n)).$$

Here, we get

$$|(T_1(x,y))(n)-(T_1(z,t))(n)|=|f_1(n,x(n))-cx(n)-g_1(n,t(n))+cz(n)|.$$

In this case we distinguish again two cases, when

$$f_1(n,x(n))-cx(n)\geq g_1(n,t(n))-cz(n)$$

and when

$$f_1(n,x(n))-cx(n)\leq g_1(n,t(n))-cz(n).$$

Thus, either

$$\begin{aligned}|(T_1(x,y))(n)-(T_1(z,t))(n)| &= f_1(n,x(n))-cx(n)-g_1(n,t(n))+cz(n) \\ &\leq f_1(n,x(n))-cx(n)-f_1(n,z(n))+cz(n) \\ &\leq |f_1(n,x(n))-f_1(n,z(n))|+c|x(n)-z(n)| \\ &\leq (L_1+c)|x(n)-z(n)| \\ &\leq \left(\max_{1\leq i\leq 2}\{L_i,K_i\}+c\right)\|(x,y)-(z,t)\|,\end{aligned}$$

or

$$\begin{aligned}
|(T_1(x,y))(n)-(T_1(z,t))(n)| &= g_1(n,t(n))-cz(n)-f_1(n,x(n))+cx(n)\\
&\le g_1(n,t(n))-cz(n)-g_1(n,y(n))+cx(n)\\
&\le |g_1(n,t(n))-g_1(n,y(n))|+c|x(n)-z(n)|\\
&\le K_1|t(n)-y(n)|+c|x(n)-z(n)|\\
&\le (K_1+c)\max_{n\in[0,\omega-1]}\{|x(n)-z(n)|,|y(n)-t(n)|\}\\
&\le \left(\max_{1\le i\le 2}\{L_i,K_i\}+c\right)\max_{n\in[0,\omega-1]}|(x(n),y(n))-(z(n),t(n))|_0\\
&= \left(\max_{1\le i\le 2}\{L_i,K_i\}+c\right)\|(x,y)-(z,t)\|.
\end{aligned}$$

Case 3. When

$$\max\{f_1(n,x(n)),g_1(n,y(n))\}=g_1(n,y(n))$$

and

$$\max\{f_1(n,z(n)),g_1(n,t(n))\}=f_1(n,z(n)).$$

It follows that

$$|(T_1(x,y))(n)-(T_1(z,t))(n)|=|g_1(n,y(n))-cx(n)-f_1(n,z(n))+cz(n)|.$$

We examine also here two cases, when

$$g_1(n,y(n))-cx(n)\ge f_1(n,z(n))-cz(n)$$

and when

$$g_1(n,y(n))-cx(n)\le f_1(n,z(n))-cz(n).$$

Thus, either

$$\begin{aligned}
|(T_1(x,y))(n)-(T_1(z,t))(n)| &= g_1(n,y(n))-cx(n)-f_1(n,z(n))+cz(n)\\
&\le g_1(n,y(n))-cx(n)-g_1(n,t(n))+cz(n)\\
&\le |g_1(n,y(n))-g_1(n,t(n))|+c|x(n)-z(n)|\\
&\le K_1|y(n)-t(n)|+c|x(n)-z(n)|\\
&\le (K_1+c)\max_{n\in[0,\omega-1]}\{|x(n)-z(n)|,|y(n)-t(n)|\}\\
&= \left(\max_{1\le i\le 2}\{L_i,K_i\}+c\right)\|(x,y)-(z,t)\|,
\end{aligned}$$

or

$$\begin{aligned}
|(T_1(x,y))(n)-(T_1(z,t))(n)| &= f_1(n,z(n))-cz(n)-g_1(n,y(n))+cx(n)\\
&\le f_1(n,z(n))-cz(n)-f_1(n,x(n))+cx(n)\\
&\le |f_1(n,z(n))-f_1(n,x(n))|+c|x(n)-z(n)|\\
&\le (L_1+c)|x(n)-z(n)|\\
&\le \left(\max_{1\le i\le 2}\{L_i,K_i\}+c\right)\|(x,y)-(z,t)\|.
\end{aligned}$$

