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On orthogonal systems of shifts of scaling function on local fields of positive characteristic

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Abstract: We present a new method for constructing an orthogonal step scaling function on local fields of positive characteristic, which generates multiresolution analysis.

Key words: Local field, scaling function, multiresolution analysis

1. Introduction

Chinese mathematicians Jiang et al. in the article [10] introduced the notion of multiresolution analysis (MRA) on local fields. For the fields $F^{(s)}$ of positive characteristic p they proved some properties and gave an algorithm for constructing wavelets for a known scaling function. Using these results they constructed "Haar MRA" and corresponding "Haar wavelets". The problem of constructing orthogonal MRA on the field $F^{(1)}$ was studied in detail in the works [6–8, 12, 14, 15].

In [11] a necessary condition and sufficient conditions for wavelet frame on local fields were given. Behera and Jahan [2] constructed the wavelet packets associated with MRA on local fields of positive characteristic. In the article [1] necessary and sufficient conditions for a function $\varphi \in L^2(F^{(s)})$ under which it is a scaling function for MRA were obtained. These conditions are as follows:

$$\sum_{k\in\mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = 1 \tag{1}$$

for a.e. ξ in unit ball $\mathcal{D}$,

$$\lim_{j\to\infty} |\hat{\varphi}(\mathfrak{p}^j\xi)| = 1 \ for\ a.e.\ \xi \in F^{(s)}, \tag{2}$$

and there exists an integral periodic function $m_0 \in L^2(\mathcal{D})$ such that

$$\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi) \ for\ a.e.\ \xi \in F^{(s)}, \tag{3}$$

where $\{u(k)\}$ is the set of shifts and $\mathfrak{p}$ is a prime element. Behera and Jahan [3] proved also that if the translates of the scaling functions of two multiresolution analysis are biorthogonal, then the associated wavelet families are also biorthogonal. The same authors [4] proved a characterization of wavelets on local fields of positive

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characteristic based on results on affine and quasi-affine frames. Therefore, to construct MRA on a local field $F^{(s)}$ we need to construct an integral periodic mask m_0 with conditions (1–3). To solve this problem, in articles [1–3, 10, 11], the prime element methods developed in [16] were used. In these articles only Haar wavelets were obtained. In the article by Lukomskii and Vodolazov [13], another method to construct integral periodic masks and corresponding scaling step functions that generate non-Haar orthogonal MRA were developed.

However, in the article [13], only the simple case of mask m_0 being elementary was considered, i.e. $m_0(\chi)$ is constant on cosets $(F_{-1}^{(s)+})^{\perp}$ and $m_0(\chi)$ takes only two values, 0 and 1. In this article, we get rid of these restrictions and specify the method of constructing the scaling function only with the condition that $|\hat{\varphi}|$ is a step function. We reduce this problem to the study of some dynamical systems and prove that its trajectory has a fixed point.

2. Basic concepts

Let p be a prime number, $s \in \mathbb{N}$, $GF(p^s)$ – finite field. Local field $F^{(s)}$ of positive characteristic p is isomorphic (Kovalski–Pontryagin theorem [9]) to the set of formal power series

$$a = \sum_{i=k}^{\infty} \mathbf{a}_i t^i, \ k \in \mathbb{Z}, \quad \mathbf{a}_i = (a_i^{(0)}, a_i^{(1)}, \dots, a_i^{(s-1)}) \in GF(p^s).$$

Addition and multiplication in the field $F^{(s)}$ are defined as the sum and product of such series, i.e. if

$$a = \sum_{i=k}^{\infty} \mathbf{a}_i t^i, \ b = \sum_{i=k}^{\infty} \mathbf{b}_i t^i,$$

then

$$a \dot{+} b = \sum_{i=k}^{\infty} (\mathbf{a}_i \dot{+} \mathbf{b}_i) t^i, \ \mathbf{a}_i \dot{+} \mathbf{b}_i = (\mathbf{a}_i + \mathbf{b}_i) \bmod p,$$

$$ab = \sum_{l=2k}^{\infty} t^l \sum_{i,j:i+j=l} (\mathbf{a}_i \mathbf{b}_j).$$

Topology in $F^{(s)}$ is defined by the base of neighborhoods of zero

$$F_n^{(s)} = \{a = \sum_{j=n}^{\infty} \mathbf{a}_j t^j | \mathbf{a}_j \in GF(p^s)\}.$$

