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Suborbital graphs for the Atkin–Lehner group

Tuncay KÖROĞLU*, **Bahadır Özgür GÜLER**, **Zeynep ŞANLI**
Department of Mathematics, Karadeniz Technical University, Turkey

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Abstract: We investigate suborbital graphs for an imprimitive action of the Atkin–Lehner group on a maximal subset of extended rational numbers on which a transitive action is also satisfied. Obtaining edge and some circuit conditions, we examine some combinatorial properties of these graphs.

Key words: Fuchsian groups, Atkin–Lehner group, group action, suborbital graphs

1. Introduction

The idea of a suborbital graph has been used mainly by finite group theorists. In [11], Jones et al. showed that this idea is also useful in the study of the modular group that is a finitely generated Fuchsian group and show that the well-known Farey graph is an example of a suborbital graph.

Then similar studies were done for related finitely generated groups.The reader is referred to [2–5,8,9,11–16] for some relevant previous work on suborbital graphs. Firstly, in [3], it was proved that the elliptic elements in $\Gamma_0(n)$ correspond to circuits in the subgraph $F_{u,n}$ of the same order and vice versa. This fact is important because it means that suborbital graphs might have a potential to clarify signature problems taking into account the order of elliptic elements are one of the invariants of signature. Note that it was seen that this relation is just provided unilaterally in [14]. Elliptic elements do not necessarily correspond to circuits of the same order. On the other hand, it is worth noting that these graphs give some number theoretical results about continued fractions and Fibonacci numbers as in [4,8,17].

In the present study, we will continue to investigate the combinatorial properties of these graphs for the Atkin–Lehner group as an important object that is studied concerning Monster groups extensively. In the final section we state that main graph $G_{u,n}$ is not a disjoint union of isomorphic copies of subgraphs different from those up to now.

2. Preliminaries

Let $PSL(2,\mathbb{R})$ denote the group of all linear fractional transformations

$$T : z \to \frac{az+b}{cz+d}\text{, where } a,b,c, \text{ and } d \text{ are real and } ad-bc=1.$$

*Correspondence: tkor@ktu.edu.tr
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In terms of matrix representation, the elements of $PSL(2,\mathbb{R})$ correspond to the matrices

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad a,b,c,d \in \mathbb{R} \text{ and } ad-bc=1.$$

This is the automorphism group of the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im(z)} > 0\}$. Γ, the modular group, is the subgroup of $PSL(2,\mathbb{R})$ such that $a,b,c,$ and d are integers. $\Gamma_0(n)$ is the subgroup of Γ with $n|c$.

In [7], the normalizer $Nor(\Gamma_0(n))$ of $\Gamma_0(n)$ in $PSL(2,\mathbb{R})$ consists exactly of matrices

$$\begin{pmatrix} ae & b/h \\ cn/h & de \end{pmatrix},$$

where $e \parallel \frac{n}{h^2}$ and h is the largest divisor of 24 for which $h^2|n$ with understandings

that the determinant e of the matrix is positive, and that $r \parallel s$ means that $r|s$ and $(r, s/r) = 1$ (r is called an exact divisor of s).

The Atkin–Lehner group $AL(\Gamma_0(n))$ is denoted by the set of transformation

$$\begin{pmatrix} ae & b \\ cn & de \end{pmatrix},$$

where $e||n$. Obviously, $AL(\Gamma_0(n))$ is a subgroup of $Nor(\Gamma_0(n))$. The elements of $AL(\Gamma_0(n))$ are called the Atkin–Lehner transformations. $AL(\Gamma_0(n))$ is a Fuchsian group whose fundamental domain has finite area, and so it has a signature consisting of the geometric invariants

$$(g; m_1, ..., m_r; s),$$

where g is the genus of the compactified quotient space, $m_1, ..., m_r$ are the periods of the elliptic elements, and s is the parabolic class number.

3. The action of $AL(\Gamma_0(n))$ on $\hat{\mathbb{Q}}(AL)$

3.1. Transitive action

In this section, we describe transitive and imprimitive action of $AL(\Gamma_0(n))$. Hence we can apply Sim's theory to obtain suborbital graphs in the next section. The action of $AL(\Gamma_0(n))$ on $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, the extended rationals, is defined by

$$\begin{pmatrix} ae & b \\ cn & de \end{pmatrix} : \frac{x}{y} \to \frac{aex+by}{cnx+dey}$$

ordinarily. It is noted that while x and y are coprime $aex+by$ and $cnx+dey$ do not have to be coprime.