Case 4. When

$$\max\{f_1(n,x(n)),g_1(n,y(n))\}=g_1(n,y(n))$$

and

$$\max\{f_1(n,z(n)),g_1(n,t(n))\}=g_1(n,t(n)),$$

i.e.

$$\begin{aligned}|(T_1(x,y))(n)-(T_1(z,t))(n)| &= |g_1(n,y(n))-cx(n)-g_1(n,t(n))+cz(n)|\\ &\leq |g_1(n,y(n))-g_1(n,t(n))|+c|x(n)-z(n)|\\ &\leq K_1|y(n)-t(n)|+c|x(n)-z(n)|\\ &\leq (K_1+c)\max_{n\in[0,\omega-1]}\{|x(n)-z(n)|,|y(n)-t(n)|\}\\ &\leq \left(\max_{1\leq i\leq 2}\{L_i,K_i\}+c\right)\|(x,y)-(z,t)\|.\end{aligned}$$

Therefore, we conclude from the above cases that for all $n\in[0,\omega-1]$,

$$|(T_1(x,y))(n)-(T_1(z,t))(n)|\leq\left(\max_{1\leq i\leq 2}\{L_i,K_i\}+c\right)\|(x,y)-(z,t)\|.$$

Thus, for all $n\in[0,\omega-1]$,

$$\begin{aligned}&\left|(c^{-w}-1)^{-1}\sum_{i=1}^{w}c^{-i}[(T_1(x,y))(n+i-1)-(T_1(z,t))(n+i-1)]\right|\\ \leq\ & |(c^{-w}-1)^{-1}|\sum_{i=1}^{w}|c|^{-i}|(T_1(x,y))(n+i-1)-(T_1(z,t))(n+i-1)|\\ \leq\ & (c^{-w}-1)^{-1}\left(\sum_{i=1}^{w}c^{-i}\right)\left(\max_{1\leq i\leq 2}\{L_i,K_i\}+c\right)\|(x,y)-(z,t)\|\\ =\ & (c^{-w}-1)^{-1}\frac{c^{-w}-1}{1-c}\left(\max_{1\leq i\leq 2}\{L_i,K_i\}+c\right)\|(x,y)-(z,t)\|\\ =\ & \frac{\max_{1\leq i\leq 2}\{L_i,K_i\}+c}{1-c}\|(x,y)-(z,t)\|.\end{aligned}$$

Similarly we show that, for all $n\in[0,\omega-1]$,

$$\left|(c^{-w}-1)^{-1}\sum_{i=1}^{w}c^{-i}[(T_2(x,y))(n+i-1)-(T_2(z,t))(n+i-1)]\right|\leq\frac{\max_{1\leq i\leq 2}\{L_i,K_i\}+c}{1-c}\|(x,y)-(z,t)\|.$$

Taking $k=\frac{\max_{1\leq i\leq 2}\{L_i,K_i\}+c}{1-c}$, it follows that

$$\|T(x,y)-T(z,t)\|\leq k\|(x,y)-(z,t)\|,$$

since $0<c\leq\frac{1-\max_{1\leq i\leq 2}\{L_i,K_i\}}{2}$, then $k<1$, which completes the proof. □

Consequently, by applying Banach fixed point theorem we get the following result.

Theorem 3 *System (1) has a unique ω-periodic solution* $\{(\widetilde{x}(n),\widetilde{y}(n))\}_{n\geq 0}$.

The following discussion deals with the global behavior of solutions of System (1).

Theorem 4 *Every solution of System (1) converges to the unique ω-periodic solution* $\{(\widetilde{x}(n),\widetilde{y}(n))\}_{n\geq 0}$.