If

$$a = \sum_{j=n}^{\infty} \mathbf{a}_j t^j, \ \mathbf{a}_n \neq \mathbf{0},$$

then by definition $\|a\| = (\frac{1}{p^s})^n$, which implies

$$F_n^{(s)} = \{x \in F^{(s)} : \|x\| \le (\frac{1}{p^s})^n\}.$$

Thus, we may consider local field $F^{(s)}$ of positive characteristic p as the field of sequences infinite in both directions

$$a = (\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \dots, \mathbf{a}_0, \mathbf{a}_1, \dots), \ \mathbf{a}_j \in GF(p^s),$$

which have only a finite number of elements $\mathbf{a}_j$ with negative j nonequal to zero, and the operations of addition and multiplication are defined by equalities

$$a \dot{+} b = ((\mathbf{a}_i \dot{+} \mathbf{b}_i))_{i\in\mathbb{Z}},$$

$$ab = (\sum_{i,j:i+j=l} (\mathbf{a}_i \mathbf{b}_j))_{l\in\mathbb{Z}}, \tag{4}$$

where "$\dot{+}$" and "$\cdot$" are respectively addition and multiplication in $GF(p^s)$. Thus,

$$\|a\| = \|(\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots)\| = (\frac{1}{p^s})^n, \text{ if } \mathbf{a}_n \neq \mathbf{0},$$

$$F_n^{(s)} = \{a = (\mathbf{a}_j)_{j\in\mathbb{Z}} : \mathbf{a}_j \in GF(p^s); \ \mathbf{a}_j = 0, \ \forall j < n\}.$$

Let us consider $F^{(s)+}$ – the additive group of the field $F^{(s)}$. Neighborhoods $F_n^{(s)}$ are compact subgroups of the group $F^{(s)+}$; we will denote them as $F_n^{(s)+}$. They have the following properties:

1) $\dots \subset F_1^{(s)+} \subset F_0^{(s)+} \subset F_{-1}^{(s)+} \dots$

2) $F_n^{(s)+}/F_{n+1}^{(s)+} \cong GF(p^s)^+$ and $\sharp(F_n^{(s)+}/F_{n+1}^{(s)+}) = p^s$.

This implies that if $s = 1$ then $F^{(1)+}$ is a Vilenkin group with the stationary generating sequence $p_n = p$. The inverse is also true: one can define multiplication in any Vilenkin group $(\mathfrak{G}, \dot{+})$ with stationary generating sequence $p_n = p$ using equality (4). Supplied with such operation $(\mathfrak{G}, \dot{+}, \cdot)$ becomes a field isomorphic to $F^{(1)}$, where $e = (\dots, 0, 0_{-1}, 1_0, 0_1, \dots)$ is a neutral element with respect to multiplication.

It was noted in [17] that the field $F^{(s)}$ can be described as a linear space over $GF(p^s)$. Using this description one may define the multiplication of element $a \in F^{(s)}$ on element $\overline{\lambda} \in GF(p^s)$ coordinatewise, i.e. $\overline{\lambda} a = (\dots \mathbf{0}_{n-1}, \overline{\lambda}\mathbf{a}_n, \overline{\lambda}\mathbf{a}_{n+1}, \dots)$, and the modulus of $\overline{\lambda} \in GF(p^s)$ can be defined as

$$|\overline{\lambda}| = \begin{cases} 1, & \overline{\lambda} \neq \mathbf{0}, \\ 0, & \overline{\lambda} = \mathbf{0}. \end{cases}$$

It was also proved there that the system $g_k \in F_k^{(s)} \setminus F_{k+1}^{(s)}$ is a basis in $F^{(s)}$, i.e. any element $a \in F^{(s)}$ can be represented as:

$$a = \sum_{k\in\mathbb{Z}} \overline{\lambda}_k g_k, \ \overline{\lambda}_k \in GF(p^s).$$

From now on we will consider $g_k = (\dots, \mathbf{0}_{k-1}, (1^{(0)}, 0^{(1)}, \dots, 0^{(s-1)})_k, \mathbf{0}_{k+1}, \dots)$. In this case $\overline{\lambda}_k = \mathbf{a}_k$.

