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An age-structured model for the transmission dynamics of hepatitis B: asymptotic analysis

Rodrigue Yves M'PIKA MASSOUKOU[1], **Aboubakari TRAORE**[2,*]
[1]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa
[2]Department of Mathematics and Computer Sciences, Cheikh Anta Diop University, Dakar, Senegal

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Abstract: In this paper, we consider the age-structured model for the transmission dynamics of Hepatitis B virus (HBV) proposed earlier in the article by Zou et al.: An age-structured model for transmission dynamics of hepatitis B. SIAM J Appl Math 2010; 70: 3121-3139, where a slight modification is made. We consider that the HBV infection processes act on a time scale different from that of the vital processes. Such a model becomes a multiple time scale model and thus it often can be significantly simplified by various asymptotic methods. We apply, as in the paper of Banasiak and M'pika Massoukou: A singularly perturbed age structured SIRS model with fast recovery. Disc Cont Dyn Sys 2014, a suitable technique of asymptotic analysis, based on the Chapman–Enskog procedure, which allows separation of scales and aggregation of variables.

Key words: Hepatitis B virus (HBV), age structure, asymptotic analysis

1. Introduction

Hepatitis B virus (HBV) is mainly transmitted through body fluids like blood, semen, and vaginal secretions. One of the most important factors influencing the probability of developing carriage of HBV is age, involving mature individuals. Thus, it can be expected that the interplay of the demographic processes with the infection mechanism will produce a nontrivial dynamics.

In this paper we consider the model for transmission dynamic of HBV introduced in [20], but slightly modified, which is formulated under the following assumptions:

(1) The latent, acute, and carrier stages are differentiated. Only acute individuals and carrier individuals are infectious.

(2) All latently infected individuals develop acute hepatitis B first.

(3) Some individuals with acute infection progress towards the carrier state and later develop immunity while others develop immunity without progressing towards the carrier state.

(4) Since the disease-induced death rate is relatively low, it is ignored.

(5) There is a possibility for treatment (or recovery) during both the acute stage of infection and the carrier state of infection.

*Correspondence: traoreabou08@gmail.com

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The population is divided into six subclasses: susceptible individuals, infected individuals but not yet infectious (latent), acutely infectious individuals, carrier individuals, individuals who have recovered from infection and are now immune, and vaccinated immune individuals, and we denote by $s(a,t)$, $l(a,t)$, $i(a,t)$, $c(a,t)$, $r(a,t)$, and $v(a,t)$ the corresponding density functions for these epidemiological age-structured classes, respectively.

Let $\beta(a)$ and $\mu(a)$ be the age-specific fertility and the age-specific mortality (or natural mortality rate) of the population, which we suppose are not affected by the disease; σ is the rate moving from latent to acute, γ_1 is the rate moving from acute to carrier, $\gamma_2(a)$ is the rate moving from carrier to vaccinated, $p(a,t)$ is the vaccination rate against HBV, $(1-\omega)$ is the proportion of births with successful vaccination, $\omega \in [0,1]$, $q(a)$ is the probability an individual fails to clear an acute infection and develops to carrier state, ψ is the rate of waning vaccine-induced immunity, and Λ is the infection rate (or force of infection).

We consider the following separable intercohort constitutive form for the force of infection [7]:

$$\Lambda(a, i(\cdot,t), c(\cdot,t)) = k(a) \int_0^{a_+} [h_1(a)i(a,t) + h_2(a)c(a,t)] \, da, \tag{1}$$

where $h_1(a)$ and $h_2(a)$ are the age-specific infectiousness corresponding to acute stage and chronic stage, respectively, $k(a)$, the age-specific contagion rate; here we assume that the two stages have the same contagion rate and a_+ is the maximum age of an individual. We assume that $h_1(a)$, $h_2(a)$, and $k(a)$ satisfy the following conditions:

$$h_j, k \in \mathrm{L}^\infty([0,a_+]), \quad h_j(a), k(a) \geq 0 \quad \text{a.e. in } [0,a_+]. \tag{2}$$

We assume that there is no proportion of perinatal infected (from carrier mothers) newborns, there is a proportion of births with successful vaccination, there is a proportion of susceptible newborns, and that an individual may become infected through contact both with acute hepatitis B individuals and chronic hepatitis B individuals. Then the dynamics of the age-structured epidemiological model for the transmission of HBV can be described by the following initial boundary value problem:

$$\begin{aligned}
\partial_t s(a,t) &= -\partial_a s(a,t) - \mu(a)s(a,t) + \psi v(a,t) - s(a,t)\Lambda(a, i(\cdot,t), c(\cdot,t)) \\
&\quad - p(a,t)s(a,t), \\
\partial_t l(a,t) &= -\partial_a l(a,t) - \mu(a)l(a,t) - \sigma l(a,t) + s(a,t)\Lambda(a, i(\cdot,t), c(\cdot,t)), \\
\partial_t i(a,t) &= -\partial_a i(a,t) - \mu(a)i(a,t) + \sigma l(a,t) - \gamma_1 i(a,t), \\
\partial_t c(a,t) &= -\partial_a c(a,t) - \mu(a)c(a,t) - \gamma_2(a)c(a,t) + q(a)\gamma_1 i(a,t), \\
\partial_t r(a,t) &= -\partial_a r(a,t) - \mu(a)r(a,t) + \gamma_2(a)c(a,t) + (1-q(a))\gamma_1 i(a,t), \\
\partial_t v(a,t) &= -\partial_a v(a,t) - \mu(a)v(a,t) - \psi v(a,t) + p(a,t)s(a,t),
\end{aligned} \tag{3}$$

with boundary conditions

$$
\begin{aligned}
s(0,t) &= \omega \int_0^{a_+} \beta(a)\left[s(a,t)+l(a,t)+i(a,t)+r(a,t)+v(a,t)+c(a,t)\right] da, \\
l(0,t) &= i(0,t) = c(0,t) = r(0,t) = 0, \\
v(0,t) &= (1-\omega) \int_0^{a_+} \left[s(a,t)+l(a,t)+i(a,t)+r(a,t)+v(a,t)+c(a,t)\right] da,
\end{aligned} \tag{4}
$$

and initial conditions

$$
\begin{aligned}
&s(a,0) = \overset{o}{s}(a), \quad l(a,0) = \overset{o}{l}(a), \quad i(a,0) = \overset{o}{i}(a), \\
&c(a,0) = \overset{o}{c}(a), \quad r(a,0) = \overset{o}{r}(a), \quad v(a,0) = \overset{o}{v}(a).
\end{aligned} \tag{5}
$$

Note that in building the age-structured (epidemiological) model (3), which takes into account both the vital and infection dynamics, we must be careful as many diseases act on different time scales than the vital process. Since we are dealing with a human population, the death and birth rates are measured in units 1/70 years, where 70 is considered to be the average life-span in the population.

Regarding the numerical values for the parameters in the model (3) provided in [19, 20], we see that $\beta = 0.0121$, $\mu = 0.00693$, $\mu_1 = 0.002$, $\psi = 0.1$, $\Lambda \approx 0.16$, $\sigma = 6/\text{year}$, $\gamma_1 = 4/\text{year}$, and $\gamma_2 = 0.025/\text{year}$. Using 70 years as unit of time in the model (3), the numerical values for σ, γ_1, and γ_2 should be multiplied by 70 and this results in $\sigma = 420$, $\gamma_1 = 280$, and $\gamma_2 = 1.75$. This shows that the processes induced by σ and γ_1 are faster than those induced by β, μ (demography processes), ψ and Λ, while the process induced by γ_2 is slightly faster than those induced by β, μ (demography processes), ψ and Λ. Here we consider the model where the duration of carriage is of the same time-scale as the duration of latency and acute infection such that these processes are faster than those induced by β, μ (demography processes), ψ, and Λ. Thus we consider

$$
\begin{aligned}
\partial_t \mathbf{u}_\epsilon &= \mathcal{S}\mathbf{u}_\epsilon + \mathcal{M}\mathbf{u}_\epsilon + \mathcal{F}(\mathbf{u}_\epsilon) + \frac{1}{\epsilon}\mathcal{C}\mathbf{u}_\epsilon, \\
\mathbf{u}_\epsilon(0,t) &= \mathcal{B}\left[\mathbf{u}_\epsilon(\cdot,t)\right], \\
\mathbf{u}_\epsilon(a,0) &= \overset{o}{\mathbf{u}},
\end{aligned} \tag{6}
$$

where $\mathbf{u}_\epsilon = (s_\epsilon, l_\epsilon, i_\epsilon, c_\epsilon, r_\epsilon, v_\epsilon)$, $\mathcal{S} = \operatorname{diag}\{-\partial_a, -\partial_a, -\partial_a, -\partial_a, -\partial_a, -\partial_a\}$ on $D(\mathcal{S}) = \mathrm{W}^{1,1}\left([0,a_+], \mathbb{R}^6\right)$,

$$
\mathcal{M}(a) = \begin{pmatrix}
-\mu(a) & 0 & 0 & 0 & 0 & \psi \\
0 & -\mu(a) & 0 & 0 & 0 & 0 \\
0 & 0 & -\mu(a) & 0 & 0 & 0 \\
0 & 0 & 0 & -\mu(a) & 0 & 0 \\
0 & 0 & 0 & 0 & -\mu(a) & 0 \\
0 & 0 & 0 & 0 & 0 & -\mu(a)-\psi
\end{pmatrix},
$$

on $D(\mathcal{M}) = \{\mathbf{u} \in \mathrm{L}^1\left([0,a_+], \mathbb{R}^6\right); \mu\mathbf{u} \in \mathrm{L}^1\left([0,a_+], \mathbb{R}^6\right)\}$ and ϵ is a small parameter reflecting the ratio of the typical time scales of the vital and epidemiological processes. Moreover, the bounded operator $\mathcal{B}$:

$\mathrm{L}^1\left([0,a_+],\mathbb{R}^6\right) \to \mathbb{R}^6$ is defined by

$$\mathcal{B}\mathbf{u} = \int_0^{a_+} B(a)\mathbf{u}(a)\,da,$$

with

$$B(a) = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{pmatrix},$$

$b_{1k} = \omega\beta(a)$ and $b_{6k} = (1-\omega)\beta(a)$, $k = 1,\ldots,6$,

$$[\mathcal{F}(\mathbf{u})](a) = \begin{pmatrix} -s(a)p(a) - s(a)\Lambda(a,i,c) \\ s(a)\Lambda(a,i,c) \\ 0 \\ 0 \\ 0 \\ s(a)p(a) \end{pmatrix}$$

and

$$[\mathcal{C}\mathbf{u}](a) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sigma & 0 & 0 & 0 & 0 \\ 0 & \sigma & -\gamma_1 & 0 & 0 & 0 \\ 0 & 0 & q(a)\gamma_1 & -\gamma_2(a) & 0 & 0 \\ 0 & 0 & (1-q(a))\gamma_1 & \gamma_2(a) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\mathbf{u}(a).$$

We aim to investigate the behavior of the solution $(s_\epsilon, l_\epsilon, i_\epsilon, c_\epsilon, r_\epsilon, v_\epsilon)$, of (6), as $\epsilon \to 0$. We shall note that if $(s_\epsilon, l_\epsilon, i_\epsilon, c_\epsilon, r_\epsilon, v_\epsilon)$ satisfies (6) then, by summing the equations in (6), it results that the total population,

$$n(a,t) = s_\epsilon(a,t) + l_\epsilon(a,t) + i_\epsilon(a,t) + c_\epsilon(a,t) + r_\epsilon(a,t) + v_\epsilon(a,t),$$

satisfies

$$\begin{aligned} \partial_t n(a,t) &= -\partial_a n(a,t) - \mu(a)n(a,t), \\ n(0,t) &= \int_0^{a_+} \beta(a)n(a,t)\,da, \\ n(a,0) &= \overset{\circ}{n}(a), \end{aligned} \tag{7}$$

and is independent of ϵ. It follows from [8] that there exists a dominant eigenvalue $\lambda_\mu \leq \overline{\beta} - \underline{\mu}$, defined as the unique real solution to

$$\int_0^{a_+} e^{-\lambda a}\beta(a)\Pi_\mu(a)\,da = 1, \tag{8}$$

where, see [2, 8, 16],

$$\Pi_\mu(a) := e^{-\int\limits_0^a \mu(s)\,ds} \tag{9}$$

is the probability of survival of an individual until age a, and a constant M such that

$$\|n(t)\| \leq M e^{\lambda_\mu t} \|\overset{\circ}{n}\|, \tag{10}$$

where λ_μ is negative, zero, or positive if and only if the *net reproduction rate* $R_\mu = \int\limits_0^{a_+} \beta(a)\Pi_\mu(a)\,da$ is, respectively, smaller than, equal to, or greater than one. As we mentioned earlier, the considered scaling is realistic for models in which there are no significant changes in the total population. Hence, throughout the paper we will assume

$$R_\mu \leq 1, \tag{11}$$

that is, $\overline{\beta} \leq \underline{\mu}$.

