Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 426 – 435
© TÜBİTAK
doi:10.3906/mat-1512-96

Approximation of analytic functions of several variables by linear k-positive operators

Tülin COŞKUN*
Department of Mathematics, Bülent Ecevit University, Zonguldak, Turkey

Received: 24.12.2015 • **Accepted/Published Online:** 25.05.2016 • **Final Version:** 03.04.2017

Abstract: We investigate the approximation of analytic functions of several variables in polydiscs by the sequences of linear k-positive operators in the Gadjiev sense.

Key words: Analytic functions, linear k-positive operators, Korovkin-type theorems

1. Introduction

The approximation of analytic functions of complex variables by linear k-positive operators was first tackled in the work of Gadjiev [5]. He introduced k-positive operators and formulated theorems of Korovkin's type for these operators in the space of analytic functions on the unit disc. He proposed a method of proving such theorems, applied further on in many other articles (e.g., [1,3,6–13,15,16]). Some of the results from [1,5–7] were included in a monograph [2,14].

In his recent article [12], Gadjiev proved very general results on convergence of the sequences of linear k-positive operators on a simply connected bounded domain within the space of analytic functions.

In this article we extend some of the result of Gadjiev to the approximation of analytic functions of several complex variables by sequences of linear k-positive operators.

2. Preliminaries

Let $\mathbb{N}$ and $\mathbb{Z}_+$ be the respective sets of positive and nonnegative numbers and $\mathbb{C}$ be the space of complex numbers. For $n \in \mathbb{N}$ let

$$S_n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| < 1,\, i = 1, 2, \cdots, n\}$$

be a polydisc in $\mathbb{C}^n$ and $A(S_n)$ be the space of analytic functions on S_n.

According to [12] let $k_i \in \mathbb{Z}_+$ for $i = 1, 2, \cdots, n$, such that the system of powers $z_1^{k_1}, z_2^{k_2}, \cdots, z_n^{k_n}$ forms a basis of $A(S_n)$ in the sense that any function $f \in A(S_n)$ can be expanded into a series in base as follows.

$$k = (k_1, k_2, \ldots, k_n),\; k+1 = (k_1+1, k_2+1, \ldots, k_n+1),\; |k| = k_1 + k_2 + \ldots + k_n,$$

*Correspondence: tulin_coskun@yahoo.de
2000 *AMS Mathematics Subject Classification:* 41A25, 41A36.
This paper is dedicated to the memory of Akif D Gadjiev.

$z^k = z_1^{k_1} \cdot z_2^{k_2} \cdots z_n^{k_n}; dz = dz_1 \cdot dz_2 \cdots dz_n$, and for any r, $0 < r < 1$

$$\int \cdots \int_{|z|=r} f(z)\, dz = \int_{|z_1|=r} \cdots \int_{|z_n|=r} f(z_1, z_2, \ldots, z_n) dz_1 \cdot dz_2 \cdots dz_n.$$

Then any function $f \in A(S_n)$ can be expanded into a series

$$f(z) = \sum_{|k|=0}^{\infty} f_k z^k \tag{2.1}$$

with

$$f_k = \frac{1}{(2\pi i)^n} \int \cdots \int_{|z|=r} \frac{f(z)\, dz}{z^{k+1}}. \tag{2.2}$$

Thus, for any linear operator T on $A(S_n)$, we have

$$Tf(z) = \sum_{|k|=0}^{\infty} z^k \sum_{|m|=0}^{\infty} T_{k,m} f_m,$$

where $T_{k,m} \in \mathbb{C}$ for multiindex k and m.

Let A^+ be the subspace of functions $f \in A(S_n)$, having nonnegative Taylor coefficients.

Definition 2.1 *Linear operator $T : A(S_n) \longrightarrow A(S_n)$ will be called* **k-positive** *if $TA^+ \subset A^+$.*

We will study the convergence of the sequences of k-positive operators

$$T_N f(z) = \sum_{|k|=0}^{\infty} z^k \sum_{|m|=0}^{\infty} T_{k,m}^{(N)} f_m \tag{2.3}$$

where N is a natural parameter.