Let us define the sets

$$H_0^{(\nu)} = \{h \in F^{(s)} : h = \mathbf{a}_{-1} g_{-1} \dot{+} \mathbf{a}_{-2} g_{-2} \dot{+} \dots \dot{+} \mathbf{a}_{-\nu} g_{-\nu}\},$$

where ν is a fixed natural number:

$$H_0 = \{h \in F^{(s)}:\ h = \mathbf{a}_{-1}g_{-1}\dot{+}\mathbf{a}_{-2}g_{-2}\dot{+}\dots\dot{+}\mathbf{a}_{-\nu}g_{-\nu},\ \nu \in \mathbb{N}\}.$$

The set H_0 is the set of shifts in $F^{(s)}$. It is an analogue of the set of nonnegative integers.

We will denote the collection of all characters of $F^{(s)+}$ as X. The set X generates a commutative group with respect to the multiplication of characters: $(\chi * \phi)(a) = \chi(a)\cdot\phi(a)$. The inverse element is defined as $\chi^{-1}(a) = \overline{\chi(a)}$, and the neutral element is $e(a) \equiv 1$.

Following [17] we define characters r_n of the group $F^{(s)+}$ in the following way. Let $x = (\dots, \mathbf{0}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}, \dots)$, $\mathbf{x}_j = (x_j^{(0)}, x_j^{(1)}, \dots, x_j^{(s-1)}) \in GF(p^s)$. The element $\mathbf{x}_j$ can be written in the form $\mathbf{x}_j = (x_{js+0}, x_{js+1}, \dots, x_{js+(s-1)})$. In this case

$$x = (\dots, 0, ..., 0, x_{ks+0}, x_{ks+1}, \dots, x_{ks+s-1}, x_{(k+1)s+0}, x_{(k+1)s+1}, \dots, x_{(k+1)s+s-1}, \dots)$$

and the collection of all such sequences x is a Vilenkin group. Thus, the equality $r_n(x) = r_{ks+l}(x) = e^{\frac{2\pi i}{p}(x_{ks+l})}$ defines the Rademacher function of $F^{(s)+}$ and every character $\chi \in X$ can be described in the following way:

$$\chi = \prod_{n\in\mathbb{Z}} r_n^{a_n}, \quad a_n = \overline{0, p-1}. \tag{5}$$

Equality (5) can be rewritten as

$$\chi = \prod_{k\in\mathbb{Z}} r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}} \tag{6}$$

and let us define

$$\mathbf{r}_k^{\mathbf{a}_k} := r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}},$$

where $\mathbf{a}_k = (a_k^{(0)}, a_k^{(1)}, \dots, a_k^{(s-1)}) \in GF(p^s)$. Then (6) takes the form

$$\chi = \prod_{k\in\mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k}. \tag{7}$$

We will refer to $\mathbf{r}_k^{(1,0,\dots,0)} = \mathbf{r}_k$ as the Rademacher functions. By definition we set

$$(\mathbf{r}_k^{\mathbf{a}_k})^{\mathbf{b}_k} = \mathbf{r}_k^{\mathbf{a}_k\mathbf{b}_k}, \quad \chi^{\mathbf{b}} = (\prod \mathbf{r}_k^{\mathbf{a}_k})^{\mathbf{b}} = \prod \mathbf{r}_k^{\mathbf{a}_k\mathbf{b}}, \quad \mathbf{a}_k, \mathbf{b}_k, \mathbf{b} \in GF(p^s).$$

The definition of the Rademacher function implies that if $\mathbf{x} = ((x_k^{(0)}, x_k^{(1)}, \dots x_k^{(s-1)}))_{k\in\mathbb{Z}}$ and $\mathbf{u} = (u^{(0)}, u^{(1)}, \dots, u^{(s-1)}) \in GF(p^s)$, then

$$(\mathbf{r}_k^{\mathbf{u}}, \mathbf{x}) = \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p} u^{(l)} x_k^{(l)}}.$$

In [17] the following properties of characters were proved:

1) $\mathbf{r}_k^{\mathbf{u}\dot{+}\mathbf{v}} = \mathbf{r}_k^{\mathbf{u}}\mathbf{r}_k^{\mathbf{v}}$, $\mathbf{u}, \mathbf{v} \in GF(p^s)$.

2) $(\mathbf{r}_k^{\mathbf{v}}, \mathbf{u}g_j) = 1\,,\ \forall k \neq j\,,\ \mathbf{u}, \mathbf{v} \in GF(p^s)\,.$

3) The set of characters of the field $F^{(s)}$ is a linear space $(X,\ *,\ \cdot^{GF(p^s)})$ over the finite field $GF(p^s)$ with multiplication being an inner operation and the power $\mathbf{u} \in GF(p^s)$ being an outer operation.