Let $\overline{w}$ be the solution to the McKendrick–von Foerster problem

$$\begin{aligned} \partial_t \overline{w}(a,t) &= -\partial_a \overline{w}(a,t) - \mu(a)\overline{w}(a,t), \\ \overline{w}(0,t) &= 0, \\ \overline{w}(a,0) &= \overset{\circ}{w}(a) = \overset{\circ}{l}(a) + \overset{\circ}{i}(a) + \overset{\circ}{c}(a) + \overset{\circ}{r}(a), \end{aligned} \tag{12}$$

and $\overline{v}$ be the solution to the McKendrick–von Foerster problem

$$\begin{aligned} \partial_t \overline{v}(a,t) &= -\partial_a \overline{v}(a,t) - \mu(a)\overline{v}(a,t) - \psi\overline{v}(a,t) \\ &\quad - p(a,t)\overline{v}(a,t) + p(a,t)(n(a,t) - \overline{w}(a,t)), \\ \overline{v}(0,t) &= (1-\omega)n(0,t), \\ \overline{v}(a,0) &= \overset{\circ}{v}(a). \end{aligned} \tag{13}$$

2. Notation, assumptions, and well-posedness results

In the sequel we consider the state space $\mathbf{X} = \mathrm{L}^1\left([0,a_+],\mathbb{R}^6\right)$. We denote by $\mathbf{X}_+ = \mathrm{L}^1\left([0,a_+],\mathbb{R}^6_+\right)$ the positive cone of $\mathbf{X}$ and $\|\cdot\|$ the suitable norm in L^1. If necessary, the notation $\|\cdot\|_X$ will be used for the norm in a specific space X. In addition, for any measurable function η on $[0,a_+]$, we introduce the notation

$$\overline{\eta} = \operatorname*{ess\,sup}_{a\in[0,a_+]} \eta(a), \quad \underline{\eta} = \operatorname*{ess\,inf}_{a\in[0,a_+]} \eta(a)$$

and we make the assumptions

A1: $\mu \in \mathrm{L}^1_{\mathrm{loc}}([0,a_+))$, $\displaystyle\int_0^{a_+} \mu(s)\,ds = \infty$ with $\underline{\mu} > 0$;

A2: $\beta \in \mathrm{L}^\infty([0,a_+])$;

A3: $q, \gamma_2 \in \mathrm{W}^{1,\infty}([0,a_+])$ with $\underline{q} > 0$, $\underline{\gamma_2} > 0$;

A4: $p \in \mathcal{C}([0,a_+] \times [0,T])$.

We make the biologically realistic assumption on the maximum age, a_+, such that $a_+ < \infty$, meaning that no individual can live indefinitely. This requires the survival probability Π_μ, defined by

$$\Pi_\mu(a) := e^{-\int\limits_0^a \mu(s)\,ds}, \tag{14}$$

to satisfy $\Pi_\mu(a_+) = 0$, which accounts for the nonintegrable singularity in **A1**. Hence, μ cannot be assumed bounded as $a \to a_+^-$ while it could be bounded in the case $a_+ = \infty$. This justifies the fact that most authors [9, 11, 14, 15, 18] consider an infinite maximum age, $a_+ = \infty$, though without any biological significance, in order to be able to handle μ after being assumed bounded. In some papers, e.g., [10], the unboundedness of μ, in the case $a_+ < \infty$, was circumvented by assuming that there is a maximum reproductive age $a_r < a_+$, so that the birth rate satisfies $\beta(a) = 0$ for $a > a_r$, and hence ignoring the postreproductive population by performing the analysis for $a \in [0, a_r]$, which causes the loss of the conservativeness of the model. The analysis of the model without any simplifying assumption in the scalar linear case was done in [8] by reducing it to an integral equation along the characteristics. It is known that the solutions obtained in this way generate a strongly continuous semigroup on $\mathrm{L}^1([0,a_+])$; see [3, 17] (though in [17] it is assumed that $a_+ = \infty$.)

The technical details required to handle the unbounded μ, on $[0,a_+]$ with $a_+ < \infty$, in the problem at hand can be found in [12]. In particular, it follows that the realization of the operator $\mathcal{A} := \mathcal{S} + \mathcal{M}$ on the domain

$$D(\mathcal{A}) = \{\mathbf{u} \in D(\mathcal{S}) \cap D(\mathcal{M});\ \mathbf{u}(0) = \mathcal{B}\mathbf{u}\} \tag{15}$$

generates a positive $\mathcal{C}_0$-semigroup, denoted by $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$. Since, for a fixed ϵ, (7) is a quadratic perturbation of the linear system, a standard argument, see [5, 12], shows that if $\mathring{\mathbf{u}} = (\mathring{s}, \mathring{l}, \mathring{i}, \mathring{c}, \mathring{r}, \mathring{v}) \in \mathbf{X}_+$, then there exists a unique global positive mild solution $t \to \mathbf{u}_\epsilon(t) = (s_\epsilon(t), l_\epsilon(t), i_\epsilon(t), c_\epsilon(t), r_\epsilon(t), v_\epsilon(t)) \in \mathcal{C}([0,\infty), \mathbf{X})$ to (7). This solution becomes a classical solution if $\mathring{\mathbf{u}} \in D(\mathcal{A})$. In such a case we obtain, in particular, that $\mathbf{u}_\epsilon$ is continuous on $[0,a_+] \times [0,T]$ for any $0 \leq T < \infty$, $\mathbf{u}_\epsilon \in D(\mathcal{S}) \cap D(\mathcal{M})$ and (7) is satisfied termwise almost everywhere on $[0,a_+] \times [0,T]$. Standard calculations, see e.g. [10], show that $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ satisfies the estimate

$$\|e^{t\mathcal{A}}\| \leq e^{(\overline{\beta} - \underline{\mu})t}. \tag{16}$$

This estimate can be improved. In fact, as mentioned earlier, if $(s_\epsilon(t), l_\epsilon(t), i_\epsilon(t),$ $c_\epsilon(t), r_\epsilon(t), v_\epsilon(t))$ is a classical solution to (6), then $n(a,t) = s_\epsilon(a,t) + l_\epsilon(a,t) + i_\epsilon(a,t) + c_\epsilon(a,t) + r_\epsilon(a,t) + v_\epsilon(a,t)$ is a classical solution to (7). We denote by $(A, D(A))$ the generator of the semigroup $\left(e^{tA}\right)_{t\geq 0}$ for (7), with domain defined analogously to (15). Hence, since $\mathring{\mathbf{u}} \geq 0$ yields $\mathbf{u}_\epsilon(t) = (s_\epsilon(t), l_\epsilon(t), i_\epsilon(t), c_\epsilon(t),$ $r_\epsilon(t), v_\epsilon(t)) \geq 0$, we see that each component of $\mathbf{u}_\epsilon$ is controlled by the ϵ-independent solution n of (7):

$$\begin{array}{lll} 0 \leq s_\epsilon(a,t) \leq n(a,t), & 0 \leq l_\epsilon(a,t) \leq n(a,t), & 0 \leq i_\epsilon(a,t) \leq n(a,t), \\ 0 \leq c_\epsilon(a,t) \leq n(a,t), & 0 \leq r_\epsilon(a,t) \leq n(a,t) \quad \text{and} & 0 \leq v_\epsilon(a,t) \leq n(a,t) \end{array} \tag{17}$$

for each $t \geq 0$ and almost every $a \in [0,a_+]$ thus satisfying the inequality (10).

3. Formal asymptotic expansion

Following the general approach of the asymptotic analysis, see e.g. [4, 6], we are looking for the so-called *hydrodynamic space* V of the singularly perturbed equation (6), which, in this case, is given by the null-space of $\mathcal{C}$. Since, in this context, a is treated as a parameter, we perform the calculations for a fixed a. Then we have

$$V = \{\mathbf{u} \in \mathbb{R}^6;\ \mathbf{u} = (u_1, 0, 0, 0, u_5, u_6), u_1, u_5, u_6 \in \mathbb{R}\} = \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\},$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are an approximate basis for V. The complementary spectral space W, called the *kinetic space*, corresponding to the eigenvalues $\lambda_4 = -\sigma$, $\lambda_5 = -\gamma_1$, and $\lambda_6 = -\gamma_2(a)$, respectively, is spanned by $\mathbf{e}_4$, $\mathbf{e}_5$, and $\mathbf{e}_6$ given by

$$\begin{pmatrix} 0 \\ 1 \\ -\frac{\sigma}{\sigma-\gamma_1} \\ \frac{\sigma q(a)\gamma_1}{(\sigma-\gamma_1)(\sigma-\gamma_2(a))} \\ \frac{\gamma_1}{\sigma-\gamma_1}\cdot\frac{(1-q(a))\sigma-\gamma_2(a)}{\sigma-\gamma_2(a)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{q(a)\gamma_1}{\gamma_1-\gamma_2(a)} \\ -\frac{(1-q(a))\gamma_1-\gamma_2(a)}{\gamma_1-\gamma_2(a)} \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix},$$

respectively. To come up with the decomposition $\mathbf{u} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 + c_4\mathbf{e}_4 + c_5\mathbf{e}_5 + c_6\mathbf{e}_6$ in the hydrodynamic and kinetic space, we have to find the coefficients c_i, $i = 1, \ldots, 6$, which yield the aggregated variables. Using standard results from linear algebra, see e.g. [6], $c_i = \mathbf{f}_i \cdot \mathbf{u}/\mathbf{f}_i \cdot \mathbf{e}_i$, where $\mathbf{f}_i$ are left eigenvectors of $\mathcal{C}$ corresponding to the eigenvectors as $\mathbf{e}_i$, $i = 1, \ldots, 6$. Here the context is more complicated as V is three dimensional and we have to choose its basis in an appropriate way. Since $\mathbf{f}_1 = (1,1,1,1,1,1)$ is a left eigenvector of $\mathcal{C}$ corresponding to the zero eigenvalue and $\mathbf{f}_1 \cdot \mathbf{u} = u_1 + u_2 + u_3 + u_4 + u_5 + u_6$, it should play a significant role in the analysis due to (7). Then, for the convenience of calculations, we take $\mathbf{e}_2 = (-1,0,0,0,0,1) \in V$, which is orthogonal to $\mathbf{f}_1$. We have some freedom in selecting $\mathbf{f}_3$: taking $\mathbf{f}_3 = (0,0,0,0,0,1)$ yields $\mathbf{e}_1 = (1,0,0,0,0,0) \in V$ and $\mathbf{e}_2 = (-1,0,0,0,1,0) \in V$. Finally $\mathbf{f}_4 = (0,1,0,0,0,0)$, $\mathbf{f}_5 = (0, \sigma/(\sigma-\gamma_1), 1, 0, 0, 0)$, $\mathbf{f}_6 = (0, \sigma q(a)\gamma_1/(\sigma-\gamma_2(a))(\gamma_1-\gamma_2(a)), q(a)\gamma_1/(\gamma_1-\gamma_2(a)), 1, 0, 0)$ so that

$$\begin{aligned}\mathbf{u} = (u_1 + u_2 + u_3 + u_4 + u_5 + u_6)\mathbf{e}_1 + (u_2 + u_3 + u_4 + u_5)\mathbf{e}_2 + u_6\mathbf{e}_3 + u_2\mathbf{e}_4 \\ + \left(\frac{\sigma}{\sigma-\gamma_1}u_2 + u_3\right)\mathbf{e}_5 + \left(\frac{q\gamma_1}{\gamma_1-\gamma_2}\cdot\frac{\sigma}{\sigma-\gamma_1}u_2 + \frac{q\gamma_1}{\gamma_1-\gamma_2}u_3 + u_4\right)\mathbf{e}_6.\end{aligned} \tag{18}$$