Definition 3.1 *$\hat{\mathbb{Q}}(AL) := \left\{\frac{a}{b} : (b,n)||n\right\}$ is a subset of $\hat{\mathbb{Q}}$.*

Theorem 3.1 *$\hat{\mathbb{Q}}(AL)$ is the maximal set on which $AL(\Gamma_0(n))$ group acts transitively.*

Proof Let us calculate the orbit of $\frac{1}{0}=\infty$ under the action of $AL(\Gamma_0(n))$.

$$\begin{pmatrix} ae & b \\ cn & de \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\frac{ae}{cn}=\frac{a}{c\frac{n}{e}}.$$

It is easily seen that a and $c\frac{n}{e}$ are coprime and $\left(c\frac{n}{e},n\right)=\frac{n}{e}\|n$.

Conversely suppose that $\frac{a_1}{b_1}\in\hat{\mathbb{Q}}$ and $b_n:=(b_1,n)\|n$. Then there exists $b_0\in\mathbb{Z}$ such that $b_1=b_0b_n$. Therefore, since $\left(n,\frac{n^2}{b_n^2}\right)=1$, there exist $a',b',c',d'\in\mathbb{Z}$ such that

$$\begin{pmatrix} a'\frac{n}{b_n} & b' \\ c'n & d'\frac{n}{b_n} \end{pmatrix}\in AL(\Gamma_0(n)).$$

Consequently, $\hat{\mathbb{Q}}(AL)=\left\{\frac{a}{b}:(b,n)||n\right\}$ □

3.2. Imprimitive action

Let us give a general discussion of primitivity of permutation groups. Let (G,Δ) be a transitive permutation group, consisting of a group G acting on a set Δ transitively. An equivalence relation $\approx$ on Δ is called G-invariant if, whenever

$$\alpha,\beta\in\Delta \text{ satisfy } \alpha\approx\beta, \text{ then } g(\alpha)\approx g(\beta) \text{ for all } g\in G.$$

The equivalence classes are called blocks, and the block containing α is denoted $[\alpha]$.

We call (G,Δ) imprimitive if Δ admits some G-invariant equivalence relation different from

- the identity relation, $\alpha\approx\beta$ iff $\alpha=\beta$.
- the universal relation, $\alpha\approx\beta$ for all $\alpha,\beta\in\Delta$.

Otherwise (G,Δ) is called primitive. These two relations are supposed to be trivial relations.

Lemma 3.1 *([6]) Let (G,Δ) be a transitive permutation group. (G,Δ) is primitive if and only if G_α, the stabilizer of $\alpha\in\Delta$, is a maximal subgroup of G for each $\alpha\in\Delta$.*

From the above lemma we see that whenever, for some α, $G_\alpha\lneq H\lneq G$, then Ω admits some G-invariant equivalence relation other than the trivial cases. Because of the transitivity, every element of Ω has the form $g(\alpha)$ for some $g\in G$. Thus one of the nontrivial G-invariant equivalence relations on Ω is given as follows:

$$g(\alpha)\approx g'(\alpha) \text{ if and only if } g'\in gH.$$

The number of blocks (equivalence classes) is the index $|G:H|$ and the block containing α is just the orbit $H(\alpha)$.

In this work we take $AL_\infty(\Gamma_0(n))$, the stabilizer of ∞ under $AL(\Gamma_0(n))$, G will be $AL(\Gamma_0(n))$ and then, since $AL_\infty(\Gamma_0(n))=\left\langle\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\rangle$,

$$AL_\infty(\Gamma_0(n)) < \Gamma_0(n) < AL(\Gamma_0(n)), n > 1.$$

In that case, we define the following $AL(\Gamma_0(n))-$ invariant equivalence relation " $\approx_{AL}$ " on $\hat{\mathbb{Q}}(AL)$. Since $AL(\Gamma_0(n))$ acts transitively on $\hat{\mathbb{Q}}(AL)$, every element of $\hat{\mathbb{Q}}(AL)$ has the form $g(\infty)$ for some $g \in AL(\Gamma_0(n))$. Thus, it is easily seen that

$$g(\infty) \approx_{AL} g'(\infty) \Longleftrightarrow g' \in g\Gamma_0(n)$$

gives an $AL(\Gamma_0(n))-$invariant imprimitive equivalence relation.