If T_N is a k-positive operator for any $N \in \mathbb{N}$, then $\sum_{|m|=0}^{\infty} T_{k,m}^{(N)} f_m \geq 0$ for all $f \in A^+$ and k. From this we obtain $T_{k,m}^{(N)} \geq 0$ for all k, m. Indeed, if $T_{k,m_0}^{(N)} < 0$ for any $m_0 = (m_{01}, m_{02}, \ldots, m_{0n})$, we take the function $\tilde{f} \in A^+$ with the nonnegative coefficients

$$\tilde{f}_m = \begin{cases} 0 & ; \quad |m| \neq |m_0| \\ 1 & ; \quad |m| = |m_0|. \end{cases}$$

Then for the function $\tilde{f}$ we obtain $\sum_{|m|=0}^{\infty} T_{k,m}^{(N)} \tilde{f}_m = T_{k,m_0}^{(N)} < 0$, but this is a contradiction to the k-positivity of T_N. This gives $T_{k,m}^{(N)} \geq 0$ as the necessary and sufficient condition for k-positivity of operators (2.3).

We will study the approximation properties of linear k-positive operators in the space $A(S_n)$. In this space, the concept of the norm can be introduced in different ways, but all these norms are equivalent and therefore $A(S_n)$ is a Frechet space (see [4]). We denote

$$\|f\|_{A(S_n),r} = \max\{|f(z)| : z = (z_1, \cdots, z_n),\ |z_j| \leq r,\ j = 1, 2, \cdots, n\}.$$

Then, obviously, the sequence $(f_N(z))$ of functions in $A(S_n)$ tends to zero in $A(S_n)$ if and only if for each $0 < r < 1$

$$\lim_{N\to\infty} \|f_N\|_{A(S_n),r} = 0.$$

The following criterion of convergence is important (for the one-dimensional case, see [4]).

Lemma 2.2 *Let $(f_N(z))$ be the sequence of functions in $A(S_n)$ such that*

$$f_N(z) = \sum_{|k|=0}^{\infty} f_{N,k} z^k, \quad \limsup_{|k|\to\infty} |f_{N,k}|^{\frac{1}{|k|}} = 1.$$

Then $f_N(z)$ tends to zero in $A(S_n)$ as $N \to \infty$ if and only if there exist sequences of positive numbers ε_N and δ_N tending to zero as $N \to \infty$ such that

$$|f_{N,k}| < \varepsilon_N (1+\delta_N)^{|k|}. \tag{2.4}$$

Proof If (2.4) holds then for any $r < 1$

$$\max_{|z_j|\le r} |f_N(z)| < \varepsilon_N \sum_{|k|=0}^{\infty} r^{|k|}(1+\delta_N)^{|k|} = \varepsilon_N \left(\frac{1}{1-r(1+\delta_N)}\right)^n,$$

since for any given $r < 1$ we can take $\delta_N < \frac{1}{r} - 1$, which is possible because $\delta_N \to 0$ by the conditions of the lemma. Therefore, $f_N(z) \to 0$ as $N \to \infty$ in $A(S_n)$.

Now, taking a null sequence δ_N such that $\varepsilon_N = \max_{|z|=\frac{1}{1+\delta_N}} |f_N(z)| \to 0$ as $N \to \infty$, we have

$$f_{N,k} = \frac{1}{(2\pi i)^n} \int_{|z|=\frac{1}{1+\delta_N}} \cdots \int \frac{f_N(z)}{z^{k+1}}\, dz.$$

Then

$$\begin{aligned} |f_{N,k}| &\le \frac{\varepsilon_N}{(2\pi)^k} \int_{|z|=\frac{1}{1+\delta_N}} \cdots \int \frac{|dz_1|\cdot|dz_2|\cdots|dz_n|}{|z_1|^{k_1+1}\cdot|z_2|^{k_2+1}\cdots|z_n|^{k_n+1}} \\ &= \frac{\varepsilon_N}{(2\pi)^k}(1+\delta_N)^{|k|+n}\frac{1}{(1+\delta_N)^n} < \varepsilon_N(1+\delta_N)^{|k|} \end{aligned}$$

and the proof is complete. □

3. Main theorems

In this section we give three theorems on approximation of analytic functions belonging to $A(S_n)$ by the sequences of linear k-positive operators.