Then we use this decomposition to change variables in (6). Accordingly, we define $n = s_\epsilon + l_\epsilon + i_\epsilon + c_\epsilon + r_\epsilon + v_\epsilon$, $w_\epsilon = l_\epsilon + i_\epsilon + c_\epsilon + r_\epsilon$, $z_\epsilon = \frac{\sigma}{\sigma-\gamma_1}l_\epsilon + i_\epsilon$, $x_\epsilon = \frac{q\gamma_1}{\gamma_1-\gamma_2}z_\epsilon + c_\epsilon$ and leave l_ϵ, v_ϵ unchanged. This yields the system of equations

$$
\begin{aligned}
\partial_t n(a,t) &= -\partial_a n(a,t) - \mu(a)n(a,t), \\
\partial_t w_\epsilon(a,t) &= -\partial_a w_\epsilon(a,t) - \mu(a)w_\epsilon(a,t) + \Lambda(a, l_\epsilon(\cdot,t), x_\epsilon(\cdot,t), z_\epsilon(\cdot,t)) \\
&\qquad \times (n(a,t) - w_\epsilon(a,t) - v_\epsilon(a,t)), \\
\partial_t z_\epsilon(a,t) &= -\partial_a z_\epsilon(a,t) - \mu(a)z_\epsilon(a,t) - \frac{\gamma_1}{\epsilon} z_\epsilon(a,t) \\
&\qquad + \frac{\sigma}{\sigma - \gamma_1}\Lambda(a, l_\epsilon(\cdot,t), x_\epsilon(\cdot,t), z_\epsilon(\cdot,t))(n(a,t) - w_\epsilon(a,t) - v_\epsilon(a,t)), \\
\partial_t x_\epsilon(a,t) &= -\partial_a x_\epsilon(a,t) - \mu(a)x_\epsilon(a,t) - \frac{\gamma_2(a)}{\epsilon} x_\epsilon(a,t) - \frac{\gamma_1}{\epsilon} \cdot \frac{\sigma q(a)}{\sigma - \gamma_1} l_\epsilon(a,t) \\
&\quad + \frac{q(a)\gamma_1}{\gamma_1 - \gamma_2(a)} \cdot \frac{\gamma_2'(a)}{\gamma_1 - \gamma_2(a)} z_\epsilon(a,t) + \frac{\sigma}{\sigma - \gamma_1} \cdot \frac{q(a)\gamma_1}{\gamma_1 - \gamma_2(a)} \\
&\qquad \times \Lambda(a, l_\epsilon(\cdot,t), x_\epsilon(\cdot,t), z_\epsilon(\cdot,t))(n(a,t) - w_\epsilon(a,t) - v_\epsilon(a,t)), \\
\partial_t l_\epsilon(a,t) &= -\partial_a l_\epsilon(a,t) - \mu(a)l_\epsilon(a,t) - \frac{\sigma}{\epsilon} l_\epsilon(a,t) \\
&\quad + \Lambda(a, l_\epsilon(\cdot,t), x_\epsilon(\cdot,t), z_\epsilon(\cdot,t))(n(a,t) - w_\epsilon(a,t) - v_\epsilon(a,t)), \\
\partial_t v_\epsilon(a,t) &= -\partial_a v_\epsilon(a,t) - \mu(a)v_\epsilon(a,t) - \psi v_\epsilon(a,t) - p(a,t)v_\epsilon(a,t) \\
&\qquad + p(a,t)(n(a,t) - w_\epsilon(a,t)),
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
n(0,t) &= \int_0^{a_+} \beta(a)n(a,t)\,da, \\
w_\epsilon(0,t) &= x_\epsilon(0,t) = z_\epsilon(0,t) = l_\epsilon(0,t) = 0, \\
v_\epsilon(0,t) &= (1-\omega)n(0,t),
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
n(a,0) &= \mathring{n}(a) = \mathring{s}(a) + \mathring{l}(a) + \mathring{i}(a) + \mathring{c}(a) + \mathring{r}(a) + \mathring{v}(a), \\
w_\epsilon(a,0) &= \mathring{w}(a) = \mathring{l}(a) + \mathring{i}(a) + \mathring{c}(a) + \mathring{r}(a), \\
z_\epsilon(a,0) &= \mathring{z}(a) = \frac{\sigma}{\sigma - \gamma_1}\mathring{l}(a) + \mathring{i}(a), \\
x_\epsilon(a,0) &= \mathring{x}(a) = \frac{q(a)\gamma_1}{\gamma_1 - \gamma_2(a)} \cdot \frac{\sigma}{\sigma - \gamma_1}\mathring{l}(a) + \frac{q(a)\gamma_1}{\gamma_1 - \gamma_2(a)}\mathring{i}(a) + \mathring{c}(a), \\
l_\epsilon(a,0) &= \mathring{l}(a), \quad v_\epsilon(a,0) = \mathring{v}(a),
\end{aligned}
\tag{21}
$$

where

$$\Lambda(a, l_\epsilon(\cdot,t), x_\epsilon(\cdot,t), z_\epsilon(\cdot,t)) = k(a)\int_0^{a_+}\Bigg[-\frac{\sigma}{\sigma-\gamma_1}h_1(a)l_\epsilon(a,t) + h_2(a)x_\epsilon(a,t) + \left(h_1(a) - \frac{q(a)\gamma_1}{\gamma_1-\gamma_2(a)}h_2(a)\right)z_\epsilon(a,t)\Bigg]\,da.$$

Therefore, the triplet $(n, w_\epsilon, v_\epsilon) \in V$ and the triplet $(z_\epsilon, x_\epsilon, l_\epsilon) \in W$. We see that the total population n decouples from the system, it is unnecessary to approximate it, and it can be treated as a known function. Therefore we shall focus on the quintuplet $(w_\epsilon, z_\epsilon, x_\epsilon, l_\epsilon, v_\epsilon)$. Let $(\overline{w}, \overline{z}, \overline{x}, \overline{l}, \overline{v})$ be the bulk approximation of the quintuplet $(w_\epsilon, z_\epsilon, x_\epsilon, l_\epsilon, v_\epsilon)$, we note that $(w_\epsilon, z_\epsilon, x_\epsilon, l_\epsilon, v_\epsilon) \approx (\overline{w}, \overline{z}, \overline{x}, \overline{l}, \overline{v})$, where $\overline{w}$ and $\overline{v}$ are defined by (12) and (13), respectively, and the approximate equality symbol $\approx$ accounts for the fact that we only consider the first terms of the asymptotic expansion. Following the Chapman–Enskog procedure, we only expand the kinetic part of the bulk approximation such that

$$\overline{z} = \overline{z}_0 + \epsilon\overline{z}_1 + \cdots,\ \overline{x} = \overline{x}_0 + \epsilon\overline{x}_1 + \cdots,\ \overline{l} = \overline{l}_0 + \epsilon\overline{l}_1 + \cdots. \tag{22}$$

Inserting (22) into the last four equations in (19), we get

$$\begin{aligned}
\partial_t\overline{z}_0 + \epsilon\partial_t\overline{z}_1 &= -\partial_a\overline{z}_0 - \epsilon\partial_a\overline{z}_1 - \mu\overline{z}_0 - \epsilon\mu\overline{z}_1 - \frac{\gamma_1}{\epsilon}\overline{z}_0 - \gamma_1\overline{z}_1 + \frac{\sigma}{\sigma-\gamma_1}\\
&\quad\times\left(\Lambda(\overline{l}_0,\overline{x}_0,\overline{z}_0) + \epsilon\Lambda(\overline{l}_1,\overline{x}_1,\overline{z}_1)\right)(n - \overline{w} - \overline{v}) + O(\epsilon^2),\\
\partial_t\overline{x}_0 + \epsilon\partial_t\overline{x}_1 &= -\partial_a\overline{x}_0 - \epsilon\partial_a\overline{x}_1 - \mu\overline{x}_0 - \epsilon\mu\overline{x}_1 - \frac{\gamma_2}{\epsilon}\overline{x}_0 - \gamma_2\overline{x}_1\\
&\quad + \frac{1}{\epsilon}\cdot\frac{q\gamma_1^2}{\gamma_1-\gamma_2}\overline{z}_0 + \frac{q\gamma_1^2}{\gamma_1-\gamma_2}\overline{z}_1 - \frac{q\gamma_1}{\gamma_1-\gamma_2}\cdot\frac{q\gamma_2'}{\gamma_1-\gamma_2}\overline{z}_0\\
&\quad - \epsilon\frac{q\gamma_1}{\gamma_1-\gamma_2}\cdot\frac{q\gamma_2'}{\gamma_1-\gamma_2}\overline{z}_1 - \frac{q}{\epsilon}\cdot\frac{\sigma\gamma_1}{\sigma-\gamma_1}\overline{l}_0 - \frac{\sigma q\gamma_1}{\sigma-\gamma_1}\overline{l}_1\\
&\quad + \frac{q\gamma_1}{\gamma_1-\gamma_2}\cdot\frac{\sigma}{\sigma-\gamma_1}\left(\Lambda(\overline{l}_0,\overline{x}_0,\overline{z}_0) + \epsilon\Lambda(\overline{l}_1,\overline{x}_1,\overline{z}_1)\right)\\
&\quad\times(n - \overline{w} - \overline{v}) + O(\epsilon^2),\\
\partial_t\overline{l}_0 + \epsilon\partial_t\overline{l}_1 &= -\partial_a\overline{l}_0 - \epsilon\partial_a\overline{l}_1 - \mu\overline{l}_0 - \epsilon\mu\overline{l}_1 + \left(\Lambda(\overline{l}_0,\overline{x}_0,\overline{z}_0) + \epsilon\Lambda(\overline{l}_1,\overline{x}_1,\overline{z}_1)\right)\\
&\quad\times(n - \overline{w} - \overline{v}) - \frac{\sigma}{\epsilon}\overline{l}_0 - \sigma\overline{l}_1 + O(\epsilon^2).
\end{aligned}$$

Comparing coefficients at like powers of ϵ and using $\Lambda(0,0,0) = 0$ we get $\overline{z}_0 = \overline{x}_0 = \overline{l}_0 = 0$, $\overline{z}_1 = \overline{x}_1 = \overline{l}_1 = 0$.

Hence we arrive at the (formal) bulk approximation

$$(n, \overline{w}, \overline{z}, \overline{x}, \overline{l}, \overline{v}) = (n, \overline{w}, 0, 0, 0, \overline{v}).$$

Note that by substituting $l_\epsilon \approx \overline{l} = 0$, $x_\epsilon \approx \overline{x} = 0$, $z_\epsilon \approx \overline{z} = 0$ in the second equation in (19) we arrive at the system, (12). Furthermore, substituting $\overline{w}$ into the last equation in (19) leads to (13). The error of the approximation,

$$(n, w_\epsilon, z_\epsilon, x_\epsilon, l_\epsilon, v_\epsilon) \approx (n, \overline{w}, 0, 0, 0, \overline{v}),$$

denoted by

$$\begin{aligned} \overline{\mathbf{E}} &= (\overline{e}_w, \overline{e}_z, \overline{e}_x, \overline{e}_l, \overline{e}_v) = (w_\epsilon - \overline{w}, z_\epsilon - \overline{z}, x_\epsilon - \overline{x}, l_\epsilon - \overline{l}, v_\epsilon - \overline{v}) \\ &= (w_\epsilon - \overline{w}, z_\epsilon, x_\epsilon, l_\epsilon, v_\epsilon - \overline{v}), \end{aligned} \tag{23}$$