Notation. s_m denotes the integers s such that $(s,n) = m$. □

Theorem 3.2 *Let* $\frac{r}{s_m}, \frac{x}{y_k} \in \hat{\mathbb{Q}}(AL)$. *Then* $\frac{r}{s_m} \approx_{AL} \frac{x}{y_k}$ *if and only if* $m = k$, *that is* $(s,n) = (y,n)$.

Proof Because of the transitivity, if $\frac{r}{s_m}$ and $\frac{x}{y_k}$ are the elements of $\hat{\mathbb{Q}}(AL)$, then $\frac{r}{s_m} = g(\infty)$ and $\frac{x}{y_k} = g'(\infty)$ for the elements $g, g' \in AL((\Gamma_0(n)))$. We see that

$$\frac{r}{s_m} \approx_{AL} \frac{x}{y_k} \Longleftrightarrow g^{-1}g' \in \Gamma_0(n)$$

Hence, the problem belongs to the same $\Gamma_0(n)$-coset for two arbitrary Atkin–Lehner transformations, which was solved (see [1,7]). The result is obvious. □

Corollary 3.1 *The number of blocks of the imprimitive action of* $AL(\Gamma_0(n))$ *on the set* $\in \hat{\mathbb{Q}}(AL)$ *is just* 2^r, *where* r *is the number of distinct primes dividing* n.

Corollary 3.2 $\hat{\mathbb{Q}}(AL)$ *is all* $\hat{\mathbb{Q}}$ *if and only if* n *is a square-free.*

Example 3.1 Case $n = 2$. *In [11], blocks are given for the simplest case. Since* $|\Gamma : \Gamma_0(2)| = 3$, *the blocks have the form*

$$[0] = \left\{\frac{even}{odd}\right\},\ [1] = \left\{\frac{odd}{odd}\right\},\ [\infty] = \left\{\frac{odd}{even}\right\}.$$

In our case, $\hat{\mathbb{Q}}(AL(\Gamma_0(2)))$ *is all* $\hat{\mathbb{Q}}$. *Since* $|AL(\Gamma_0(2)) : \Gamma_0(2)| = 2$, *the blocks have the form*

$$[0] = \left\{\frac{even}{odd}, \frac{odd}{odd}\right\},\ [\infty] = \left\{\frac{odd}{even}\right\}.$$

Case $n = 2 \cdot 3 \cdot 5$. *By above corollary,* $\hat{\mathbb{Q}}(AL(\Gamma_0(30)))$ *is all* $\hat{\mathbb{Q}}$ *and the blocks are*

$$\left[\frac{1}{1}\right], \left[\frac{1}{2}\right], \left[\frac{1}{3}\right], \left[\frac{1}{5}\right], \left[\frac{1}{2\cdot 3}\right], \left[\frac{1}{2\cdot 5}\right], \left[\frac{1}{3\cdot 5}\right], \left[\frac{1}{2\cdot 3\cdot 5}\right].$$

The last one is the block $\left[\frac{1}{0}\right] = [\infty]$.

Corollary 3.3 *The index* $|AL(\Gamma_0(n)) : \Gamma_0(n)|$ *is equal to* 2^r, *which is also the number of blocks.*

4. Suborbital graphs and edge conditions

$AL(\Gamma_0(n))$ acts on $\hat{\mathbb{Q}}(AL) \times \hat{\mathbb{Q}}(AL)$ by $g(\alpha,\beta) = (g(\alpha), g(\beta))(g \in AL(\Gamma_0(n)), \alpha, \beta \in \hat{\mathbb{Q}}(AL))$. The orbits of this action are called *suborbitals* of $AL(\Gamma_0(n))$. From the suborbital $O(\alpha,\beta)$ containing (α,β) we can form the *suborbital graph* $G(\alpha,\beta)$; its vertices are the elements of $\hat{\mathbb{Q}}(AL)$, and there is a directed edge from k to l, denoted by $k \to l$, if $(k,l) \in O(\alpha,\beta)$. Because of the elements, we draw this edge as a hyperbolic geodesic in $\mathbb{H}$.