Proof Let $\{(x(n),y(n))\}_{n\geq 0}$ be an arbitrary solution of System (1). We define the sequences

$$z(n)=x(n)-\widetilde{x}(n),\ t(n)=y(n)-\widetilde{y}(n),\ n\geq 0.$$

Clearly, we have $\{(x(n),y(n))\}_{n\geq 0}$ converges to $\{(\widetilde{x}(n),\widetilde{y}(n))\}_{n\geq 0}$ if and only if $\{(z(n),t(n))\}_{n\geq 0}$ converges to $(0,0)$ as $n\longrightarrow\infty$. Thus, it is sufficient to show that

$$\lim_{n\to\infty} z(n)=\lim_{n\to\infty} t(n)=0.$$

The sequence $\{(z(n),t(n))\}_{n\geq 0}$ satisfies the two equations:

$$z(n+1)=\max\{f_1(n,z(n)+\widetilde{x}(n)),g_1(n,t(n)+\widetilde{y}(n))\}-\max\{f_1(n,\widetilde{x}(n)),g_1(n,\widetilde{y}(n))\} \tag{3}$$

and

$$t(n+1)=\max\{f_2(n,z(n)+\widetilde{x}(n)),g_2(n,t(n)+\widetilde{y}(n))\}-\max\{f_2(n,\widetilde{x}(n)),g_2(n,\widetilde{y}(n))\}. \tag{4}$$

At this point we will show that for all $n\geq 0$,

$$|z(n+1)|\leq L_1|z(n)| \ \text{ or } \ |z(n+1)|\leq K_1|t(n)|, \tag{5}$$

and in a similar way we can also prove that for all $n\geq 0$,

$$|t(n+1)|\leq L_2|z(n)| \ \text{ or } \ |t(n+1)|\leq K_2|t(n)|. \tag{6}$$

From (3), we distinguish four cases:

Case 1. When

$$\max\{f_1(n,z(n)+\widetilde{x}(n)),g_1(n,t(n)+\widetilde{y}(n))\}=f_1(n,z(n)+\widetilde{x}(n))$$

and

$$\max\{f_1(n,\widetilde{x}(n)),g_1(n,\widetilde{y}(n))\}=f_1(n,\widetilde{x}(n)).$$

Thus, (2) implies

$$|z(n+1)|=|f_1(n,z(n)+\widetilde{x}(n))-f_1(n,\widetilde{x}(n))|\leq L_1|z(n)|.$$

Case 2. When

$$\max\{f_1(n,z(n)+\widetilde{x}(n)),g_1(n,t(n)+\widetilde{y}(n))\}=f_1(n,z(n)+\widetilde{x}(n))$$

and

$$\max\{f_1(n,\widetilde{x}(n)),g_1(n,\widetilde{y}(n))\}=g_1(n,\widetilde{y}(n)),$$

i.e.

$$z(n+1)=f_1(n,z(n)+\widetilde{x}(n))-g_1(n,\widetilde{y}(n)).$$

In this case we also have two cases: $z(n+1)\geq 0$ and $z(n+1)<0$. Therefore, if $z(n+1)\geq 0$, then we have

$$z(n+1)=f_1(n,z(n)+\widetilde{x}(n))-g_1(n,\widetilde{y}(n))\leq f_1(n,z(n)+\widetilde{x}(n))-f_1(n,\widetilde{x}(n));$$

hence

$$|z(n+1)| \leq |f_1(n, z(n) + \widetilde{x}(n)) - f_1(n, \widetilde{x}(n))| \leq L_1|z(n)|.$$

Now, if $z(n+1) < 0$, then we have

$$-z(n+1) = g_1(n, \widetilde{y}(n)) - f_1(n, z(n) + \widetilde{x}(n)) \leq g_1(n, \widetilde{y}(n)) - g_1(n, t(n) + \widetilde{y}(n));$$

thus

$$|z(n+1)| \leq |g_1(n, \widetilde{y}(n)) - g_1(n, t(n) + \widetilde{y}(n))| \leq K_1|t(n)|.$$

Case 3. When

$$\max\{f_1(n, z(n) + \widetilde{x}(n)), g_1(n, t(n) + \widetilde{y}(n))\} = g_1(n, t(n) + \widetilde{y}(n))$$

and

$$\max\{f_1(n, \widetilde{x}(n)), g_1(n, \widetilde{y}(n))\} = f_1(n, \widetilde{x}(n)),$$

i.e.