satisfies

$$\begin{aligned} \partial_t \overline{e}_w &= -\partial_a \overline{e}_w - \mu \overline{e}_w - \Lambda(\overline{e}_l, \overline{e}_x, \overline{e}_z)(\overline{e}_w + \overline{e}_v) + \Lambda(\overline{e}_l, \overline{e}_x, \overline{e}_z)(n - \overline{w} - \overline{v}), \\ \partial_t \overline{e}_z &= -\partial_a \overline{e}_z - \mu \overline{e}_z - \frac{\gamma_1}{\epsilon}\overline{e}_z - \frac{\sigma}{\sigma - \gamma_1}\Lambda(\overline{e}_l, \overline{e}_x, \overline{e}_z)(\overline{e}_w + \overline{e}_v) \\ &\quad + \frac{\sigma}{\sigma - \gamma_1}\Lambda(\overline{e}_l, \overline{e}_x, \overline{e}_z)(n - \overline{w} - \overline{v}), \\ \partial_t \overline{e}_x &= -\partial_a \overline{e}_x - \mu \overline{e}_x - \frac{\gamma_2}{\epsilon}\overline{e}_x - \frac{1}{\epsilon}\cdot\frac{\sigma\gamma_1}{\sigma - \gamma_1}\overline{e}_l + \frac{q\gamma_1}{\gamma_1 - \gamma_2}\cdot\frac{\gamma_2'}{\gamma_1 - \gamma_2}\overline{e}_z \\ &\quad - \frac{\sigma}{\sigma - \gamma_1}\cdot\frac{q\gamma_1}{\gamma_1 - \gamma_2}\Lambda(\overline{e}_l, \overline{e}_x, \overline{e}_z)(\overline{e}_w + \overline{e}_v) \\ &\quad + \frac{\sigma}{\sigma - \gamma_1}\cdot\frac{q\gamma_1}{\gamma_1 - \gamma_2}\Lambda(\overline{e}_l, \overline{e}_x, \overline{e}_z)(n - \overline{w} - \overline{v}), \\ \partial_t \overline{e}_l &= -\partial_a \overline{e}_l - \mu \overline{e}_l - \frac{\sigma}{\epsilon}\overline{e}_l - \Lambda(\overline{e}_l, \overline{e}_x, \overline{e}_z)(\overline{e}_w + \overline{e}_v) \\ &\quad + \Lambda(\overline{e}_l, \overline{e}_x, \overline{e}_z)(n - \overline{w} - \overline{v}), \\ \partial_t \overline{e}_v &= -\partial_a \overline{e}_v - \mu \overline{e}_v - (p + \psi)\overline{e}_v - p\overline{e}_w, \end{aligned} \tag{24}$$

with the boundary condition

$$\overline{e}_w(0,t) = \overline{e}_z(0,t) = \overline{e}_x(0,t) = \overline{e}_l(0,t) = \overline{e}_v(0,t) = 0, \tag{25}$$

and the initial condition

$$\begin{aligned} &\overline{e}_w(a,0) = 0, \quad \overline{e}_z(a,0) = \frac{\sigma}{\sigma - \gamma_1}\mathring{l}(a) + \mathring{i}(a), \\ &\overline{e}_x(a,0) = \frac{q\gamma_1}{\gamma_1 - \gamma_2(a)}\cdot\frac{\sigma}{\sigma - \gamma_1}\mathring{l}(a) + \frac{q\gamma_1}{\gamma_1 - \gamma_2(a)}\mathring{i}(a) + \mathring{c}(a), \\ &\overline{e}_l(a,0) = \mathring{l}(a), \quad \overline{e}_v(a,0) = 0. \end{aligned} \tag{26}$$

We see that the initial condition is of order 1; therefore the error cannot be of order ϵ. To remedy the situation we have to introduce layer corrections that will take care of the transient phenomena occurring close to $t = 0$, namely the initial layer.

We carry out the initial layer correction by blowing up time according to $\tau = t/\epsilon$ and looking for the approximation

$$(w_\epsilon(t), z_\epsilon(t), x_\epsilon(t), l_\epsilon(t), v_\epsilon(t)) \approx (\overline{w}(t), \widetilde{z}(\tau), \widetilde{x}(\tau), \widetilde{l}(\tau), \overline{v}(t)),$$

where we anticipate that it is unnecessary to introduce the initial layer for w_ϵ and v_ϵ as $\overline{w}$ and $\overline{v}$ satisfy the exact initial condition. We insert the formal expansion

$$\widetilde{z} = \widetilde{z}_0 + \epsilon\widetilde{z}_1 + \cdots, \ \widetilde{x} = \widetilde{x}_0 + \epsilon\widetilde{x}_1 + \cdots, \ \widetilde{l} = \widetilde{l}_0 + \epsilon\widetilde{l}_1 + \cdots$$

and rescale time, due to $\partial_t = \epsilon^{-1}\partial_\tau$, in the third, fourth, and fifth equations in (19).

Comparing coefficients at like powers of ϵ, the equations for the terms at ϵ^{-1} level are

$$\partial_\tau \widetilde{z}_0 = -\gamma_1 \widetilde{z}_0, \quad \partial_\tau \widetilde{l}_0 = -\sigma \widetilde{l}_0,$$
$$\partial_\tau \widetilde{x}_0 = -\gamma_2(a)\widetilde{x}_0 - \frac{\sigma\gamma_1}{\sigma-\gamma_1}\widetilde{l}_0$$

which, subject to the initial condition

$$\widetilde{z}_0(0) = \frac{\sigma}{\sigma-\gamma_1}\overset{\circ}{l} + \overset{\circ}{i}, \quad \widetilde{l}_0(0) = \overset{\circ}{l},$$
$$\widetilde{x}_0(0) = \frac{q(a)\gamma_1}{\gamma_1-\gamma_2(a)} \cdot \frac{\sigma}{\sigma-\gamma_1}\overset{\circ}{l} + \frac{q(a)\gamma_1}{\gamma_1-\gamma_2(a)}\overset{\circ}{i} + \overset{\circ}{c},$$

yield

$$\begin{aligned}\widetilde{z}_0(a,t/\epsilon) &= \left(\frac{\sigma}{\sigma-\gamma_1}\overset{\circ}{l}(a) + \overset{\circ}{i}(a)\right)e^{-\frac{\gamma_1}{\epsilon}t}, \quad \widetilde{l}_0(a,t/\epsilon) = \overset{\circ}{l}(a)e^{-\frac{\sigma}{\epsilon}t},\\
\widetilde{x}_0(a,t/\epsilon) &= \left(\frac{q(a)\gamma_1}{\gamma_1-\gamma_2(a)} \cdot \frac{\sigma}{\sigma-\gamma_1}\overset{\circ}{l}(a) + \frac{q(a)\gamma_1}{\gamma_1-\gamma_2(a)}\overset{\circ}{i}(a) + \overset{\circ}{c}(a)\right)e^{-\frac{\gamma_2(a)}{\epsilon}t}\\
&\quad + \frac{\sigma}{\sigma-\gamma_1} \cdot \frac{\overset{\circ}{l}(a)}{\sigma-\gamma_2(a)}\left(e^{-\frac{\sigma}{\epsilon}t} - e^{-\frac{\gamma_2(a)}{\epsilon}t}\right).\end{aligned} \tag{27}$$

The new error is given by

$$\begin{aligned}\widetilde{\mathbf{E}} = (\widetilde{e}_w, \widetilde{e}_z, \widetilde{e}_x, \widetilde{e}_l, \widetilde{e}_v) &= (w_\epsilon - \overline{w}, z_\epsilon - \widetilde{z}_0, x_\epsilon - \widetilde{x}_0, l_\epsilon - \widetilde{l}_0, v_\epsilon - \overline{v})\\
&= (\overline{e}_w, \overline{e}_z - \widetilde{z}_0, \overline{e}_x - \widetilde{x}_0, \overline{e}_l - \widetilde{l}_0, \overline{e}_v).\end{aligned} \tag{28}$$

Since $\overset{\circ}{l}, \overset{\circ}{i}, \overset{\circ}{c}, \overset{\circ}{v} \in \mathrm{W}^{1,1}([0,a_+])$ and $\mu\overset{\circ}{l}, \mu\overset{\circ}{i}, \mu\overset{\circ}{c}, \mu\overset{\circ}{v} \in \mathrm{L}^1([0,a_+])$, the error equation for $\widetilde{E}$ can be obtained from (24), (25), and (26) by expressing $\overline{e}_w$, $\overline{e}_z$, $\overline{e}_x$, $\overline{e}_l$, and $\overline{e}_v$ in terms of $\widetilde{e}_w$, $\widetilde{e}_z$, $\widetilde{e}_x$, $\widetilde{e}_l$, and $\widetilde{e}_v$, according to (28). We get

$$\begin{aligned}\partial_t \widetilde{e}_w &= -\partial_a \widetilde{e}_w - \mu\widetilde{e}_w - \Lambda(\widetilde{e}_l, \widetilde{e}_x, \widetilde{e}_z)(\widetilde{e}_w + \widetilde{e}_v) - \Lambda(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0)(\widetilde{e}_w + \widetilde{e}_v)\\
&\quad + \Lambda(\widetilde{e}_l, \widetilde{e}_x, \widetilde{e}_z)(n - \overline{w} - \overline{v}) - \Lambda(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0)(n - \overline{w} - \overline{v}),\\
\partial_t \widetilde{e}_z &= -\partial_a \widetilde{e}_z - \mu\widetilde{e}_z - \frac{\gamma_1}{\epsilon}\widetilde{e}_z - \frac{\sigma}{\sigma-\gamma_1}\Lambda(\widetilde{e}_l, \widetilde{e}_x, \widetilde{e}_z)(\widetilde{e}_w + \widetilde{e}_v)\\
&\quad - \frac{\sigma}{\sigma-\gamma_1}\Lambda(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0)(\widetilde{e}_w + \widetilde{e}_v) + \frac{\sigma}{\sigma-\gamma_1}\Lambda(\widetilde{e}_l, \widetilde{e}_x, \widetilde{e}_z)(n - \overline{w} - \overline{v})\\
&\quad - \frac{\sigma}{\sigma-\gamma_1}\Lambda(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0)(n - \overline{w} - \overline{v}) - \partial_a \widetilde{z}_0 - \mu\widetilde{z}_0,\end{aligned}$$

$$
\begin{aligned}
\partial_t\widetilde{e}_x &= -\partial_a\widetilde{e}_x - \mu\widetilde{e}_x - \frac{\gamma_2}{\epsilon}\widetilde{e}_x - \frac{1}{\epsilon}\cdot\frac{\sigma\gamma_1}{\sigma-\gamma_1}\widetilde{e}_l + \frac{q\gamma_1}{\gamma_1-\gamma_2}\cdot\frac{\gamma_2^{'}}{\gamma_1-\gamma_2}\widetilde{e}_z \\
&\quad - \frac{\sigma}{\sigma-\gamma_1}\cdot\frac{q\gamma_1}{\gamma_1-\gamma_2}\Big(\Lambda(\widetilde{e}_l,\widetilde{e}_x,\widetilde{e}_z) + \Lambda(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0)\Big)(\widetilde{e}_w+\widetilde{e}_v) \\
&\quad + \frac{\sigma}{\sigma-\gamma_1}\cdot\frac{q\gamma_1}{\gamma_1-\gamma_2}\Big(\Lambda(\widetilde{e}_l,\widetilde{e}_x,\widetilde{e}_z) - \Lambda(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0)\Big)(n-\overline{w}-\overline{v}) \\
&\quad + \frac{q\gamma_1}{\gamma_1-\gamma_2}\cdot\frac{\gamma_2^{'}}{\gamma_1-\gamma_2}\widetilde{z}_0 - \partial_a\widetilde{x}_0 - \mu\widetilde{x}_0, \\
\partial_t\widetilde{e}_l &= -\partial_a\widetilde{e}_l - \mu\widetilde{e}_l - \frac{\sigma}{\epsilon}\widetilde{e}_l - \Lambda(\widetilde{e}_l,\widetilde{e}_x,\widetilde{e}_z)(\widetilde{e}_w+\widetilde{e}_v) - \Lambda(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0)(\widetilde{e}_w+\widetilde{e}_v) \\
&\quad + \Lambda(\widetilde{e}_l,\widetilde{e}_x,\widetilde{e}_z)(n-\overline{w}-\overline{v}) - \Lambda(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0)(n-\overline{w}-\overline{v}) - \partial_a\widetilde{l}_0 - \mu\widetilde{l}_0, \\
\partial_t\widetilde{e}_v &= -\partial_a\widetilde{e}_v - \mu\widetilde{e}_v - (p+\psi)\widetilde{e}_v - p\widetilde{e}_w,
\end{aligned}
\tag{29}
$$

with boundary condition

$$
\begin{aligned}
&\widetilde{e}_w(0,t) = 0, \quad \widetilde{e}_z(0,t) = -\widetilde{z}_0(0,t/\epsilon), \quad \widetilde{e}_x(0,t) = -\widetilde{x}_0(0,t/\epsilon), \\
&\widetilde{e}_l(0,t) = -\widetilde{l}_0(0,t/\epsilon), \quad \widetilde{e}_v(0,t) = 0,
\end{aligned}
\tag{30}
$$

as we assumed that the initial condition for the original problem satisfies
$(\overset{\circ}{s},\overset{\circ}{l},\overset{\circ}{i},\overset{\circ}{c},\overset{\circ}{r},\overset{\circ}{v}) \in D(\mathcal{A})$, and initial condition

$$
\widetilde{e}_w(a,0) = \widetilde{e}_z(a,0) = \widetilde{e}_x(a,0) = \widetilde{e}_l(a,0) = \widetilde{e}_v(a,0) = 0. \tag{31}
$$