4) The sequence of Rademacher functions $(\mathbf{r}_k)$ is a basis in the space $(X,\ *,\ \cdot^{GF(p^s)})$.

5) Any sequence of characters $\chi_k \in (F_{k+1}^{(s)})^\perp \setminus (F_k^{(s)})^\perp$ is also a basis in the space $(X,\ *,\ \cdot^{GF(p^s)})$, where $F_n^{(s)\perp}$ is the annihilator of $F_n^{(s)+}$.

The dilation operator $\mathcal{A}$ in local field $F^{(s)}$ can be defined as $\mathcal{A}x := \sum_{n=-\infty}^{+\infty} \mathbf{a}_n g_{n-1}$, where $x = \sum_{n=-\infty}^{+\infty} \mathbf{a}_n g_n \in F^{(s)}$. In the group of characters it is defined as $(\chi\mathcal{A}, x) = (\chi, \mathcal{A}x)$.

3. Scaling function and MRA

We will consider a case of scaling function φ, which generates an orthogonal MRA, being a step function. The set of step functions constant on cosets of a subgroup $F_M^{(s)}$ with the support $\mathrm{supp}(\varphi) \subset F_{-N}^{(s)}$ will be denoted as $\mathfrak{D}_M(F_{-N}^{(s)})$, $M, N \in \mathbb{N}$. Similarly, $\mathfrak{D}_{-N}(F_M^{(s)\perp})$ is a set of step functions, constant on the cosets of a subgroup $F_{-N}^{(s)\perp}$ with the support $\mathrm{supp}(\varphi) \subset F_M^{(s)\perp}$. If $\varphi \in \mathfrak{D}_M(F_{-N}^{(s)})$ generates an orthogonal MRA, it satisfies the refinement equation $\varphi(x) = \sum_{h \in H_0^{(N+1)}} \beta_h \varphi(\mathcal{A}x \dot{-} h)$, which can be rewritten in a frequency from

$$\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}), \tag{8}$$

where

$$m_0(\chi) = \frac{1}{p^s} \sum_{h \in H_0^{(N+1)}} \beta_h \overline{(\chi\mathcal{A}^{-1}, h)} \tag{9}$$

is the mask of equation (8).

For the step functions in the article [13] condition (3) and orthogonality condition (1) are rewritten in the terms of Rademacher functions.

1) If $\hat{\varphi}(\chi) \in \mathfrak{D}_{-N}(F_M^{(s)\perp})$ is a solution of refinement equation (8) and the system of shifts $(\varphi(x \dot{-} h))_{h \in H_0}$ is orthonormal, then φ generates an orthogonal MRA.

2) If $\hat{\varphi}(\chi) \in \mathfrak{D}_{-N}(F_M^{(s)\perp})$, then the system of shifts $(\varphi(x \dot{-} h))_{h \in H_0}$ will be orthonormal iff for any $\mathbf{a}_{-N}, \mathbf{a}_{-N+1}, \ldots, \mathbf{a}_{-1} \in GF(p^s)$

$$\sum_{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{M-1} \in GF(p^s)} |\hat{\varphi}(F_{-N}^{(s)\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \ldots \mathbf{r}_0^{\mathbf{a}_0} \ldots \mathbf{r}_{M-1}^{\mathbf{a}_{M-1}})|^2 = 1. \tag{10}$$

Thus, to construct an orthogonal MRA one must construct a function $\hat{\varphi}(\chi) \in \mathfrak{D}_{-N}(F_M^{(s)\perp})$, which is a solution of refinement equation (8) and which satisfies conditions (10). Satisfying both conditions is the main difficulty of this problem.

As was already mentioned in the introduction, a method for construction of a scaling function that generates non-Haar orthogonal MRA was specified in [13]. It is constructed by the means of some tree and results in a function such that $|\hat{\varphi}|$ takes two values only: 0 and 1. A more general case will be presented in the next section.

4. Construction of orthogonal scaling function

Definition 4.1. *Let $F^{(s)}$ be a local field of positive characteristic p, and N is a natural number. Then by N-valid tree we mean a tree oriented from leaves to root and satisfying these conditions:*

1)Every vertex is an element of $GF(p^s)$, i.e has the form $\mathbf{a}_i = (a_i^{(0)}, a_i^{(1)}, \dots, a_i^{(s-1)})$, $a_i^{(j)} = \overline{0, p-1}$.