Definition 3.1 *$A_g(S_n)$ be the subspace of functions $f(z) \in A(S_n)$, for which*

$$|f_k| \le M g_{|k|}, \quad k = (k_1, k_2, \ldots, k_n) \tag{3.1}$$

where f_k are the Taylor coefficients of function f andM is a constant, depending on f only and $g_r \ge 1$ increasing sequences for $r \in \mathbb{Z}_+$, such that $\limsup_{r\to\infty} \sqrt[r]{g_r} = 1$.

Let, for any $r \in \mathbb{N}$,

$$\triangle_g(r) := \min\{\sqrt{g_{r+1}} - \sqrt{g_r}; \sqrt{g_r} - \sqrt{g_{r-1}}\}. \tag{3.2}$$

Obviously $\triangle_g^2(r) < g_r$.

Now we assume that the condition

$$\lim_{r\to\infty} (\sqrt{g_r} - \sqrt{g_{r-1}})^{\frac{1}{r}} = 1 \tag{3.3}$$

holds and consider the functions

$$g_\nu(z) = \sum_{|k|=0}^{\infty} g_{|k|}^{\frac{\nu}{2}} z^k \tag{3.4}$$

and

$$\mu_{\nu,j}(z) = \sum_{|k|=0}^{\infty} k_j^{\nu} z^k, \quad j = 1, 2, \ldots, n. \tag{3.5}$$

Theorem 3.2 *Let $g_\nu(z)$ and $\mu_{\nu,j}(z)$ be the functions defined in (3.4) and (3.5) and T_N be the sequences of linear k-positive operators acting from $A(S_n)$ to $A(S_n)$. If*

$$\lim_{N\to\infty} \|T_N g_\nu(z) - g_\nu(z)\|_{A(S_n),r} = 0, \quad \nu = 0, 1, 2, \tag{3.6}$$

$$\lim_{N\to\infty} \|T_N \mu_{\nu,j}(z) - \mu_{\nu,j}(z)\|_{A(S_n),r} = 0, \quad \nu = 1, 2; \quad j = 1, 2, \ldots, n, \tag{3.7}$$

then

$$\lim_{N\to\infty} \|T_N f(z) - f(z)\|_{A(S_n),r} = 0$$

for each function $f \in A_g(S_n)$.

Proof Let $f \in A(S_n)$. Then by the definition of norm in $A(S_n)$, for each $r < 1$

$$\|T_N f(z) - f(z)\|_{A(S_n),r} \leq \sum_{|k|=0}^{\infty} r^{|k|} \sum_{|m|=0}^{\infty} T_{k,m}^{(N)} |f_m - f_k|$$

$$+ \sum_{|k|=0}^{\infty} r^{|k|} |f_k| \; |\sum_{|m|=0}^{\infty} T_{k,m}^{(N)} - 1| = I'_N + I''_N. \tag{3.8}$$

From condition (3.6) with $\nu = 0$ and Lemma 2.2 it follows that

$$|\sum_{|m|=0}^{\infty} T_{k,m}^{(N)} - 1| < \varepsilon_N (1 + \delta_N)^{|k|}$$

and therefore $I''_N \to 0$ as $N \to \infty$.