We see that at the boundary we still have terms that are of order lower than ϵ except for $\widetilde{e}_w(0,t) = 0$ and $\widetilde{e}_v(0,t) = 0$. Fortunately, to eliminate this initial layer contribution on the boundary, we need to introduce the corner layer by simultaneously rescaling time and age according to $\tau = t/\epsilon$ and $\alpha = a/\epsilon$. Unfortunately, the standard approach to the corner layer, [3], will not suffice here as the corner layer equations will not incorporate the multiplication by μ. Thus the classical corner layer will not belong to $D(\mathcal{M})$ and we will not be able to substitute the error terms into the equations, as in (29)–(31). To remedy the problem, we define the corner corrector to be the solution to

$$
\begin{aligned}
\partial_t\breve{z} &= -\partial_a\breve{z} - \mu\breve{z} - \frac{\gamma_1}{\epsilon}\breve{z}, \\
\partial_t\breve{x} &= -\partial_a\breve{x} - \mu\breve{x} - \frac{\gamma_2}{\epsilon}\breve{x} - \frac{\gamma_1}{\epsilon}\cdot\frac{q\sigma}{\sigma-\gamma_1}\breve{l}, \\
\partial_t\breve{l} &= -\partial_a\breve{l} - \mu\breve{l} - \frac{\sigma}{\epsilon}\breve{l},
\end{aligned}
\tag{32}
$$

with boundary condition

$$
\breve{z}(0,t) = -\widetilde{z}_0(0,t/\epsilon), \quad \breve{x}(0,t) = -\widetilde{x}_0(0,t/\epsilon), \quad \breve{l}(0,t) = -\widetilde{l}_0(0,t/\epsilon), \tag{33}
$$

and initial condition

$$
\breve{z}(a,0) = c_\epsilon e^{-\frac{a}{\epsilon}}, \quad \breve{x}(a,0) = d_\epsilon e^{-\frac{a}{\epsilon}}, \quad \breve{l}(a,0) = h_\epsilon e^{-\frac{a}{\epsilon}}, \tag{34}
$$

where c_ϵ, d_ϵ, and h_ϵ are constants obtained from the equality of the boundary and the initial condition at $(a,t)=(0,0)$, such that the classical solvability of the problem with inhomogeneous boundary condition holds. Hence,

$$\begin{aligned} c_\epsilon &= -\widetilde{z}_0(0,0) = -\frac{\sigma}{\sigma-\gamma_1}\overset{\circ}{l}(0) - \overset{\circ}{i}(0), \quad h_\epsilon = -\overset{\circ}{l}(0), \\ d_\epsilon &= -\widetilde{x}_0(0,0) = -\frac{q(0)\gamma_1}{\gamma_1-\gamma_2(0)}\cdot\frac{\sigma}{\sigma-\gamma_1}\overset{\circ}{l}(0) - \frac{q(0)\gamma_1}{\gamma_1-\gamma_2(0)}\overset{\circ}{i}(0) - \overset{\circ}{c}(0) \end{aligned} \tag{35}$$

as we assumed that the initial condition for the original problem satisfies
$(\overset{\circ}{s},\overset{\circ}{l},\overset{\circ}{i},\overset{\circ}{c},\overset{\circ}{r},\overset{\circ}{v}) \in D(\mathcal{A})$.

The necessary estimates of the corner layer solution will be provided later. Here, assuming that it is sufficiently regular, we consider the new approximation

$$(w_\epsilon, z_\epsilon, x_\epsilon, l_\epsilon, v_\epsilon) \approx (\overline{w}, \widetilde{z}_0 + \breve{l}, \widetilde{x}_0 + \breve{i}, \widetilde{l}_0 + \breve{c}, \overline{v}).$$

The corresponding error,

$$\begin{aligned} \breve{E} &= (\breve{e}_w, \breve{e}_z, \breve{e}_x, \breve{e}_l, \breve{e}_v) \\ &= (w_\epsilon - \overline{w}, z_\epsilon - \widetilde{z}_0 - \breve{z}, x_\epsilon - \widetilde{x}_0 - \breve{x}, l_\epsilon - \widetilde{l}_0 - \breve{l}, v_\epsilon - \overline{v}) \\ &= (\widetilde{e}_w, \widetilde{e}_z - \breve{z}, \widetilde{e}_x - \breve{x}, \widetilde{e}_l - \breve{l}, \widetilde{e}_v), \end{aligned}$$

satisfies the system

$$\begin{aligned} \partial_t \breve{e}_w &= -\partial_a \breve{e}_w - \mu \breve{e}_w - \Lambda(\breve{e}_l, \breve{e}_x, \breve{e}_z)\,(\breve{e}_w + \breve{e}_v) - \Lambda(\breve{l}, \breve{x}, \breve{z})\,(\breve{e}_w + \breve{e}_v) \\ &\quad + \Lambda(\breve{e}_l, \breve{e}_x, \breve{e}_z)\,(n - \overline{w} - \overline{v}) - \Lambda(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0)\,(\breve{e}_w + \breve{e}_v) \\ &\quad - \Lambda(\breve{l}, \breve{x}, \breve{z})\breve{w} + \Lambda(\breve{l}, \breve{x}, \breve{z})\,(n - \overline{w} - \overline{v}) \\ &\quad + \Lambda(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0)\,(n - \overline{w} - \overline{v}), \\ \partial_t \breve{e}_z &= -\partial_a \breve{e}_z - \mu \breve{e}_z - \frac{\gamma_1}{\epsilon}\breve{e}_z - \frac{\sigma}{\sigma-\gamma_1}\Lambda(\breve{e}_l, \breve{e}_x, \breve{e}_z)\,(\breve{e}_w + \breve{e}_v) \\ &\quad - \frac{\sigma}{\sigma-\gamma_1}\Lambda(\breve{l}, \breve{x}, \breve{z})\,(\breve{e}_w + \breve{e}_v) + \frac{\sigma}{\sigma-\gamma_1}\Lambda(\breve{e}_l, \breve{e}_x, \breve{e}_z)\,(n - \overline{w} - \overline{v}) \\ &\quad - \frac{\sigma}{\sigma-\gamma_1}\Lambda(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0)\,(\breve{e}_w + \breve{e}_v) + \frac{\sigma}{\sigma-\gamma_1}\Lambda(\breve{l}, \breve{x}, \breve{z})\,(n - \overline{w} - \overline{v}) \\ &\quad + \frac{\sigma}{\sigma-\gamma_1}\Lambda(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0)\,(n - \overline{w} - \overline{v}) - \partial_a \widetilde{z}_0 - \mu \widetilde{z}_0, \end{aligned}$$

$$
\begin{aligned}
\partial_t \breve{e}_x &= -\partial_a \breve{e}_x - \mu \breve{e}_x - \frac{\gamma_2}{\epsilon}\breve{e}_x - \frac{1}{\epsilon}\cdot\frac{\sigma\gamma_1}{\sigma-\gamma_1}\breve{e}_l + \frac{q\gamma_1}{\gamma_1-\gamma_2}\cdot\frac{\gamma_2^{'}}{\gamma_1-\gamma_2}\breve{e}_z \\
&\quad - \frac{\sigma}{\sigma-\gamma_1}\cdot\frac{q\gamma_1}{\gamma_1-\gamma_2}\Big(\Lambda(\breve{e}_l,\breve{e}_x,\breve{e}_z) + \Lambda(\breve{l},\breve{x},\breve{z})\Big)(\breve{e}_w+\breve{e}_v) \\
&\qquad + \frac{\sigma}{\sigma-\gamma_1}\cdot\frac{q\gamma_1}{\gamma_1-\gamma_2}\Big(\Lambda(\breve{e}_l,\breve{e}_x,\breve{e}_z)(n-\overline{w}-\overline{v}) - \Lambda(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0)(\breve{e}_w+\breve{e}_v)\Big) \\
&\qquad\quad + \frac{\sigma}{\sigma-\gamma_1}\cdot\frac{q\gamma_1}{\gamma_1-\gamma_2}\Big(\Lambda(\breve{l},\breve{x},\breve{z}) + \Lambda(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0)\Big)(n-\overline{w}-\overline{v}) \\
&\qquad\qquad + \frac{q\gamma_1}{\gamma_1-\gamma_2}\cdot\frac{\gamma_2^{'}}{\gamma_1-\gamma_2}(\breve{z}+\widetilde{z}_0) - \frac{1}{\epsilon}\cdot\frac{\sigma\gamma_1}{\sigma-\gamma_1}\breve{l} - \partial_a\widetilde{x}_0 - \mu\widetilde{x}_0, \\
\partial_t \breve{e}_l &= -\partial_a \breve{e}_l - \mu\breve{e}_l - \frac{\sigma}{\epsilon}\breve{e}_l - \Lambda(\breve{e}_l,\breve{e}_x,\breve{e}_z)(\breve{e}_w+\breve{e}_v) - \Lambda(\breve{l},\breve{x},\breve{z})(\breve{e}_w+\breve{e}_v) \\
&\quad + \Big(\Lambda(\breve{e}_l,\breve{e}_x,\breve{e}_z) + \Lambda(\breve{l},\breve{x},\breve{z})\Big)(n-\overline{w}-\overline{v}) - \Lambda(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0)(\breve{e}_w+\breve{e}_v) \\
&\qquad + \Lambda(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0)(n-\overline{w}-\overline{v}) - \partial_a\widetilde{l}_0 - \mu\widetilde{l}_0, \\
\partial_t \breve{e}_v &= -\partial_a\breve{e}_v - \mu\breve{e}_v - (p+\psi)\breve{e}_v - p\breve{e}_w,
\end{aligned} \tag{36}
$$

with boundary condition

$$
\breve{e}_w(0,t) = \breve{e}_z(0,t) = \breve{e}_x(0,t) = \breve{e}_l(0,t) = \breve{e}_v(0,t) = 0, \tag{37}
$$

and initial condition

$$
\begin{aligned}
&\breve{e}_w(a,0) = 0, \quad \breve{e}_z(a,0) = -c_\epsilon e^{-\frac{a}{\epsilon}}, \quad \breve{e}_x(a,0) = -d_\epsilon e^{-\frac{a}{\epsilon}}, \\
&\breve{e}_l(a,0) = -h_\epsilon e^{-\frac{a}{\epsilon}}, \quad \breve{e}_v(a,0) = 0.
\end{aligned} \tag{38}
$$

4. Corner corrector estimates

Lemma 4.1 *Let $M_\theta = \mu + \theta/\epsilon$, where θ is a constant. If η is the solution of*

$$
\partial_t\eta = -\partial_a\eta - M_\theta\eta, \quad \eta(0,t) = -\delta_1 e^{-\frac{\theta}{\epsilon}t}, \quad \eta(a,0) = -\delta_2 e^{-\frac{a}{\epsilon}},
$$

then there exists a nonzero constant C_η such that

$$
\|\eta(t)\| \le \epsilon C_\eta e^{-\frac{\theta}{2\epsilon}t}. \tag{39}
$$

Proof The solution of (39) is given by

$$
\eta(a,t) = \begin{cases} \delta_2 e^{-\frac{a-t}{\epsilon}} \dfrac{\Pi_{M_\theta}(a)}{\Pi_{M_\theta}(a-t)} & \text{if } a > t, \\[2ex] \delta_1 e^{-\frac{\theta}{\epsilon}(t-a)}\Pi_{M_\theta}(a) & \text{if } a < t, \end{cases}
$$

where Π_{M_θ} is defined by (9) with μ replaced by M_θ. Using the above expression of $\eta(a,t)$, we get

$$\begin{aligned}
\|\eta(t)\| &\le |\delta| \left(\int_0^{\min\{t,a_+\}} e^{-\frac{\theta}{2\epsilon}(t-a)} \Pi_{M_\theta}(a)\, da + e^{\frac{t}{\epsilon}} \int_{\min\{t,a_+\}}^{a_+} e^{-\frac{a}{\epsilon}} \frac{\Pi_{M_\theta}(a)}{\Pi_{M_\theta}(a-t)}\, da \right) \\
&\le |\delta| e^{-\frac{\theta}{2\epsilon}t} \left(\int_0^{\min\{t,a_+\}} e^{-\frac{\theta}{2\epsilon}a} e^{-\int_0^a \mu(\tau)\, d\tau} da + e^{\frac{t}{\epsilon}} \int_{\min\{t,a_+\}}^{a_+} e^{-\frac{a}{\epsilon} - \int_{a-t}^{a} \mu(\tau)\, d\tau} da \right) \\
&\le |\delta| e^{-\frac{\theta}{2\epsilon}t} \left(\int_0^{\min\{t,a_+\}} e^{-\frac{\theta}{2\epsilon}a}\, da + e^{\frac{t}{\epsilon}} \int_{\min\{t,a_+\}}^{a_+} e^{-\frac{a}{\epsilon}}\, da \right) \\
&\le \epsilon e^{-\frac{\theta}{2\epsilon}t} |\delta| \left(\frac{2}{\theta} + e^{-\frac{\min\{t,a_+\}-t}{\epsilon}} - e^{-\frac{a_+-t}{\epsilon}} \right) = \epsilon K_\theta e^{-\frac{\theta}{2\epsilon}t},
\end{aligned}$$

where $|\delta| = \max\{|\delta_1|, |\delta_2|\}$, and we arrive at (39) □

Now we want to specify (39) to $\breve{z}$ and $\breve{l}$, solution to (32)–(34), where $\delta_1 = \delta_2$. From Lemma 4.1, we have the following result:

Proposition 4.2 *Let $\breve{z}$ and $\breve{l}$ be the solution to (32)–(34). Then there are constants K_z and K_l, depending on the coefficients and the $\mathrm{W}^{1,1}(\mathbb{R}_+)$ norm of the initial condition $\overset{\circ}{i}$ and $\overset{\circ}{l}$, such that for any $t \in \mathbb{R}_+$,*

$$\|\breve{z}(t)\| \le \epsilon K_z e^{-\frac{\gamma_1}{2\epsilon}t}, \tag{40}$$

$$\|\breve{l}(t)\| \le \epsilon K_l e^{-\frac{\sigma}{2\epsilon}t},. \tag{41}$$

Moreover, we also have the following result:

Proposition 4.3 *Let $0 < \underline{\gamma_2} < \gamma_2 < \overline{\gamma_2} < \gamma_1 < \sigma$. Let $\breve{x}_1$ be the solution to (32)–(34). Then there is a constant K_x, depending on the coefficients and on the $\mathrm{W}^{1,1}([0,a_+])$ norm of the initial conditions $\overset{\circ}{c}$, $\overset{\circ}{i}$ and $\overset{\circ}{l}$, such that for any $t \in \mathbb{R}_+$,*

$$\|\breve{x}(t)\| \le \epsilon K_x e^{-\frac{\gamma_2}{\epsilon}t}. \tag{42}$$

Proof Let $\breve{x}$ be the solution to (32)–(34). We get

$$\breve{x}(a,t) = \begin{cases} \zeta_1 e^{-\frac{a-t}{\epsilon}} \dfrac{\Pi_{M_{\gamma_2}}(a)}{\Pi_{M_{\gamma_2}}(a-t)} + \dfrac{\Pi_{M_{\gamma_2}}(a)}{\epsilon} \displaystyle\int_0^t \dfrac{F(a-t+\tau,\tau)}{\Pi_{M_{\gamma_2}}(a-t+\tau)}\, d\tau & \text{if } a > t, \\ \zeta_2 e^{-\frac{\gamma_2(a)}{\epsilon}(t-a)} \Pi_{M_{\gamma_2}}(a) + \dfrac{\Pi_{M_{\gamma_2}}(a)}{\epsilon} \displaystyle\int_0^a \dfrac{F(\tau,t-a+\tau)}{\Pi_{M_{\gamma_2}}(\tau)}\, d\tau & \text{if } a < t, \end{cases}$$

where $\zeta_1 = -\widetilde{x}_0(0,0)$, $\zeta_2 = -\widetilde{x}_0(0,0) - \dfrac{\sigma}{\sigma-\gamma_1} \cdot \dfrac{\overset{\circ}{l}(0)}{\sigma - \gamma_2(0)} \left(e^{-\frac{\sigma-\gamma_2(0)}{\epsilon}(t-a)} - 1 \right)$ and $F(a,t) = -\dfrac{\sigma\gamma_1}{\sigma-\gamma_1} q(a)\breve{l}(a,t)$.

It follows that

$$
\begin{aligned}
\|\breve{x}(t)\| \le |\zeta| &\left(\int\limits_0^{\min\{t,a_+\}} e^{-\frac{\gamma_2}{2\epsilon}(t-a)} \Pi_{M_{\gamma_2}}(a)\, da + e^{\frac{t}{\epsilon}} \int\limits_{\min\{t,a_+\}}^{a_+} e^{-\frac{a}{\epsilon}} \frac{\Pi_{M_{\gamma_2}}(a)}{\Pi_{M_{\gamma_2}}(a-t)}\, da \right) \\
&+ \frac{1}{\epsilon} \int\limits_0^{\min\{t,a_+\}} \Pi_{M_{\gamma_2}}(a) \left(\int\limits_0^a \frac{|F(\tau, t-a+\tau)|}{\Pi_{M_{\gamma_2}}(\tau)}\, d\tau \right) da \\
&+ \frac{1}{\epsilon} \int\limits_{\min\{t,a_+\}}^{a_+} \Pi_{M_{\gamma_2}}(a) \left(\int\limits_0^t \frac{|F(a-t+\tau, \tau)|}{\Pi_{M_{\gamma_2}}(a-t+\tau)}\, d\tau \right) da \\
&\le |\zeta| e^{-\frac{\gamma_2}{2\epsilon}t} \left(\int\limits_0^{\min\{t,a_+\}} e^{-\frac{\gamma_2}{2\epsilon}a} e^{-\int\limits_0^a \mu(\tau)\, d\tau}\, da + e^{\frac{t}{\epsilon}} \int\limits_{\min\{t,a_+\}}^{a_+} e^{-\frac{a}{\epsilon} - \int\limits_{a-t}^{a} \mu(\tau)\, d\tau}\, da \right) \\
&+ \frac{\sigma \overline{q} \gamma_1}{\sigma - \gamma_1} \int\limits_0^{\min\{t,a_+\}} e^{-\frac{\gamma_2}{\epsilon}a} \left(\int\limits_0^a e^{\frac{\gamma_2}{\epsilon}\tau} |\breve{l}(\tau, t-a+\tau)|\, d\tau \right) da \\
&+ \frac{\sigma \overline{q} \gamma_1}{\sigma - \gamma_1} \int\limits_{\min\{t,a_+\}}^{a_+} e^{-\frac{\gamma_2}{\epsilon}t} \left(\int\limits_0^t e^{\frac{\gamma_2}{\epsilon}\tau} |\breve{l}(a-t+\tau, \tau)|\, d\tau \right) da \\
&\le |\zeta| e^{-\frac{\gamma_2}{2\epsilon}t} \left(\int\limits_0^{\min\{t,a_+\}} e^{-\frac{\gamma_2}{2\epsilon}a}\, da + e^{\frac{t}{\epsilon}} \int\limits_{\min\{t,a_+\}}^{a_+} e^{-\frac{a}{\epsilon}}\, da \right) \\
&+ \frac{\sigma \overline{q} \gamma_1}{\sigma - \gamma_1} \int\limits_0^{\min\{t,a_+\}} e^{\frac{\gamma_2}{\epsilon}\tau} |\breve{l}(\tau, t-a+\tau)| \left(\int\limits_\tau^{\min\{t,a_+\}} e^{-\frac{\gamma_2}{\epsilon}a}\, da \right) d\tau \\
&+ \frac{\sigma \overline{q} \gamma_1}{\sigma - \gamma_1} \int\limits_{a-t}^{a_+} e^{\frac{\gamma_2}{\epsilon}\tau} |\breve{l}(\tau, t-a+\tau)| \left(\int\limits_\tau^{a_+} e^{-\frac{\gamma_2}{\epsilon}a}\, da \right) d\tau \\
&\le \epsilon e^{-\frac{\gamma_2}{2\epsilon}} |\zeta| \left(\frac{2}{\underline{\gamma_2}} + e^{-\frac{\min\{t,a_+\}-t}{\epsilon}} - e^{-\frac{a_+-t}{\epsilon}} \right) + \frac{2\sigma \overline{q} \gamma_1}{\sigma - \gamma_1} \|\breve{l}(t)\| \\
&= \epsilon K_{\gamma_2} e^{-\frac{\gamma_2}{2\epsilon}} + \frac{2\sigma \overline{q} \gamma_1}{\sigma - \gamma_1} \|\breve{l}(t)\|,
\end{aligned}
$$

where $|\zeta| = \max\{|\zeta_1|, |\zeta_2|\}$. Using (41), we get

$$
\begin{aligned}
\|\breve{x}(t)\| &\le \epsilon K_{\gamma_2} e^{-\frac{\gamma_2}{2\epsilon}t} + \epsilon \frac{2\sigma \overline{q} \gamma_1}{\sigma - \gamma_1} K_l e^{-\frac{\sigma}{2\epsilon}t} \\
&\le \epsilon \left(K_{\gamma_2} + \frac{2\sigma \overline{q} \gamma_1}{\sigma - \gamma_1} K_l \right) e^{-\frac{\gamma_2}{2\epsilon}t} = \epsilon K_x e^{-\frac{\gamma_2}{2\epsilon}t}.
\end{aligned}
$$

□

5. Main result

Theorem 5.1 *Let the coefficients of the problem* (6) *satisfy* **A1**–**A4** *together with* (11), *and* $(\overset{\circ}{s}, \overset{\circ}{l}, \overset{\circ}{i}, \overset{\circ}{c}, \overset{\circ}{r}, \overset{\circ}{v})$ *be such that* $(s_\epsilon, l_\epsilon, i_\epsilon, c_\epsilon, r_\epsilon, v_\epsilon)$ *is a classical solution to* (6). *Then there exist constants* $C_1, C_2, C_3, C_4, C_5, C_6$, *depending only on the coefficients of the problem and the* $D(\mathcal{S}) \cap D(\mathcal{M})$ *norm of the initial conditions, such that for all sufficiently small* $\epsilon > 0$

$$\|s_\epsilon(t) - (n(t) - \overline{w}(t) - \overline{v}(t))\| \leq \epsilon C_1, \tag{43}$$

$$\|l_\epsilon(t) - \overset{\circ}{l} e^{-\frac{\sigma}{\epsilon}t}\| \leq \epsilon C_2, \tag{44}$$

$$\left\| i_\epsilon(t) - \left(\overset{\circ}{i} e^{-\frac{\gamma_1}{\epsilon}t} + \frac{\sigma \overset{\circ}{l}}{\sigma - \gamma_1} \left(e^{-\frac{\gamma_1}{\epsilon}t} - e^{-\frac{\sigma}{\epsilon}t} \right) \right) \right\| \leq \epsilon C_3, \tag{45}$$

$$\begin{aligned} \Bigg\| c_\epsilon(t) - \Bigg(\overset{\circ}{c} e^{-\frac{\gamma_2}{\epsilon}t} - \frac{q\gamma_1^2}{(\gamma_1 - \gamma_2)^2} \left(\frac{\sigma \overset{\circ}{l}}{\sigma - \gamma_1} + \overset{\circ}{i} \right) \left(e^{-\frac{\gamma_2}{\epsilon}t} - e^{-\frac{\gamma_1}{\epsilon}t} \right) \\ - \frac{\sigma \overset{\circ}{l}}{(\sigma - \gamma_1)(\sigma - \gamma_2)} \left(e^{-\frac{\gamma_2}{\epsilon}t} - e^{-\frac{\sigma}{\epsilon}t} \right) \Bigg) \Bigg\| \leq \epsilon C_4, \end{aligned} \tag{46}$$

$$\begin{aligned} \Bigg\| r_\epsilon(t) - \Bigg(\overline{w} - \frac{\gamma_1 \overset{\circ}{l}}{\sigma - \gamma_1} e^{-\frac{\sigma}{\epsilon}t} - \frac{(1-q)\gamma_1 - \gamma_2}{\gamma_1 - \gamma_2} \left(\frac{\sigma \overset{\circ}{l}}{\sigma - \gamma_1} + \overset{\circ}{i} \right) e^{-\frac{\gamma_1}{\epsilon}t} \\ - \frac{\sigma \overset{\circ}{l}}{(\sigma - \gamma_1)(\gamma_1 - \gamma_2)} \left(e^{-\frac{\gamma_2}{\epsilon}t} - e^{-\frac{\sigma}{\epsilon}t} \right) \\ - \left(\frac{q\sigma\gamma_1 \overset{\circ}{l}}{(\sigma - \gamma_1)(\gamma_1 - \gamma_2)} + \frac{q\gamma_1 \overset{\circ}{i}}{\gamma_1 - \gamma_2} + \overset{\circ}{c} \right) e^{-\frac{\gamma_1}{\epsilon}t} \\ - \frac{q\gamma_1^2}{(\gamma_1 - \gamma_2)^2} \left(\frac{\sigma \overset{\circ}{l}}{\sigma - \gamma_1} + \overset{\circ}{i} \right) \left(e^{-\frac{\gamma_2}{\epsilon}t} - e^{-\frac{\gamma_1}{\epsilon}t} \right) \Bigg) \Bigg\| \leq \epsilon C_5, \end{aligned} \tag{47}$$

$$\|v_\epsilon(t) - \overline{v}(t)\| \leq \epsilon C_6. \tag{48}$$

Proof We shall simplify the notation and subsequent calculations by introducing the rescaled errors $\breve{e}_w = \epsilon d$, $\breve{e}_z = \epsilon f$, $\breve{e}_x = \epsilon g$, $\breve{e}_l = \epsilon h$ and $\breve{e}_v = \epsilon j$; that is,

$$\begin{aligned} &w_\epsilon = \overline{w} + \epsilon d, \quad z_\epsilon = \widetilde{z}_0 + \breve{z} + \epsilon f, \quad x_\epsilon = \widetilde{x}_0 + \breve{x} + \epsilon g, \\ &l_\epsilon = \widetilde{l}_0 + \breve{l} + \epsilon h, \quad v_\epsilon = \overline{v} + \epsilon j. \end{aligned} \tag{49}$$