We now investigate these suborbital graphs. Since $AL(\Gamma_0(n))$ acts transitively on $\hat{\mathbb{Q}}(AL)$, each suborbital $O(\alpha,\beta)$ contains a pair (∞, v) for some $v \in \hat{\mathbb{Q}}(AL)$. Let $v = \frac{u}{m}$, $m \| n$. Then the suborbital $O(\infty, \frac{u}{m})$ will be denoted by $O_{u,m}$ and related suborbital graph by $G_{u,m}$.

Theorem 4.1 *There is an edge* $\dfrac{r}{s_e} \to \dfrac{x}{y_f} \in G_{u,m} \iff$

(i) $x \equiv r(n/e)u \bmod \left(\dfrac{m}{(n/e,m)}\right)$, $y_f \equiv us_e \dfrac{n/e}{(n/e,m)} \bmod \left(\dfrac{n/e}{(n/e,m)}m\right)$, $ry_f - xs_e = \dfrac{m}{(n/e,m)}$ *or,*

(ii) $x \equiv -r(n/e)u \bmod \left(\dfrac{m}{(n/e,m)}\right)$, $y_f \equiv -us_e \dfrac{n/e}{(n/e,m)} \bmod \left(\dfrac{n/e}{(n/e,m)}m\right)$, $ry_f - xs_e = -\dfrac{m}{(n/e,m)}$.

Proof Let $\dfrac{r}{s_e} \to \dfrac{x}{y_f} \in G_{u,m}$. Then some element $\begin{pmatrix} at & b \\ cn & dt \end{pmatrix} \in AL(\Gamma_0(N))$ sends $\dfrac{1}{0}$ to $\dfrac{r}{s_e}$ and $\dfrac{u}{m}$ to $\dfrac{x}{y_f}$. $\begin{pmatrix} at & b \\ cn & dt \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \dfrac{at}{cn} = \dfrac{a}{c\frac{n}{t}}$. Since $adt - bc\frac{n}{t} = 1$, a and $c\frac{n}{t}$ are coprime. Therefore $\dfrac{a}{c\frac{n}{t}} = \dfrac{r}{s_e}\left(\text{or } \dfrac{-r}{-s_e}\right)$. Thus $a = r$, $n\frac{n}{t} = s_e$. It can be easily seen that $\frac{n}{t} = e$. On the other hand,

$$\begin{pmatrix} a\frac{n}{e} & b \\ cn & d\frac{n}{e} \end{pmatrix}\begin{pmatrix} u \\ m \end{pmatrix} = \frac{a\frac{n}{e}u + bm}{cnu + d\frac{n}{e}m} = \frac{a\frac{n}{e}u + bm}{\frac{n}{e}(ceu + dm)}.$$

Since $\left(a\frac{n}{e}u + bm, ceu + dm\right) = 1$ and $\left(a\frac{\frac{n}{e}}{(\frac{n}{e},m)} + b\frac{m}{(\frac{n}{e},m)}, \frac{n}{e}\right) = 1$, then

$\dfrac{a\frac{\frac{n}{e}}{(\frac{n}{e},m)} + b\frac{m}{(\frac{n}{e},m)}}{\frac{\frac{n}{e}}{(\frac{n}{e},m)}(ceu + dm)}$ is a reduced fraction.

$x = a\dfrac{\frac{n}{e}}{(\frac{n}{e},m)}u + b\dfrac{m}{(\frac{n}{e},m)}$ and $y_f = \dfrac{\frac{n}{e}}{(\frac{n}{e},m)}(ceu + dm)$. Thus, $x \equiv r(\frac{n}{e})u \bmod \left(\dfrac{m}{(\frac{n}{e},m)}\right)$ and $y_f \equiv us_e\dfrac{\frac{n}{e}}{(\frac{n}{e},m)} \bmod \left(\dfrac{\frac{n}{e}}{(\frac{n}{e},m)}m\right)$. Furthermore,

$$\begin{pmatrix} r & x \\ s_e & y_f \end{pmatrix} = \begin{pmatrix} a & a\dfrac{\frac{n}{e}}{(\frac{n}{e},m)}u + b\dfrac{m}{(\frac{n}{e},m)} \\ ce & cu\dfrac{n}{(\frac{n}{e},m)}u + \dfrac{d\frac{n}{e}m}{(\frac{n}{e},m)} \end{pmatrix}.$$

Taking the determinant, we get that
$ry_f - xs_e = \dfrac{m}{(\frac{n}{e},m)}$. This gives (a). If we take the minus sign, (b) can, likewise, be obtained.