Consider I'_N. Since $f \in A_g(S_n)$ we can write by (3.1) and the properties of g_r the estimate

$$\begin{aligned} |f_m - f_k| &\leq 2M g_{|m|} g_{|k|} \leq 4M g_{|k|}((\sqrt{g_{|m|}} - \sqrt{g_{|k|}})^2 + g_{|k|}) \\ &\leq 4M g_{|k|}^2 (\frac{(\sqrt{g_{|m|}} - \sqrt{g_{|k|}})^2}{\triangle_g^2(|k|)} + 1) \end{aligned}$$

where $\triangle_g^2$ is defined in (3.2). Therefore, if $|m| \neq |k|$ we can write

$$|f_m - f_k| \leq 8Mg_{|k|}^2 \frac{(\sqrt{g_{|m|}} - \sqrt{g_{|k|}})^2}{\triangle_g^2(|k|)}.$$

If $|m| = |k|$ and $m \neq k$ then there exists at least one number j, for which $(m_j - k_j)^2 > 1$. Therefore, if $|m| = |k|$,

$$|f_m - f_k| \leq 2Mg_{|m|}g_{|k|} < 2Mg_{|k|}^2 \sum_{j=1}^{n}(m_j - k_j)^2.$$

The last two inequalities for $|f_m - f_k|$ gives us that for all k and m

$$|f_m - f_k| \leq 8Mg_{|k|}^2 \{\frac{(\sqrt{g_{|m|}} - \sqrt{g_{|k|}})^2}{\triangle_g^2(|k|)} + \sum_{j=1}^{n}(m_j - k_j)^2\}. \tag{3.9}$$

Using (3.9) in the definition of I_N' in (3.8) we can write

$$\begin{aligned} I_N' \quad &\leq \quad 8M \sum_{|k|=0}^{\infty} r^{|k|} \{\frac{g_{|k|}^2}{\triangle_g^2(|k|)} \sum_{|m|=0}^{\infty} T_{k,m}^{(N)} (\sqrt{g_{|m|}} - \sqrt{g_{|k|}})^2 \\ &+ \quad \sum_{j=1}^{n} \sum_{|m|=0}^{\infty} T_{k,m}^{(N)} (m_j - k_j)^2\}. \end{aligned}$$

The conditions (3.6) and Lemma 2.2 give

$$\sum_{|m|=0}^{\infty} (\sqrt{g_{|k|}} - \sqrt{g_{|m|}})^2 T_{k,m}^{(N)} \leq 4\varepsilon_N (1 + \delta_N)^{|k|} g_{|k|}$$

and the conditions (3.6) in $\nu = 0$ and (3.7) in view of Lemma 2.2 give

$$\sum_{j=1}^{n} \sum_{|m|=0}^{\infty} (m_j - k_j)^2 T_{k,m}^{(N)} \leq \varepsilon_N (1 + \delta_N)^{|k|} n (1 + |k|)^2.$$

Therefore,

$$I_N' < 32M\varepsilon_N \sum_{|k|=0}^{\infty} r^{|k|} (1 + \delta_N)^{|k|} \{\frac{g_{|k|}^3}{\triangle_g^2(|k|)} + n \cdot (1 + |k|)^2\},$$

and from the properties on $g_{|k|}$, it follows that $I_N' \to 0$ as $N \to \infty$. The proof is complete. □

Now we give a second theorem on approximation of functions in the subspace $A_g(S_n)$ by k-positive operators under the condition that $g_{|k|} \geq 1$ for all k.

Consider the following test functions. Let

$$
\begin{aligned}
p_0(z) &= \sum_{|k|=0}^{\infty} g_{|k|} z^k, \\
p_{1,j}(z) &= \sum_{|k|=0}^{\infty} k_j g_{|k|} z^k, \quad j = 1, 2, \cdots, nm \\
p_{2,j}(z) &= \sum_{|k|=0}^{\infty} k_j^2 g_{|k|} z^k, \quad j = 1, 2, \cdots, n.
\end{aligned}
$$