This converts the system (36)–(38) into

$$\begin{aligned} \partial_t d = &-\partial_a d - \mu d - (d + j) F(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0, \breve{l}, \breve{x}, \breve{z}) - \epsilon(d + j)\Lambda(h, g, f) \\ &+ \Lambda(h, g, f) G(n, \overline{w}, \overline{v}) + \frac{1}{\epsilon} F(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0, \breve{l}, \breve{x}, \breve{z}) G(n, \overline{w}, \overline{v}), \end{aligned}$$

$$
\begin{aligned}
\partial_t f &= -\partial_a f - \mu f - \frac{\gamma_1}{\epsilon} f - \epsilon \frac{\sigma}{\sigma-\gamma_1}(d+j)\Lambda(h,g,f) - \frac{\sigma}{\sigma-\gamma_1}(d+j) \\
&\quad \times F(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0,\breve{l},\breve{x},\breve{z}) + \frac{\sigma}{\sigma-\gamma_1}\Lambda(h,g,f)G(n,\overline{w},\overline{v}) \\
&\quad + \frac{1}{\epsilon}\left(-\partial_a\widetilde{z}_0 - \mu\widetilde{z}_0 + \frac{\sigma}{\sigma-\gamma_1}F(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0,\breve{l},\breve{x},\breve{z})G(n,\overline{w},\overline{v})\right), \\
\partial_t g &= -\partial_a g - \mu g - \frac{\gamma_2}{\epsilon} g - \frac{1}{\epsilon}\cdot\frac{\sigma}{\sigma-\gamma_1}h + \frac{q\gamma_1}{\gamma_1-\gamma_2}\cdot\frac{\gamma_2^{'}}{\gamma_1-\gamma_2}f, \\
&\quad - \epsilon\frac{\sigma}{\sigma-\gamma_1}\cdot\frac{q\gamma_1}{\gamma_1-\gamma_2}(d+j)\Lambda(h,g,f) - \frac{\sigma}{\sigma-\gamma_1}\cdot\frac{q\gamma_1}{\gamma_1-\gamma_2} \\
&\quad \times\left((d+j)F(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0,\breve{l},\breve{x},\breve{z}) - \Lambda(h,g,f)G(n,\overline{w},\overline{v})\right) \\
&\quad + \frac{1}{\epsilon}\left(\frac{\sigma}{\sigma-\gamma_1}\cdot\frac{q\gamma_1}{\gamma_1-\gamma_2}F(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0,\breve{l},\breve{x},\breve{z})G(n,\overline{w},\overline{v})\right. \\
&\quad \left. + \frac{q\gamma_1}{\gamma_1-\gamma_2}\cdot\frac{\gamma_2^{'}}{\gamma_1-\gamma_2}(\widetilde{z}_0+\breve{z}) - \partial_a\widetilde{x}_0 - \mu\widetilde{x}_0\right) - \frac{1}{\epsilon^2}\cdot\frac{\sigma\gamma_1}{\sigma-\gamma_1}\breve{l}, \\
\partial_t h &= -\partial_a h - \mu h - \frac{\sigma}{\epsilon}h - \epsilon(d+j)\Lambda(h,g,f) - (d+j)F(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0,\breve{l},\breve{x},\breve{z}) \\
&\quad + \Lambda(h,g,f)G(n,\overline{w},\overline{v}) + \frac{1}{\epsilon}\Big(- \partial_a\widetilde{l}_0 - \mu\widetilde{l}_0 \\
&\quad F(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0,\breve{l},\breve{x},\breve{z})G(n,\overline{w},\overline{v})\Big), \\
\partial_t j &= -\partial_a j - \mu j - (p+\psi)j - pd,
\end{aligned}
\tag{50}
$$

with boundary condition

$$
d(0,t) = f(0,t) = g(0,t) = h(0,t) = j(0,t) = 0, \tag{51}
$$

and with initial condition

$$
\begin{aligned}
&d(a,0) = 0, \quad f(a,0) = -\frac{c_\epsilon}{\epsilon}e^{-\frac{a}{\epsilon}}, \quad g(a,0) = -\frac{d_\epsilon}{\epsilon}e^{-\frac{a}{\epsilon}}, \\
&h(a,0) = -\frac{h_\epsilon}{\epsilon}e^{-\frac{a}{\epsilon}}, \quad j(a,0) = 0,
\end{aligned}
\tag{52}
$$

where

$$
\begin{aligned}
F(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0,\breve{l},\breve{x},\breve{z}) &= \Lambda(\widetilde{l}_0,\widetilde{x}_0,\widetilde{z}_0) + \Lambda(\breve{l},\breve{x},\breve{z}), \\
G(n,\overline{w},\overline{v}) &= n - \overline{w} - \overline{v}.
\end{aligned}
$$

In the sequel we consider (11), as mentioned earlier. In this case $\lambda_\mu \leq 0$ and n and $\overline{w}$ satisfy (10). Furthermore, we shall need the estimate for the $\overline{v}$ solution to the scalar McKendrick problem (13). For this, we first consider the auxiliary problem

$$
\begin{aligned}
\partial_t \phi &= -\partial_a \phi - \mu\phi - \psi\phi, \\
\phi(0,t) &= (1-\omega)\int_0^{a_+} \beta(a)\phi(a)\,da, \\
\phi(a,0) &= \overset{\circ}{\phi}(a).
\end{aligned}
\tag{53}
$$

Using the fact that $\left(e^{tA}\right)_{t\geq 0}$ is a positive semigroup, we get

$$
\|\phi(t)\| \leq M e^{-\psi t}\|\overset{\circ}{\phi}\|, \quad t \geq 0. \tag{54}
$$

Next we apply the Duhamel formula to the Equation (13). Taking the norm and using (54) lead to

$$
\begin{aligned}
e^{\psi t}\|\overline{v}(t)\| \leq M\|\overset{\circ}{v}\| &+ \int_0^{a_+}\left(\int_0^t e^{\psi s}|p(a,s)||\overline{v}(a,s)|\,ds\right)da \\
&+ \int_0^{a_+}\left(\int_0^t e^{\psi s}|p(a,s)|\Big(|n(a,s)| + |\overline{w}(a,s)|\Big)\,ds\right)da.
\end{aligned}
\tag{55}
$$

Since $p \in \mathcal{C}\left([0,a_+]\times[0,T]\right)$, see **A4**, we can find a positive real number δ_0 such that if $0 < a+t < \delta_0$ then $|p(a,t) - p(0,0)| < \epsilon_0$. We see that

$$
|p(a,t)| \leq \epsilon_0 + c_0, \tag{56}
$$

where $c_0 = |p(0,0)|$ and hence, from (55), we get

$$
\begin{aligned}
e^{\psi t}\|\overline{v}(t)\| \leq M\|\overset{\circ}{v}\| &+ (\epsilon_0 + c_0)\int_0^t e^{\psi s}\|\overline{v}(s)\|\,ds \\
&+ (\epsilon_0 + c_0)\int_0^t e^{\psi s}\Big(\|n(s)\| + \|\overline{w}(s)\|\Big)\,ds.
\end{aligned}
$$

This leads, by (10), to

$$e^{\psi t}\|\overline{v}(t)\| \leq M\|\overset{\circ}{v}\| + (\epsilon_0 + c_0)\int_0^t e^{\psi s}\|\overline{v}(s)\|\, ds$$

$$+ M(\epsilon_0 + c_0)\int_0^t e^{\psi s}\left(\|\overset{\circ}{n}\| + \|\overset{\circ}{w}\|\right) ds,$$

$$\leq M\left(1 + (\epsilon_0 + c_0)\int_0^t e^{\psi s}\, ds\right)\|\overset{\circ}{n}\|$$

$$+ (\epsilon_0 + c_0)\int_0^t e^{\psi s}\|\overline{v}(s)\|\, ds,$$

$$\leq M(1 + \epsilon_0 + c_0)e^{\psi t}\|\overset{\circ}{n}\| + (\epsilon_0 + c_0)\int_0^t e^{\psi s}\|\overline{v}(s)\|\, ds$$

and the Gronwall inequality gives

$$\|\overline{v}(t)\| \leq M(1 + \epsilon_0 + c_0)\left(1 + \frac{\epsilon_0 + c_0}{\epsilon_0 + c_0 - \psi}e^{(\epsilon_0 + c_0 - \psi)t}\right)\|\overset{\circ}{n}\| \leq K_v\|\overset{\circ}{n}\|, \tag{57}$$

with $\psi \neq \epsilon_0 + c_0$.

Then, (49), by (10) with (17), (27), (40), (41), (42), and (57), yields

$$\|\epsilon d(t)\| \leq C_d\|\overset{\circ}{n}\|_{W^{1,1}([0,a_+])}, \tag{58}$$

$$\|\epsilon f(t)\| \leq C_f\|\overset{\circ}{n}\|_{W^{1,1}([0,a_+])}\left(1 + e^{-\frac{\gamma_1}{\epsilon}t} + \epsilon e^{-\frac{\gamma_1}{2\epsilon}t}\right), \tag{59}$$

$$\|\epsilon g(t)\| \leq C_g\|\overset{\circ}{n}\|_{W^{1,1}([0,a_+])}\left(1 + e^{-\frac{\gamma_2}{\epsilon}t} + \epsilon e^{-\frac{\gamma_2}{2\epsilon}t}\right), \tag{60}$$

$$\|\epsilon h(t)\| \leq C_h\|\overset{\circ}{n}\|_{W^{1,1}([0,a_+])}\left(1 + e^{-\frac{\sigma}{\epsilon}t} + \epsilon e^{-\frac{\sigma}{2\epsilon}t}\right), \tag{61}$$

$$\|\epsilon j(t)\| \leq C_j\|\overset{\circ}{n}\|_{W^{1,1}([0,a_+])}, \tag{62}$$

for some constants C_d, C_f, C_g, C_h, and C_j.