Conversely suppose that (a) holds; then there exists some integers b and d such that $x = r(\frac{\frac{n}{e}}{(\frac{n}{e},m)})u + b\frac{m}{(\frac{n}{e},m)}$, $y_f = s_e u(\frac{\frac{n}{e}}{(\frac{n}{e},m)})u + d\frac{\frac{n}{e}}{(\frac{n}{e},m)}m$. Hence the element $\begin{pmatrix} r\frac{n}{e} & x \\ s_e\frac{n}{e} & d\frac{n}{e} \end{pmatrix}$ is in $AL((\Gamma_0(n)))$, and sends $\frac{1}{0}$ to $\frac{r}{s_e}$ and $\frac{u}{m}$ to $\frac{x}{y_f}$ as follows:

$$\begin{pmatrix} r & b \\ s_e & d\frac{n}{e} \end{pmatrix} \begin{pmatrix} 1 & (\frac{\frac{n}{e}}{(\frac{n}{e},m)})u \\ 0 & (\frac{m}{(\frac{n}{e},m)}) \end{pmatrix} = \begin{pmatrix} r & ru\frac{\frac{n}{e}}{(\frac{n}{e},m)}u + b\frac{m}{(\frac{n}{e},m)} \\ s_e & s_e u\frac{n}{(\frac{n}{e},m)}u + d\frac{\frac{n}{e}}{(\frac{n}{e},m)}m \end{pmatrix} = \begin{pmatrix} r & x \\ s_e & y_f \end{pmatrix}.$$

From (a), the right side matrix has determinant 1; therefore $\begin{pmatrix} r & b \\ s_e & d\frac{n}{e} \end{pmatrix}$ is a desired transformation belonging to $AL(\Gamma_0(n))$. If (b) holds, we again get the proof. □

Corollary 4.1 *There is an edge* $\frac{r}{s_m} \to \frac{x}{y_m} \in G_{u,m} \Longleftrightarrow$

(i) $x \equiv r\frac{n}{m}u \ mod\,(m)$, $y_m \equiv 0 mod\,(n)$, $ry_m - xs_m = m$ *or,*

(ii) $x \equiv -r\frac{n}{m}u \ mod\,(m)$, $y_m \equiv 0 mod\,(n)$, $ry_m - xs_m = -m$.

Corollary 4.2 *If* $e = f = m = n$, *there is an edge* $\frac{r}{s_n} \to \frac{x}{y_n} \in G_{u,n} \Longleftrightarrow$

(i) $x \equiv ru \ mod\,(m)$, $y_m \equiv us_n \ mod\,(n)$, $ry_n - xs_n = n$ *or,*

(ii) $x \equiv -ru \ mod\,(m)$, $y_m \equiv -us_n \ mod\,(n)$, $ry_n - xs_n = n$.

Proof In this case, desired elements in $AL(\Gamma_0(n))$ come from the group $\Gamma_0(n)$, since $\frac{n}{e} = 1$. □

5. Circuits and triangles

Let us give some standard notations as definitions.

Definition 5.1 *Let* $\nu_1, \nu_2, \cdots \to \nu_k$ *be different vertices in* $G_{u,m}$.

(i) *The configuration* $\nu_1 \to \nu_2 \to \cdots \to \nu_k \to \nu_1$ *is called a directed circuit or a closed path.*

(ii) *An anti-directed circuit will denote a configuration like the above with at least one arrow (not all) reversed.*

(iii) *If* $k = 2$, *the circuit* $\nu_1 \to \nu_2 \to \nu_1$, *is called a self-paired edge.*

(iv) *If* $k = 3$, *the circuit, directed or not, is called a triangle.*

Theorem 5.1 *Let* $F_{u,n}$ *be the subgraph of* $G_{u,n}$ *whose vertices are the block* $\begin{bmatrix} 1 \\ n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; *then* $F_{u,n}$ *contains*

(i) *a self-paired edge if and only if* $u^2 \equiv -1(modn)$.