$$z(n+1) = g_1(n, t(n) + \widetilde{y}(n)) - f_1(n, \widetilde{x}(n)).$$

We also have two cases here: $z(n+1) \geq 0$ and $z(n+1) < 0$. If $z(n+1) \geq 0$, then

$$z(n+1) = g_1(n, t(n) + \widetilde{y}(n)) - f_1(n, \widetilde{x}(n)) \leq g_1(n, t(n) + \widetilde{y}(n)) - g_1(n, \widetilde{y}(n)),$$

and so

$$|z(n+1)| \leq |g_1(n, t(n) + \widetilde{y}(n)) - g_1(n, \widetilde{y}(n))| \leq K_1|t(n)|.$$

If $z(n+1) < 0$, then

$$-z(n+1) = f_1(n, \widetilde{x}(n)) - g_1(n, t(n) + \widetilde{y}(n)) \leq f_1(n, \widetilde{x}(n)) - f_1(n, z(n) + \widetilde{x}(n)),$$

which implies that

$$|z(n+1)| \leq |f_1(n, \widetilde{x}(n)) - f_1(n, z(n) + \widetilde{x}(n))| \leq L_1|z(n)|.$$

Case 4. When

$$\max\{f_1(n, z(n) + \widetilde{x}(n)), g_1(n, t(n) + \widetilde{y}(n))\} = g_1(n, t(n) + \widetilde{y}(n))$$

and

$$\max\{f_1(n, \widetilde{x}(n)), g_1(n, \widetilde{y}(n))\} = g_1(n, \widetilde{y}(n)),$$

we obtain

$$|z(n+1)| = |g_1(n, t(n) + \widetilde{y}(n)) - g_1(n, \widetilde{y}(n))| \leq K_1|t(n)|.$$

Next we show by induction that for all $n \in \mathbb{N}$ there exist sequences (a_n), (b_n), (c_n), (d_n), (α_n), (β_n), (γ_n), and (λ_n) of natural numbers such that

$$a_n + b_n + c_n + d_n = \alpha_n + \beta_n + \gamma_n + \lambda_n = n, \text{ for all } n \geq 0,$$

with

$$|z(n)| \leq L_1^{a_n} L_2^{b_n} K_1^{c_n} K_2^{d_n} \xi \tag{7}$$

and

$$|t(n)| \leq L_1^{\alpha_n} L_2^{\beta_n} K_1^{\gamma_n} K_2^{\lambda_n} \rho, \tag{8}$$

where $\xi, \rho \in \{z(0), t(0)\}$. It is clear that the result holds for $n = 0$. Now, for $n \geq 1$, we suppose that the assumptions hold for n and we prove that they are also true for $n+1$. We have from (5) two cases:
Either,

$$|z(n+1)| \leq L_1|z(n)| \leq L_1^{a_n+1} L_2^{b_n} K_1^{c_n} K_2^{d_n} \xi.$$

By putting $a_{n+1} = a_n + 1$, $b_{n+1} = b_n$, $c_{n+1} = b_n$, $d_{n+1} = b_n$, we have

$$a_{n+1} + b_{n+1} + c_{n+1} + d_{n+1} = a_n + 1 + b_n + c_n + d_n = n+1$$

and

$$|z(n+1)| \leq L_1^{a_{n+1}} L_2^{b_{n+1}} K_1^{c_{n+1}} K_2^{d_{n+1}} \xi$$

as desired. Or,

$$|z(n+1)| \leq K_1|t(n)| \leq L_1^{\alpha_n} L_2^{\beta_n} K_1^{\gamma_n+1} K_2^{\lambda_n} \rho.$$

By putting $a_{n+1} = \alpha_n$, $b_{n+1} = \beta_n$, $c_{n+1} = \gamma_n + 1$, $d_{n+1} = \lambda_n$, we have then

$$a_{n+1} + b_{n+1} + c_{n+1} + d_{n+1} = \alpha_n + \beta_n + \gamma_n + 1 + \lambda_n = n+1$$

and for $\xi = \rho$, we get

$$|z(n+1)| \leq L_1^{a_{n+1}} L_2^{b_{n+1}} K_1^{c_{n+1}} K_2^{d_{n+1}} \xi.$$

In a similar way we complete the proof for $t(n+1)$ by using (6).