2)The root and all vertices of level $N-1$ are equal to the zero element of $GF(p^s)$: $\mathbf{0} = (0^{(0)}, 0^{(1)}, \dots, 0^{(s-1)})$.

3)Any path $(\mathbf{a}_k \to \mathbf{a}_{k+1} \to \dots \to \mathbf{a}_{k+N-1})$ of length $N-1$ appears in the tree exactly one time.

Let us choose N-valid tree T and construct a scaling function using it.

1) We will use this tree T to construct new tree $\tilde{T}$. Every vertex of the tree $\tilde{T}$ is a vector of N elements each being an element of $GF(p^s)$: $\mathbf{A} = (\mathbf{a}_N, \mathbf{a}_{N-1}, \dots, \mathbf{a}_1)$. Such vertices are constructed in the following way: if a tree T has a path of length $N-1$ starting from $\mathbf{a}_N$

$$\mathbf{a}_N \to \mathbf{a}_{N-1} \to \dots \to \mathbf{a}_1,$$

then in $\tilde{T}$ we will have a vertex with the value equal to the array of N elements $(\mathbf{a}_N, \mathbf{a}_{N-1}, \dots \ \dots, \mathbf{a}_1)$. Due to condition 3) of N-validity of tree T each such array corresponds to the unique vertex of the new tree $\tilde{T}$. Thus, the root of $\tilde{T}$ is an N-dimensional vector with all elements equal to the zero of $GF(p^s)$ $\mathbf{O} = (\mathbf{0}, \mathbf{0}, \dots, \mathbf{0})$. Vertices of level 1 in tree $\tilde{T}$ are N-dimensional vectors, which have all their elements, except the first one, equal to the zero of $GF(p^s)$: $(\mathbf{a}_i, \mathbf{0}, \dots, \mathbf{0})$, where $\mathbf{a}_i$ is some vertex of level N in tree T. Vertices of level 2 in tree $\tilde{T}$ are N-dimensional vectors: $(\mathbf{a}_{i_2}, \mathbf{a}_{i_1}, \mathbf{0}, \dots, \mathbf{0})$, where $\mathbf{a}_{i_2}$ and $\mathbf{a}_{i_1}$ are some vertices of levels $N+1$ and N of tree T respectively, which are connected. We should note that in this example $\mathbf{a}_{i_1} \neq \mathbf{0}$, but $\mathbf{a}_{i_2}$ may be a zero element of $GF(p^s)$. Thus, in $\tilde{T}$ connected vertices have the form $(\mathbf{a}_{i_N}, \mathbf{a}_{i_{N-1}}, \dots, \mathbf{a}_{i_1}) \to (\mathbf{a}_{i_{N-1}}, \dots, \mathbf{a}_{i_1}, \mathbf{a}_{i_0})$. However, not all vertices satisfying this condition will be connected. Arcs are taken from the original tree T. If we denote $height(T) = H$, $height(\tilde{T}) = \tilde{H}$, then obviously $\tilde{H} = H - N + 1$.

2) Now we will construct a directed graph Γ using $\tilde{T}$. We connect each vertex $\mathbf{A}_N = (\mathbf{a}_N, \mathbf{a}_{N-1}, \dots, \mathbf{a}_1)$ of $\tilde{T}$ to each vertex of lesser level of the form $(\mathbf{a}_{N-1}, \dots, \mathbf{a}_1, \mathbf{a}_0)$, i.e. having the first $(N-1)$ elements equal to the last $(N-1)$ elements of vertex $\mathbf{A}_N$. The vertices, to which $\mathbf{A}_N$ is connected, we will denote by $(\mathbf{a}_{N-1}, \dots, \mathbf{a}_1, \tilde{\mathbf{a}}_0)$, i.e. $\mathbf{a}_0 \in \{\tilde{\mathbf{a}}_0\}$ iff the vertex $\mathbf{A}_N$ is connected to $(\mathbf{a}_{N-1}, \dots, \mathbf{a}_1, \mathbf{a}_0)$ in digraph Γ.