(ii) *a directed triangle if and only if* $u^2 \pm u + 1 \equiv 0(modn)$.

Proof Theorem 5.11 in [11] and Corollary 4.3 give the result. □

Unlike the results in [11], there is a different situation as below:

Corollary 5.1 *If* $\frac{r}{s_e} \to \frac{x}{y_f} \in G_{u,m}$ *then the vertices* $\frac{r}{s_e}$ *and* $\frac{x}{y_f}$ *might not be in the same block.*

Proof Taking $n = 2^2 3^2$, $\frac{1}{2^2} \to \frac{17}{2^3 3^2} \in G_{1,2^2}$ but $\frac{1}{2^2}$ and $\frac{17}{2^3 3^2}$ are not in the same block. It is clear that $\left|AL(\Gamma_0(2^2 3^2)) : \Gamma_0(2^2 3^2)\right| = 4$ and

$$\hat{\mathbb{Q}}(AL(\Gamma_0(2^2 3^2))) = \left[\frac{1}{1}\right] \cup \left[\frac{1}{2^2}\right] \cup \left[\frac{1}{3^2}\right] \cup \left[\frac{1}{2^2 3^2}\right].$$

It is easily seen that $\frac{1}{2^2} \in \left[\frac{1}{2^2}\right]$ and $\frac{17}{2^3 3^2} \in \left[\frac{1}{2^2 3^2}\right]$. □

Theorem 5.2 *There is a self-paired edge whose vertices are in the same block in* $G_{u,m}$ *if and only if* $m = n$.

Proof By definition 5.1, let $\frac{r}{s_m} \to \frac{x}{y_m} \to \frac{r}{s_m}$ be a self-paired edge. Since $m \| n$, assume that $n = km$ such that $(k, m) = 1$. Without loss of generality, we suppose that $s = \alpha m, y = \beta m$ for some $\alpha, \beta \in \mathbb{Z}$. Since $(s, n) = m$ and $(y, n) = m$, then $(\alpha, k) = 1$ and $(\beta, k) = 1$. By applying corollary 4.2 to the first edge, we get $k|\beta$, which contradicts $(\beta, k) = 1$.

Hence, the case assuming $m = n$ turns into the case in [11]. For example, all edges in $G_{1,2}$ are self-paired. □

Theorem 5.3 $G_{u,m}$ *is self-paired if and only if* $u^2 \equiv -1(modm)$.

Proof $G_{u,m}$ is self-paired means that the pair $(\infty, \frac{u}{m})$ is sent to $(\frac{u}{m}, \infty)$ by $AL((\Gamma_0(n)))$. Since $\begin{pmatrix} at & b \\ cn & dt \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{u}{m}$, then $a = u$, $c\frac{n}{t} = m$. From the equation $\begin{pmatrix} ut & b \\ cn & dt \end{pmatrix} \begin{pmatrix} u \\ m \end{pmatrix} = \frac{1}{0}$, we see that $\begin{pmatrix} ut & b \\ cn & -ut \end{pmatrix}$ is a desired element. Taking the determinant $-u^2 - bcm = 1$, the result is obvious. □

Theorem 5.4 *There are no triangle vertices that are in same block in* $G_{u,m}$.

Proof Suppose that $G_{u,m}$ contains a triangle. Because of the transitive action, we may suppose that it has the form $\infty \to u \to v \to \infty$. By Corollary 4.2 we have that denominators of u and v must be equal to m and that denominator of u must be equivalent to n. Thus, it must be $m = n$. However, there would be a contradiction of the condition $ry_n - xs_n = n$ from the second edge. □

Theorem 5.5 *If* n *is square-free, there is no triangle in* $G_{u,m}$.

Proof Without loss of generality, we give the proof for $n = p_1p_2$ where p_1, p_2 are coprime. In this case, the number of blocks is 4. We can write the blocks as

$$\left[\frac{1}{1}\right] \cup \left[\frac{1}{p_1}\right] \cup \left[\frac{1}{p_2}\right] \cup \left[\frac{1}{p_1p_2}\right]$$

Since $m \| n$, without loss of generality, let $m = p_1$. Hence, the graph will be shown by G_{u,p_1}. Because of the transitivity, we may assume that the triangle has the form $\infty \to u \to v \to \infty$.