Observe that, as $n \to +\infty$, there must be at least one of the sequences (a_n), (b_n), (c_n), and (d_n) that approaches $+\infty$, and for at least one of the sequences (α_n), (β_n), (γ_n), and (λ_n) the limit is also $+\infty$. Therefore, from (7) and (8) we see that $z(n) \to 0$ and $t(n) \to 0$ as $n \to \infty$ and the proof is complete. □

3. Examples

In this section, we present some examples to show the usefulness of the results obtained in the previous section.

Example 1 *Consider the system of difference equations*

$$\begin{cases} x(n+1) = & \max\left\{\frac{A_n}{B_n + x(n)},\, e^{-\lambda_n y(n)}\right\} \\ \\ y(n+1) = & \max\left\{e^{-\sigma_n x(n)},\, \frac{C_n}{D_n + y(n)}\right\} \end{cases}, n = 0, 1, 2, \ldots, \tag{9}$$

where (A_n), (B_n), (C_n), (D_n), (λ_n), and (σ_n) are positive periodic sequences with period ω, where

$$\max_{n \in [0, \omega-1]} \left\{\frac{A_n}{B_n^2},\, \frac{C_n}{D_n^2},\, \sigma_n,\, \lambda_n\right\} < 1,$$

and the initial values $x(0), y(0)$ are nonnegative real numbers. Clearly, this system is in the form (1) with $f_1(n,u) = \frac{A_n}{B_n+u}$, $f_2(n,u) = e^{-\sigma_n u}$, $g_1(n,u) = e^{-\lambda_n u}$, and $g_2(n,u) = \frac{C_n}{D_n+u}$. These functions are continuous in u and we have

$$\frac{\partial f_1}{\partial u}(n,u) = -\frac{A_n}{(B_n+u)^2},\ \frac{\partial f_2}{\partial u}(n,u) = -\sigma_n e^{-\sigma_n u},\ \frac{\partial g_1}{\partial u}(n,u) = -\lambda_n e^{-\lambda_n u},\ \frac{\partial g_2}{\partial u}(n,u) = -\frac{C_n}{(D_n+u)^2}.$$

Observe that

$$\left|\frac{\partial f_1}{\partial u}(n,u)\right| \le \frac{A_n}{B_n^2},\ \left|\frac{\partial f_2}{\partial u}(n,u)\right| \le \sigma_n,\ \left|\frac{\partial g_1}{\partial u}(n,u)\right| \le \lambda_n,\ \left|\frac{\partial g_2}{\partial u}(n,u)\right| \le \frac{C_n}{D_n^2}.$$

Hence, by using the mean value theorem, we can prove that

$$|f_1(n,u)-f_1(n,v)| \le \max_{n\in[0,\omega-1]}\left\{\frac{A_n}{B_n^2}\right\}|u-v|,\ |f_2(n,u)-f_2(n,v)| \le \max_{n\in[0,\omega-1]}\{\sigma_n\}|u-v|$$

and

$$|g_1(n,u)-g_1(n,v)| \le \max_{n\in[0,\omega-1]}\{\lambda_n\}|u-v|,\ |g_2(n,u)-g_2(n,v)| \le \max_{n\in[0,\omega-1]}\left\{\frac{C_n}{D_n^2}\right\}|u-v|.$$

Thus, from Theorems (3) and (4) it follows that System (9) has a unique ω-periodic solution that attracts all the solutions of this system.