3) Let us denote

$$\lambda_{\mathbf{a}_{-N}, \mathbf{a}_{-N+1}, \dots, \mathbf{a}_{-1}, \mathbf{a}_0} = |m_0({F^{(s)}}^{\perp}_{-N} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \mathbf{r}_{-N+1}^{\mathbf{a}_{-N+1}} \dots \mathbf{r}_{-1}^{\mathbf{a}_{-1}} \mathbf{r}_0^{\mathbf{a}_0})|^2,$$

i.e. $\lambda_{\mathbf{a}_{-N}, \mathbf{a}_{-N+1}, \dots, \mathbf{a}_{-1}, \mathbf{a}_0}$ is an $(N+1)$-dimensional array, enumerated by the elements of $GF(p^s)$.

If the vertex $(\mathbf{a}_N, \mathbf{a}_{N-1}, \dots, \mathbf{a}_1)$ of graph Γ is connected to the vertices $(\mathbf{a}_{N-1}, \mathbf{a}_{N-2} \dots, \mathbf{a}_1, \ \tilde{\mathbf{a}}_0)$ then we define the values of the mask in the way satisfying the condition

$$\sum_{\tilde{\mathbf{a}}_0} \lambda_{\mathbf{a}_{-N}, \mathbf{a}_{-N+1}, \dots, \mathbf{a}_{-1}, \tilde{\mathbf{a}}_0} = 1 \text{ and } \lambda_{\mathbf{a}_{-N}, \mathbf{a}_{-N+1}, \dots, \mathbf{a}_{-1}, \mathbf{a}_0} = 0 \text{ for any } \mathbf{a}_0 \notin \{\tilde{\mathbf{a}}_0\}. \tag{11}$$

Also, let us define $m_0({F^{(s)}_{-N}}^{\perp}) = 1$, which implies $\lambda_{\mathbf{0}, \mathbf{0}, \dots, \mathbf{0}} = 1$.

To present the main result we will need some extra notation. First, we must note that the orthonormality condition (10) for the system of shifts of $\varphi(x)$ can be rewritten as follows: for any $\mathbf{a}_{-N}, \mathbf{a}_{-N+1}, \dots, \mathbf{a}_{-1} \in$

$GF(p^s)$

$$1 = \sum_{\mathbf{a}_0,\mathbf{a}_1,\ldots,\mathbf{a}_{M-1}\in GF(p^s)} |\hat{\varphi}(F_{-N}^{(s)\,\perp}\mathbf{r}_{-N}^{\mathbf{a}_{-N}}\ldots\mathbf{r}_{-1}^{\mathbf{a}_{-1}}\mathbf{r}_0^{\mathbf{a}_0}\ldots\mathbf{r}_{M-1}^{\mathbf{a}_{M-1}})|^2 =$$

$$\sum_{\mathbf{a}_0\in GF(p^s)} \lambda_{\mathbf{a}_{-N},\mathbf{a}_{-N+1},\ldots,\mathbf{a}_0} \sum_{\mathbf{a}_1\in GF(p^s)} \lambda_{\mathbf{a}_{-N+1},\mathbf{a}_{-N+2},\ldots,\mathbf{a}_1}\cdots$$

$$\cdots \sum_{\mathbf{a}_{M-2}\in GF(p^s)} \lambda_{\mathbf{a}_{M-N-2},\mathbf{a}_{M-N-1},\ldots,\mathbf{a}_{M-2}}$$

$$\sum_{\mathbf{a}_{M-1}\in GF(p^s)} \lambda_{\mathbf{a}_{M-N-1},\mathbf{a}_{M-N},\ldots,\mathbf{a}_{M-1}}\lambda_{\mathbf{a}_{M-N},\mathbf{a}_{M-N+1},\ldots,\mathbf{a}_{M-1},\mathbf{0}}\cdots\lambda_{\mathbf{a}_{M-1},\mathbf{0},\ldots,\mathbf{0}}. \tag{12}$$

Let us then define a sequence of N-dimensional arrays $A^{(n)} = (a^{(n)}_{\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_N})_{\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_N\in GF(p^s)}$ recurrently by giving the relations of their components:

$$a^{(0)}_{\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_N} = \lambda_{\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_N,\mathbf{0}}\lambda_{\mathbf{i}_2,\mathbf{i}_3,\ldots,\mathbf{i}_N,\mathbf{0},\mathbf{0}}\cdots\lambda_{\mathbf{i}_N,\mathbf{0},\ldots,\mathbf{0}}, \tag{13}$$

$$a^{(n)}_{\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_N} = \sum_{\mathbf{j}\in GF(p^s)} \lambda_{\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_N,\mathbf{j}}\, a^{(n-1)}_{\mathbf{i}_2,\mathbf{i}_3,\ldots,\mathbf{i}_N,\mathbf{j}}. \tag{14}$$

We will say that the element $a^{(s)}_{\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_N}$ corresponds to vertex $(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_N)$.