For any function of a, or a constant, θ, we consider the following auxiliary problem:

$$\partial_t\varrho = -\partial_a\varrho - \mu\varrho - \frac{\theta}{\epsilon}\varrho, \quad \varrho(0,t) = 0, \quad \varrho(a,0) = \overset{\circ}{\varrho}(a). \tag{63}$$

Using the fact that $\left(e^{tA}\right)_{t\geq 0}$ is a positive semigroup, it is easy to see that then

$$\|\varrho(t)\| \leq Me^{-\frac{\theta}{\epsilon}t}\|\overset{\circ}{\varrho}\|, \qquad t \geq 0, \tag{64}$$

where, as mentioned earlier, $\underline{\theta} = \min_{a\in[0,a_+]} \theta(a)$. Note that the assumption on R_μ is also sufficient here since $\omega \in [0,1]$. Using this estimate and the Duhamel formula, from (50), we have

$$\|d(t)\| \leq M \int_0^t \Big\| -(d+j)F(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0, \breve{l}, \breve{x}, \breve{z}) - \epsilon(d+j)\Lambda(h,g,f) + \Lambda(h,g,f)G(n,\overline{w},\overline{v}) + \frac{1}{\epsilon}F(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0, \breve{l}, \breve{x}, \breve{z})G(n,\overline{w},\overline{v}) \Big\| \, ds, \tag{65}$$

$$\|f(t)\| \leq Me^{-\frac{\gamma_1}{\epsilon}t}|c_\epsilon| + M\int_0^t e^{-\frac{\gamma_1}{\epsilon}(t-s)} \Big\| -\epsilon\frac{\sigma}{\sigma-\gamma_1}(d+j)\Lambda(h,g,f) - \frac{\sigma}{\sigma-\gamma_1}(d+j)F(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0, \breve{l}, \breve{x}, \breve{z}) + \frac{\sigma}{\sigma-\gamma_1}\Lambda(h,g,f)G(n,\overline{w},\overline{v}) + \frac{1}{\epsilon}\left(-\partial_a\widetilde{z}_0 - \mu\widetilde{z}_0 + \frac{\sigma}{\sigma-\gamma_1}F(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0, \breve{l}, \breve{x}, \breve{z})G(n,\overline{w},\overline{v})\right) \Big\| \, ds, \tag{66}$$

$$\begin{aligned}\|g(t)\| \leq{}& Me^{-\frac{\gamma_2}{\epsilon}t}|d_\epsilon| + M\int_0^t e^{-\frac{\gamma_2}{\epsilon}(t-s)} \Big\| -\frac{1}{\epsilon}\cdot\frac{\sigma}{\sigma-\gamma_1}h + \frac{q\gamma_1}{\gamma_1-\gamma_2}\cdot\frac{\gamma_2^{'}}{\gamma_1-\gamma_2}f \\ &- \epsilon\frac{\sigma}{\sigma-\gamma_1}\cdot\frac{q\gamma_1}{\gamma_1-\gamma_2}(d+j)\Lambda(h,g,f) - \frac{\sigma}{\sigma-\gamma_1}\cdot\frac{q\gamma_1}{\gamma_1-\gamma_2} \\ &\times\Big((d+j)F(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0, \breve{l}, \breve{x}, \breve{z}) - \Lambda(h,g,f)G(n,\overline{w},\overline{v})\Big) \\ &+ \frac{1}{\epsilon}\Big(\frac{\sigma}{\sigma-\gamma_1}\cdot\frac{q\gamma_1}{\gamma_1-\gamma_2}F(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0, \breve{l}, \breve{x}, \breve{z})G(n,\overline{w},\overline{v}) + \frac{q\gamma_1}{\gamma_1-\gamma_2} \\ &\times\frac{\gamma_2^{'}}{\gamma_1-\gamma_2}(\widetilde{z}_0+\breve{z}) - \partial_a\widetilde{x}_0 - \mu\widetilde{x}_0\Big) - \frac{1}{\epsilon^2}\cdot\frac{\sigma\gamma_1}{\sigma-\gamma_1}\breve{l}\Big\| \, ds,\end{aligned} \tag{67}$$

$$\|h(t)\| \leq Me^{-\frac{\sigma}{\epsilon}t}|h_\epsilon| + M\int_0^t e^{-\frac{\sigma}{\epsilon}(t-s)} \Big\| -\epsilon(d+j)\Lambda(h,g,f) - (d+j)F(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0, \breve{l}, \breve{x}, \breve{z}) + \Lambda(h,g,f)G(n,\overline{w},\overline{v}) + \frac{1}{\epsilon}\Big(-\partial_a\widetilde{l}_0 - \mu\widetilde{l}_0 + F(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0, \breve{l}, \breve{x}, \breve{z})G(n,\overline{w},\overline{v})\Big) \Big\| \, ds, \tag{68}$$

$$\|j(t)\| \leq M\int_0^t e^{-\psi(t-s)}\|-pj - pd\| \, ds. \tag{69}$$

Next, by (59), (60), and (61), we obtain

$$\|\epsilon\Lambda(h,g,f)\| \leq k_1\left(1 + e^{-\frac{\gamma_1}{\epsilon}t} + e^{-\frac{\gamma_2}{\epsilon}t} + e^{-\frac{\sigma}{\epsilon}t} + \epsilon e^{-\frac{\gamma_1}{\epsilon}t} + \epsilon e^{-\frac{\gamma_2}{\epsilon}t} + \epsilon e^{-\frac{\sigma}{\epsilon}t}\right). \tag{70}$$

Further, by (27), (40), (41), and (42),

$$\|F(\widetilde{l}_0, \widetilde{x}_0, \widetilde{z}_0, \breve{l}, \breve{x}, \breve{z})\| \leq k_2 e^{-\frac{\gamma_2}{\epsilon}t}(1+\epsilon)\left(1 + e^{-\frac{\gamma_1-\gamma_2}{\epsilon}t} + e^{-\frac{\sigma-\gamma_2}{\epsilon}t}\right). \tag{71}$$

Next, by (10) and (57),

$$\|G(n, \overline{w}, \overline{v})\| \leq k_3. \tag{72}$$

Finally, by (27),

$$\|\partial_a \widetilde{l}_0 + \mu \widetilde{l}_0\| \leq k_4 e^{-\frac{\sigma}{\epsilon}t}, \quad \|\partial_a \widetilde{x}_0 + \mu \widetilde{x}_0\| \leq k_5 e^{-\frac{\gamma_2}{\epsilon}t}, \quad \|\partial_a \widetilde{z}_0 + \mu \widetilde{z}_0\| \leq k_6 e^{-\frac{\gamma_1}{\epsilon}t}, \tag{73}$$

where k_4, k_5, and k_6 depend, in particular, on L^1 norm of $\partial_a \overset{\circ}{l}$ and $\mu \overset{\circ}{l}$, $\partial_a \overset{\circ}{x}$ and $\mu \overset{\circ}{x}$ and $\partial_a \overset{\circ}{z}$ and $\mu \overset{\circ}{z}$, respectively; that is, on the $D(\mathcal{A})$ norm of the initial condition.

The error estimates (50)–(52) will be completed below. Firstly, we see that, by (58), (59), (60), (71), (72), and (73) and by defining $H(t) = e^{\frac{\sigma}{\epsilon}t}\|h(t)\|$, (68) can be written as

$$H(t) \leq K_1 \int_0^t H(s)\, ds + \Phi_1(t),$$

where K_1 is a constant and $0 < \Phi_1(t) \leq k_7(e^{\frac{\sigma}{\epsilon}t} + 1)$, and the Gronwall inequality gives

$$H(t) \leq k_7(e^{\frac{\sigma}{\epsilon}t} + 1) + K_1 k_7 e^{K_1 t} \int_0^t e^{-K_1 s}(e^{\frac{\sigma}{\epsilon}s} + 1)\, ds \leq k_8 e^{\frac{\sigma}{\epsilon}t} + k_9 e^{K_1 t}$$

and hence

$$\|h(t)\| \leq k_8 + k_9 e^{\left(K_1 - \frac{\sigma}{\epsilon}\right)t} \leq k_{10}. \tag{74}$$

Secondly, we see that, by (58), (62), (70), (71), (72), (73), and (74) and by defining $F(t) = e^{\frac{\gamma_1}{\epsilon}t}\|f(t)\|$, (66) can be written as

$$F(t) \leq K_2 \int_0^t F(s)\, ds + \Phi_2(t),$$

where K_2 is a constant and $0 < \Phi_2(t) \leq k_{11}(e^{\frac{\gamma_1}{\epsilon}t} + 1)$, and the Gronwall inequality gives

$$F(t) \leq k_{11}(e^{\frac{\gamma_1}{\epsilon}t} + 1) + K_2 k_{11} e^{K_2 t} \int_0^t e^{-K_2 s}(e^{\frac{\gamma_1}{\epsilon}s} + 1)\, ds \leq k_{12} e^{\frac{\gamma_1}{\epsilon}t} + k_{13} e^{K_1 t}$$

and hence

$$\|f(t)\| \leq k_{12} + k_{13} e^{\left(K_1 - \frac{\gamma_1}{\epsilon}\right)t} \leq k_{14}. \tag{75}$$

Thirdly, we see that, by (27), (40), (41), (42), (58), (59), (62), (70), (71), (72), (73), and (74) and by defining $G(t) = e^{\frac{\gamma_2}{\epsilon}t}\|g(t)\|$, (67) can be written as

$$G(t) \leq K_3 \int_0^t G(s)\, ds + \Phi_3(t),$$

where K_3 is a constant and $0 < \Phi_3(t) \leq k_{15}(e^{\frac{\gamma_2}{\epsilon}t} + 1)$, and the Gronwall inequality gives

$$G(t) \leq k_{15}(e^{\frac{\gamma_2}{\epsilon}t} + 1) + K_3 k_{15} e^{K_3 t} \int_0^t e^{-K_3 s}(e^{\frac{\gamma_2}{\epsilon}s} + 1)\, ds \leq k_{16} e^{\frac{\gamma_2}{\epsilon}t} + k_{17} e^{K_3 t}$$

and hence

$$\|g(t)\| \leq k_{16} + k_{17} e^{\left(K_3 - \frac{\gamma_2}{\epsilon}\right)t} \leq k_{18}. \tag{76}$$

Next, we see that, by (27), (40), (41), (42), (70), (71), (72), (73), (74), and (75), (65) can be written as

$$\|d(t)\| \leq k_{19} \int_0^t e^{-\frac{\gamma_2}{2\epsilon}s} \|d(s)\|\, ds + \frac{k_{20}}{\epsilon} \int_0^t e^{-\frac{\gamma_2}{2\epsilon}s}\, ds + k_{21}.$$

Taking the maximum of t over $[0, \infty)$, we get

$$\begin{aligned}\max_{t \in [0,\infty)} \|d(t)\| &\leq k_{19} \int_0^\infty e^{-\frac{\gamma_2}{2\epsilon}s} \|d(s)\|\, ds + \frac{k_{20}}{\epsilon} \int_0^\infty e^{-\frac{\gamma_2}{2\epsilon}s}\, ds + k_{21} \\ &\leq \epsilon k_{22} \max_{t \in [0,\infty)} \|d(t)\| + \epsilon k_{23} + k_{21},\end{aligned}$$

which implies that $\max_{t \in [0,\infty)} \|d(t)\| \leq \dfrac{k_{21} + \epsilon k_{23}}{1 - \epsilon k_{22}} =: k_{24} < \infty$. Hence

$$\|d(t)\| \leq k_{24}. \tag{77}$$

Finally, we see that, by (56) and (77) and by defining $J(t) = e^{\psi t}\|j(t)\|$, (69) can be written as

$$J(t) \leq K_4 \int_0^t J(s)\, ds + \Phi_4(t),$$

where K_4 is a constant and $0 < \Phi_4(t) \leq k_{25}\left(e^{\psi t} - 1\right)$, and the Gronwall inequality gives

$$J(t) \leq k_{25}\left(e^{\psi t} - 1\right) + K_4 k_{25} e^{K_4 t} \int_0^t e^{-K_4 s}\left(e^{\psi s} - 1\right)\, ds$$

and hence

$$\|j(t)\| \leq \frac{\psi k_{25}}{\psi - K_4} =: k_{26}, \tag{78}$$

where $\psi > K_4$. This completes the proof. □

6. Conclusion

In this paper we proposed an age-structured epidemiological model for the transmission of HBV; then an asymptotic analysis of a singularly perturbed for such a model was performed. The Chapman–Enskog procedure, as an asymptotic method, was used for the mathematical analysis. This asymptotic method was developed in [1, 3, 6, 12] for age-structured epidemiological models and in [13] for the Carleman model of the Boltzmann equation, has enabled a systematic aggregation of variables, and has yielded a good approximated formula with layers correctors, namely initial layer and corner layer correctors. Note that a special corner layer equation was used instead of the standard one, and since the corner layer corrector decays exponentially fast in both a/ϵ and t/ϵ, while the initial layer corrector only does so in t/ϵ, this was neglected because of the $\mathrm{L}([0,a_+],\mathbb{R}^6)$-norm and therefore we obtained the approximation in terms of the initial layer corrector.

In summary, the result obtained shows that the solution of the nonlinear problem (6) can be approximated by the solution of five scalar linear problems and explicitly given initial layer corrector, uniformly for any time. The result is very interesting as the error estimates are uniform on the whole infinite time interval, in contrast to the typical result based on the Tikhonov theorem and classical asymptotic expansions.

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