Example 2 *Consider the system of difference equations*

$$\begin{cases} x(n+1) = & \max\{\alpha_n\cos(x(n)),\ \beta_n\sin(y(n))\} \\ y(n+1) = & \max\{\gamma_n\sin(x(n)),\ \lambda_n\cos(y(n))\} \end{cases},n=0,1,2,\dots, \tag{10}$$

where (α_n), (β_n), (γ_n), and (λ_n) are ω-periodic sequences of nonnegative numbers, where

$$\max_{n\in[0,\omega-1]}\{\alpha_n,\ \beta_n,\ \gamma_n,\ \lambda_n\} < 1,$$

and the initial values $x(0), y(0)$ are real numbers. It is clear that this system is in the form (1) with $f_1(n,u) = \alpha_n\cos u$, $f_2(n,u) = \gamma_n\sin u$, $g_1(n,u) = \beta_n\sin u$, and $g_2(n,u) = \lambda_n\cos u$. Observing that

$$\left|\frac{\partial f_1}{\partial u}(n,u)\right| \le \alpha_n,\ \left|\frac{\partial f_2}{\partial u}(n,u)\right| \le \gamma_n,\ \left|\frac{\partial g_1}{\partial u}(n,u)\right| \le \beta_n,\ \left|\frac{\partial g_2}{\partial u}(n,u)\right| \le \lambda_n.$$

Hence, by the mean value theorem we get

$$|f_1(n,u)-f_1(n,v)| \le \max_{n\in[0,\omega-1]}\{\alpha_n\}|u-v|,\ |f_2(n,u)-f_2(n,v)| \le \max_{n\in[0,\omega-1]}\{\gamma_n\}|u-v|$$

and

$$|g_1(n,u)-g_1(n,v)| \le \max_{n\in[0,\omega-1]}\{\beta_n\}|u-v|,\ |g_2(n,u)-g_2(n,v)| \le \max_{n\in[0,\omega-1]}\{\lambda_n\}|u-v|.$$

Then System (10) has a unique ω-periodic solution that attracts all the solutions of this system.

Example 3 *Consider the system of difference equations*

$$\begin{cases} x(n+1) = & \max\{\alpha_n,\ \beta_n y(n)\} \\ y(n+1) = & \max\{\gamma_n x(n),\ \lambda_n\} \end{cases},n=0,1,2,\dots, \tag{11}$$

where (α_n), (β_n), (γ_n), and (λ_n) are ω-periodic sequences in $[0,1)$, and the initial values $x(0), y(0)$ are real numbers.

Suppose in addition that

$$\beta_n \lambda_{n-1} \leq \alpha_n,\ \gamma_n \alpha_{n-1} \leq \lambda_n,\ \text{for all } n \geq 1. \tag{12}$$

One can easily see that System (11) satisfies the hypothesis in our main results and so it follows that this system has a unique periodic solution with period ω*. Furthermore, every solution* $\{(x(n), y(n))\}_{n\geq 0}$ *of System (11) converges to this periodic solution.*

Now we want to find the explicit form of this ω*-periodic solution. We have the following property:*

Lemma 3 *Every solution of System (11) that satisfies*

$$\gamma_{n_0} x(n_0) \leq \lambda_{n_0},\ \beta_{n_0} y(n_0) \leq \alpha_{n_0},\quad \text{for some}\ \ n_0 \geq 0, \tag{13}$$

is eventually periodic with period ω *and we have*

$$\{(x(n), y(n))\}_{n\geq n_0+1} = \{(\alpha_{n-1}, \lambda_{n-1})\}_{n\geq n_0+1}. \tag{14}$$

Proof We prove (14) by induction. The case when $n = n_0 + 1$ is obvious from Assumption (13). Now we take $n > n_0 + 1$ and we suppose that (14) holds for n. Clearly, we have

$$x_{n+1} = \max\{\alpha_n,\, \beta_n y(n)\} = \max\{\alpha_n,\, \beta_n \lambda_{n-1}\} = \alpha_n$$

and

$$y_{n+1} = \max\{\gamma_n x(n),\, \lambda_n\} = \max\{\gamma_n \alpha_{n-1},\, \lambda_n\} = \lambda_n.$$