Theorem 3.3 *If the sequence T_N of linear k-positive operators from $A(S_n)$ to $A(S_n)$ satisfies the conditions*

$$\lim_{N\to\infty} \|T_N p_0(z) - p_0(z)\|_{A(S_n),r} = 0$$

and

$$\lim_{N\to\infty} \|T_N p_{\nu,j}(z) - p_{\nu,j}(z)\|_{A(S_n),r} = 0, \quad \nu = 1, 2; \quad j = 1, 2, \ldots, n, \tag{3.10}$$

then for each function $f \in A_g(S_n)$

$$\lim_{N\to\infty} \|T_N f(z) - f(z)\|_{A(S_n),r} = 0.$$

Proof By condition (3.10) and Lemma 2.2

$$| \sum_{|m|=0}^{\infty} g_{|m|} T_{k,m}^{(N)} - g_{|k|}| < \varepsilon_N (1 + \delta_N)^{|k|} \tag{3.11}$$

$$| \sum_{|m|=0}^{\infty} m_j g_{|m|} T_{k,m}^{(N)} - k_j g_{|k|}| < \varepsilon_N (1 + \delta_N)^{|k|}$$

$$| \sum_{|m|=0}^{\infty} m_j^2 g_{|m|} T_{k,m}^{(N)} - k_j^2 g_{|k|}| < \varepsilon_N (1 + \delta_N)^{|k|}.$$

Then we can write

$$\sum_{|m|=0}^{\infty} \sum_{j=1}^{n} (m_j - k_j)^2 g_{|m|} T_{k,m}^{(N)} < \varepsilon_N (1 + \delta_N)^{|k|} n (1 + |k|)^2. \tag{3.12}$$

Since $g_k \geq 1$, for all m and k and $\sum_{j=1}^{n} (m_j - k_j)^2 > 1$ we can write

$$|f_m - f_k| \leq 2 g_{|m|} g_{|k|} \sum_{j=1}^{n} (m_j - k_j)^2. \tag{3.13}$$

Now, as in the proof of Theorem 2.2, we can write inequality (3.8).

$$
\begin{aligned}
\|T_N f(z) - f(z)\|_{A(S_n),r} \quad &\leq \quad \sum_{|k|=0}^{\infty} r^{|k|} \sum_{|m|=0}^{\infty} T_{k,m}^{(N)} |f_m - f_k| g_{|m|} \\
&+ \sum_{|k|=0}^{\infty} r^{|k|} |f_k| \; | \sum_{|m|=0}^{\infty} g_{|m|} T_{k,m}^{(N)} - g_{|k|}|.
\end{aligned}
$$

Using (3.11) for the estimate of the second term in the right-hand side and (3.13)–(3.12) for the first term, we obtain

$$\begin{aligned}\|T_N f(z) - f(z)\|_{A(S_n),r} &\leq \varepsilon_N n \sum_{|k|=0}^{\infty} r^{|k|}(1+\delta_N)^{|k|}(1+|k|)^2 \\ &+ \varepsilon_N \sum_{|k|=0}^{\infty} r^{|k|}|f_k|(1+\delta_N)^{|k|}\end{aligned}$$

and therefore

$$\lim_{N\to\infty} \|T_N f(z) - f(z)\|_{A(S_n),r} = 0$$

for each function $f \in A_g(S_n)$.

The proof is complete. □

Consider now a special case.

Definition 3.4 *The subspace $\widetilde{A}(S_n)$ is the set of analytic functions in polydisc S_n for which*

$$|f_k| \leq M(1+|k|),$$

where M is a constant depending on f only.