From now on, we use the symbols in Theorem 4.1. From the first edge, since the denominator of the first vertex is equal to p_1p_2, then e is equal to p_1p_2. Hence, we get the denominator of the second vertex is equal to p_1 by the condition $ry_f - xs_e = \dfrac{m}{(n/e, m)}$. Furthermore, the numerator of the second vertex is equal to $u + kp_1$ for some integer k.

On the other hand, the third vertex may belong to the blocks

$$\left[\frac{1}{1}\right], \left[\frac{1}{p_2}\right], \left[\frac{1}{p_1p_2}\right]$$

because of the denominator of the second vertex. Otherwise, applying Theorem 4.1 to the second edge, it gives a contradiction. Taking the second vertex as $\dfrac{u + kp_1}{p_1}$, we apply Theorem 4.1 to the second edge. We see that denominator of the third vertex is congruent to 0 modulo p_1p_2. In this case, it gives a contradiction applying Theorem 4.1 to the third edge. □

It is well known that there are some elliptic elements of order 3 in the Atkin–Lehner group. At this point, taking into account the relation elliptic elements in group and circuits in the related graph, it is reasonable to conjecture that

Conjecture 5.1 *$G_{u,m}$ has a triangle if and only if n is not a square-free integer.*

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References

[1] Akbaş M, Singerman D. The normalizer of $\Gamma_0(N)$ in $PSL(2, R)$. Glasgow Math J 1990; 32: 317-327.

[2] Akbaş M, Başkan T. Suborbital graphs for the normalizer of $\Gamma_0(N)$. Turk J Math 1996; 20: 379-387.

[3] Akbaş M. On suborbital graphs for the modular group. Bull London Math Soc 2001; 33: 647-652.

[4] Akbaş M, Kör T, Kesicioglu Y. Disconnectedness of the subgraph F^3 for the group Γ^3. J Inequal Appl 2013; 283: 7 pp.

[5] Beşenk M, Güler BO, Değer AH, Kesicioğlu Y. Circuit lengths of graphs for the Picard group. J Inequal Appl 2013; 106: 8 pp.

[6] Bigg NL, White AT. Permutation groups and combinatorial structures. London Math Soc Lecture Note Series; 33, Cambridge, UK: CUP, 1979.

[7] Conway JH, Norton SP. Monstrous moonshine. Bull London Math Soc 1977; 11: 308-339.

[8] Deger AH, Beşenk M, Güler BO. On suborbital graphs and related continued fractions. Appl Math Comput 2011; 218: 746-750.

[9] Güler BO, Beşenk M, Değer AH, Kader S. Elliptic elements and circuits in suborbital graphs. Hacet J Math Stat 2011; 40: 203-210.

[10] Jones GA, Singerman D. Complex Functions: an Algebraic and Geometric Viewpoint. Cambridge, UK: CUP, 1987.

[11] Jones GA, Singerman D, Wicks K. The modular group and generalized Farey graphs. London Math Soc Lecture Note Series 1991; 160: 316-338.

[12] Kader S, Güler BO, Değer AH. Suborbital graphs for a special subgroup of the normalizer. Iran J Sci Technol A 2010; 34: 305-312.

[13] Kader S, Güler BO. On suborbital graphs for the extended Modular group $\hat{\Gamma}$. Graphs and Combinatorics 2013; 29: 1813-1825.

[14] Kesicioglu Y, Akbaş M, Beşenk M. Connectedness of a suborbital graph for congruence subgroups. J Inequal Appl 2013; 117: 7 pp.

[15] Keskin R. Suborbital graphs for the normalizer of $\Gamma_0(m)$. Eur J Combin 2006; 27: 193-206.

[16] Keskin R, Demirturk B. On suborbital graphs for the normalizer of $\Gamma_0(N)$. Electron J Combin 2009; 16: 116, 18 pp.

[17] Sarma R, Kushwaha S, Krishnan R. Continued fractions arising from $F_{1,2}$. J Number Theory 2015; 154: 179-200.

[18] Sims CC. Graphs and finite permutation groups. Math Z 1967; 95: 76-86.