Therefore, (14) is also true for $n + 1$ and thus the result. □

Consequently, the sequence $\{(x(n), y(n))\}_{n\geq 0}$*, where*

$$x(0) = \alpha_{\omega-1}, y(0) = \lambda_{\omega-1} \text{ and } x(n) = \alpha_{n-1},\, y(n) = \lambda_{n-1},\quad \text{for all } n \geq 1,$$

is the unique ω*-periodic solution of System (11).*

To confirm this result let us consider a numerical example as follows:
Let $\omega = 3$*, and the sequences*

$$\alpha_n = \begin{cases} 0.6, & \text{if}\quad n = 3k, \\ 0.9, & \text{if}\quad n = 3k+1, \\ 0.8, & \text{if}\quad n = 3k+2, \end{cases} \qquad \beta_n = \begin{cases} 0.55, & \text{if}\quad n = 3k, \\ 0.4, & \text{if}\quad n = 3k+1, \\ 0.35, & \text{if}\quad n = 3k+2, \end{cases}$$

and

$$\gamma_n = \begin{cases} 0.5, & \text{if}\quad n = 3k, \\ 0.62, & \text{if}\quad n = 3k+1, \\ 0.85, & \text{if}\quad n = 3k+2, \end{cases} \qquad \lambda_n = \begin{cases} 0.75, & \text{if}\quad n = 3k, \\ 0.85, & \text{if}\quad n = 3k+1, \\ 0.96, & \text{if}\quad n = 3k+2, \end{cases}$$

with $n, k \in \mathbb{N}$*. Observe that*

$$\beta_n \leq \alpha_n,\, \gamma_n \leq \lambda_n,\ \text{for all } n \geq 0,$$

which implies that (12) is satisfied. Then the unique periodic solution with period three is

$$\{(\widetilde{x}(n), \widetilde{y}(n))\}_{n\geq 0} = \{(0.8, 0.96), (0.6, 0.75), (0.9, 0.85), (0.8, 0.96), (0.6, 0.75), (0.9, 0.85), \ldots\}.$$

We consider here two arbitrary solutions of the system. If we take $(x(0), y(0)) = (19.6.10^3, 5.10^3)$*, Table 1 shows the convergence of the solution* $\{(x(n), y(n))\}_{n\geq 0}$ *to the periodic solution* $\{(\widetilde{x}(n), \widetilde{y}(n))\}_{n\geq 0}$*. Similarly, for* $(x(0), y(0)) = (10, 56)$*, the convergence of the solution is presented in Table 2.*

Table 1.

n	$x(n) - \widetilde{x}(n)$	$y(n) - \widetilde{y}(n)$
0	19599.2	4999.04
1	2749.40	9799.25
2	3919.10	1704.1500
3	595.950000	3331.0400
4	1832.000000	297.6250000
5	118.4500000	1135.362000
6	396.8742000	100.4875000
7	55.19612500	198.0871000
8	78.63484000	33.74359750
9	11.30775912	66.64461400
10	36.58253770	5.303879560
11	1.521551824	22.20317337
12	7.268610680	1.098319050
13	0.532075478	3.284305340
14	0.713722136	0.
15	0.	0.411663816
16	0.1544150988	0.
17	0.	0.
18	0.	0.
19	0.	0.

Table 2.

n	$x(n) - \widetilde{x}(n)$	$y(n) - \widetilde{y}(n)$
0	9.2	55.04
1	30.20	4.25
2	1.10	18.2460
3	5.883600	0.7400
4	0.335000	2.5918000
5	0.43672000	0.
6	0.	0.176212000
7	0.0249166000	0.
8	0.	0.
9	0.	0.
10	0.	0.

4. Conclusion

This paper is concerned with the behavior of solutions of a functional first-order nonautonomous max-type system of difference equations. More precisely, we have shown that if the real-valued functions that define our system are contraction mappings, then this last has a periodic solution that attracts all the other solutions. Firstly, the contraction property enabled us to use Banach fixed point theorem to show the existence and uniqueness of a periodic solution. Secondly, the attractivity of this periodic solution was investigated. In addition, to illustrate these results, three examples were provided.

Finally, we note that the results obtained here about the proposed system also hold for both of the min-type analogue system and the mixed ones by following the same procedures of the proofs with simple modifications.

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