Let $f \in \widetilde{A}(S_n)$. Then for $|m| > |k|$

$$\begin{aligned}|f_m - f_k| &\leq M(2+|m|+|k|) \\ &\leq 2M(1+(|m|-|k|)+|k|) \\ &\leq 4M(1+|k|)(|m|-|k|)^2.\end{aligned}$$

If $|m| < |k|$ then

$$|f_m - f_k| \leq 2M(1+|k|) \leq 2M(1+|k|)(|m|-|k|)^2.$$

Finally, if $|m| = |k|$ and $m \neq k$ then

$$|f_m - f_k| \leq 2M(1+|k|)\sum_{j=1}^{n}(m_j-k_j)^2.$$

Thus, the Taylor coefficients f_k of any function $f \in \widetilde{A}(S_n)$ satisfy the inequality

$$|f_m - f_k| \leq 4M(1+|k|)\{(|m|-|k|)^2 + \sum_{j=1}^{n}(m_j-k_j)^2\} \tag{3.14}$$

for all m and k. Using this inequality we can prove the approximation theorem in the space $\widetilde{A}(S_n)$.

Let

$$\varphi_t(z) = \sum_{|k|=0}^{\infty} k^t z^t, \quad |t| \leq 2 \tag{3.15}$$

where $k^t = k_1^{t_1}k_2^{t_2}\cdots k_n^{t_n}$.

Theorem 3.5 *Let T_N be the sequence of linear k-positive operators from $A(S_n)$ to $A(S_n)$. Then the conditions*

$$\lim_{N\to\infty}\|T_N\varphi_t(z)-\varphi_t(z)\|_{A(S_n),r}=0,\quad |t|\le 2 \tag{3.16}$$

are necessary and sufficient such that

$$\lim_{N\to\infty}\|T_Nf(z)-f(z)\|_{A(S_n),r}=0$$

for any function $f\in\widetilde{A}(S_n)$.

Proof As in the proof of Theorem 2.2 and Theorem 2.3, we can write the inequality

$$\|T_Nf(z)-f(z)\|_{A(S_n),r}\le\sum_{|k|=0}^{\infty}r^{|k|}\sum_{|m|=0}^{\infty}T_{k,m}^{(N)}|f_m-f_k|$$

$$+\sum_{|k|=0}^{\infty}r^{|k|}|f_k|\;|\sum_{|m|=0}^{\infty}T_{k,m}^{(N)}-1|=\jmath'_N+\jmath''_N. \tag{3.17}$$

Taking in (3.16) $t=0$ and using (3.15) we see that by Lemma 2.2

$$|\sum_{|m|=0}^{\infty}T_{k,m}^{(N)}-1|<\varepsilon_N(1+\delta_N)^{|k|}$$

and therefore $\lim_{N\to\infty}\jmath''_N=0$, and it is sufficient to estimate only the first term in (3.17). Using (3.14) we obtain

$$\jmath'_N\le 4M\sum_{|k|=0}^{\infty}r^{|k|}(1+|k|)\sum_{|m|=0}^{\infty}\{(|m|-|k|)^2+\sum_{j=1}^{n}(m_j-k_j)^2\}T_{k,m}^{(N)}.$$

Obviously

$$\begin{aligned}\sum_{|m|=0}^{\infty}(|m|-|k|)^2T_{k,m}^{(N)} &= [\sum_{|m|=0}^{\infty}|m|^2T_{k,m}^{(N)}-|k|^2]\\ &+ 2|k|[|k|-\sum_{|m|=0}^{\infty}|m|T_{k,m}^{(N)}]+|k|^2[\sum_{|m|=0}^{\infty}T_{k,m}^{(N)}-1].\end{aligned}$$

Using the equality

$$|m|^2=m_1^2+m_2^2+\cdots m_n^2+2m_1m_2+\cdots+2m_{n-1}m_n$$

we see that

$$\begin{aligned}\sum_{|m|=0}^{\infty}|m|^2T_{k,m}^{(N)} &= \sum_{|m|=0}^{\infty}m_1^2T_{k,m}^{(N)}+\cdots+\sum_{|m|=0}^{\infty}m_n^2T_{k,m}^{(N)}\\ &+ 2\sum_{|m|=0}^{\infty}m_1m_2T_{k,m}^{(N)}+\cdots+2\sum_{|m|=0}^{\infty}m_{n-1}m_nT_{k,m}^{(N)}.\end{aligned}$$

Taking in (3.15) successively

$$\begin{aligned} t_1 &= 2,\ t_j = 0 \ \text{ if } j \neq 1 \\ t_2 &= 2,\ t_j = 0 \ \text{ if } j \neq 2 \\ &\vdots \\ t_n &= 2,\ t_j = 0 \ \text{ if } j \neq n \end{aligned}$$

and then

$$\begin{aligned} t_1 &= t_2 = 1, \ \text{ and } t_j = 0, \quad j \neq 1,2 \\ t_2 &= t_3 = 1, \ \text{ and } t_j = 0, \quad j \neq 2,3 \\ &\vdots \\ t_{n-1} &= t_n = 1, \ \text{ and } t_j = 0, \quad j \neq n-1,n \end{aligned}$$

we complete the proof using conditions (3.16) and Lemma 2.2. □

Acknowledgment

The author is thankful to the referees for their attentive reading of the manuscript and for useful comments.

References

[1] Ahadov RA. On the convergence of sequences of linear operators in a space of functions that are analytic in the disc. Izv Akad Nauk Azerbaidzhan SSR Ser Fiz Tekhn Mat Nauk 2 1981; 1: 67-71 (in Russian).

[2] Altomare F, Campiti M. Korovkin-Type Approximation Theory and Its Applications. de Gruyter Studies in Mathematics, Vol. 17. Berlin, Germany: Walter de Gruyter, 1994.

[3] Duman O. Statistical approximation theorems by k-positive linear operators. Arch Math (Basel) 2006; 86: 569-576.

[4] Evgrafov MA. The method of near systems in the space of analytic functions and its application to interpolation. Tr Mosk Mat Obs 1956; 5: 89-201 (in Russian).

[5] Gadjiev AD. Linear k-positive operators in a space of regular functions, and theorems of P.P. Korovkin type. Izv Akad Nauk Azerb SSR Ser Fiz Tekh Mat Nauk 1974; 5: 49-53 (in Russian).

[6] Gadjiev AD. The convergence problem for a sequence of positive linear operators on unbounded sets and theorems analogous of that P.P. Korovkin. Soviet Dokl Akad Nauk SSSR 1974; 218: 1001-1004 (in Russian).

[7] Gadjiev AD. Theorems of Korovkin type. Mat Zametki 1976; 20: 781-786 (in Russian).

[8] Gadjiev AD, Aliev RA. Approximation of analytical functions by k-positive linear operators in the closed domain. Positivity 2014; 18: 439-447.

[9] Gadjiev AD, Cetin N. Approximation of analytical functions by sequences of linear operators. Filomat 2014; 28: 99-106.

[10] Gadjiev AD, Duman O, Ghorbanalizadeh AM. Ideal convergence of k-positive linear operators. J Funct Spaces Appl 2012; 2012: 178316.

[11] Gadjiev AD, Orhan C. Some approximation theorems via statistical convergence. Rocky Mountain J Math 2002; 32, 1: 129-138.

[12] Gadjiev AD, Ghorbanalizadeh AM. Approximation of analytical functions by sequences of k-positive linear operators. J Approx Theory 2010; 162: 1245-1255.

[13] Gadjiev AD, Ghorbanalizadeh AM. On an approximation processes in the space of analytical functions. Cent Eur J Math 2010; 8: 389-398.

[14] Hacısalihoğlu HH, Gadjiev AD. On Convergence of the Sequences of Linear Positive Operators. Ankara, Turkey: Ankara University Press, 1995 (in Turkish).

[15] Ispir N. Convergence of sequences of k-positive linear operators in subspaces of the space of analytic functions. Hacettepe Bulletin of Natural Sciences and Engineering Series B 1999; 28: 47-53.

[16] Ispir N, Atakut C. On the convergence of a sequence of positive linear operators on the space of m-multiple complex sequences. Hacettepe Bulletin of Natural Sciences and Engineering Series B 2000; 29: 47-54.