In new notation the sum

$$\sum_{\mathbf{a}_{M-1}\in GF(p^s)} \lambda_{\mathbf{a}_{M-N-1},\mathbf{a}_{M-N},\ldots,\mathbf{a}_{M-1}}\lambda_{\mathbf{a}_{M-N},\mathbf{a}_{M-N+1},\ldots,\mathbf{a}_{M-1},\mathbf{0}}\cdots\lambda_{\mathbf{a}_{M-1},\mathbf{0},\ldots,\mathbf{0}}$$

from (12) defines elements of the array $A^{(1)}$. The sum

$$\sum_{\mathbf{a}_{M-2}\in GF(p^s)} \lambda_{\mathbf{a}_{M-N-2},\mathbf{a}_{M-N-1},\ldots,\mathbf{a}_{M-2}}$$

$$\sum_{\mathbf{a}_{M-1}\in GF(p^s)} \lambda_{\mathbf{a}_{M-N-1},\mathbf{a}_{M-N},\ldots,\mathbf{a}_{M-1}}\lambda_{\mathbf{a}_{M-N},\mathbf{a}_{M-N+1},\ldots,\mathbf{a}_{M-1},\mathbf{0}}\cdots\lambda_{\mathbf{a}_{M-1},\mathbf{0},\ldots,\mathbf{0}}$$

defines the elements of the array $A^{(2)}$, and so on. The whole sum specified in (12) defines elements of array $A^{(M)}$. Using new notation, orthonormality condition (12) can be reformulated in the following way: the system of shifts of the function $\varphi(x) \in \mathfrak{D}_M(F_{-N}^{(s)})$ is orthonormal if and only if for any $\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_N$: $a^{(M)}_{\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_N} = 1$, in other words, iff an array $A^{(M)}$ has all its elements equal to 1.

Lemma 4.1. *The components of $A^{(0)}$ corresponding to vertices of level $l \le N$ in the tree $\tilde{T}$ are equal to 1.*

Proof First, let us notice that any vertex of $\tilde{T}$ of level $l \le N$ has the form $(\mathbf{a}_l,\mathbf{a}_{l-1},\ldots,\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0})$, $\mathbf{a}_1 \neq \mathbf{0}$. Indeed, if a vertex has level l in $\tilde{T}$, then the first element of the vector - the vertex of T - is of level $l+N-1$ in T and is the beginning of the following path directed to root: $(\mathbf{a}_l \to \mathbf{a}_{l-1} \to \cdots \to \mathbf{a}_1 \to \mathbf{0} \to \cdots \to \mathbf{0})$, where $\mathbf{a}_1$ is a vertex of level N and is nonzero by the N-validity condition.

We will prove the lemma by induction on l. Let $l = 0$. Thus, we consider the root of $\tilde{T}$. The root has the form $(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})$. By construction $\lambda_{\mathbf{0},\mathbf{0},\ldots,\mathbf{0}} = 1$. Its corresponding element of array $A^{(0)}$ is $a^{(0)}_{\mathbf{0},\mathbf{0},\ldots,\mathbf{0}}$. Let us substitute $\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_N = \mathbf{0}$ into (13). We obtain

$$a^{(0)}_{\mathbf{0},\mathbf{0},\ldots,\mathbf{0}} = \lambda_{\mathbf{0},\mathbf{0},\ldots,\mathbf{0}}\lambda_{\mathbf{0},\mathbf{0},\ldots,\mathbf{0}}\ldots\lambda_{\mathbf{0},\mathbf{0},\ldots,\mathbf{0}} = 1.$$

Now we prove that if any vertex of level $l = k-1 < N$ satisfies the condition $a^{(0)}_{\mathbf{a}_{k-1},\mathbf{a}_{k-2},\ldots,\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}} = 1$, then such a condition is also satisfied by any vertex of level $l = k \leq N$ of the tree $\tilde{T}$. Using (13) and substituting $\mathbf{i}_1 = \mathbf{a}_{k-1},\ \mathbf{i}_2 = \mathbf{a}_{k-2},\ \ldots,\ \mathbf{i}_{k-1} = \mathbf{a}_1 \neq \mathbf{0},\ \mathbf{i}_k = \mathbf{0},\ \ldots,\ \mathbf{i}_N = \mathbf{0}$, we rewrite the induction hypothesis:

$$a^{(0)}_{\mathbf{a}_{k-1},\mathbf{a}_{k-2},\ldots,\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}} = \lambda_{\mathbf{a}_{k-1},\mathbf{a}_{k-2},\ldots,\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}}\lambda_{\mathbf{a}_{k-2},\mathbf{a}_{k-3},\ldots,\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}}\ldots\lambda_{\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}}\lambda_{\mathbf{0},\mathbf{0},\ldots,\mathbf{0}}\ldots$$

$$\ldots\lambda_{\mathbf{0},\mathbf{0},\ldots,\mathbf{0}} = \lambda_{\mathbf{a}_{k-1},\mathbf{a}_{k-2},\ldots,\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}}\lambda_{\mathbf{a}_{k-2},\mathbf{a}_{k-3},\ldots,\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}}\ldots\lambda_{\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}} = 1.$$

Here we omit $\lambda_{\mathbf{0},\mathbf{0},\ldots,\mathbf{0}} = 1$. Now, let

$$\mathbf{A}_k = (\mathbf{a}_k, \mathbf{a}_{k-1}, \ldots, \mathbf{a}_1, \mathbf{0}, \ldots, \mathbf{0}), \quad \mathbf{a}_1 \neq \mathbf{0}$$

be a vertex of level k of $\tilde{T}$.

Let this vertex be connected to the vertex $\mathbf{A}_{k-1} = (\mathbf{a}_{k-1}, \ldots, \mathbf{a}_1, \mathbf{0}, \ldots, \mathbf{0})$ of level $k-1$ in $\tilde{T}$. Then it can be shown that the vertex $\mathbf{A}_k$ is only connected to the vertex $\mathbf{A}_{k-1}$ in digraph Γ also.

First, let us prove that in graph Γ the vertex $\mathbf{A}_k$ is not connected to any other vertex, which has level $k-1$ in $\tilde{T}$. We will prove the fact by contradiction. Assume that $\mathbf{B}_{k-1} = (\mathbf{b}_{k-1}, \ldots, \mathbf{b}_1 \neq \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})$ is another vertex that has level $k-1$ in $\tilde{T}$ and that $\mathbf{A}_k$ is connected to $\mathbf{A}_{k-1}$ and $\mathbf{B}_{k-1}$ in graph Γ. By construction, if $\mathbf{A}_k$ is connected to $\mathbf{B}_{k-1}$ then for any $i = \overline{1, k-1}, \quad \mathbf{a}_i = \mathbf{b}_i$, which implies vertices $\mathbf{A}_{k-1}$ and $\mathbf{B}_{k-1}$ being identical, which contradicts the uniqueness of the vertices in $\tilde{T}$ and Γ. Thus, there is only one vertex, which is of level $(k-1)$ in $\tilde{T}$ and to which $\mathbf{A}_k$ is connected in graph Γ.

Secondly, we prove that in Γ the vertex $\mathbf{A}_k$ is not connected to any vertex that has level strictly less than $k-1$ in tree $\tilde{T}$. Let $n > 1$, $\mathbf{B}_{k-n} = (\mathbf{b}_{k-n}, \ldots, \mathbf{b}_1, \mathbf{0}, \ldots, \mathbf{0})$ be an arbitrary vertex of level $(k-n)$ in $\tilde{T}$. By construction of Γ, for the vertex $\mathbf{A}_k$ to be connected to $\mathbf{B}_{k-n}$ it is necessary for the equality $\mathbf{a}_1 = \mathbf{0}$ to hold, which is impossible by assumption $\mathbf{a}_1 \neq \mathbf{0}$. Thus, we proved that the vertex $\mathbf{A}_k$ is connected only to $\mathbf{A}_{k-1}$ in Γ.

By construction that means that $\lambda_{\mathbf{a}_k,\ldots,\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}} = 1$. Thus, substituting $\mathbf{i}_1 = \mathbf{a}_k, \mathbf{i}_2 = \mathbf{a}_{k-1}, \ldots, \mathbf{i}_k = \mathbf{a}_1, \mathbf{i}_{k+1} = \mathbf{0}, \ldots, \mathbf{i}_N = \mathbf{0}$ into (13) and using the induction hypothesis we obtain

$$a^{(0)}_{\mathbf{a}_k,\ldots,\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}} = \lambda_{\mathbf{a}_k,\ldots,\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}}\lambda_{\mathbf{a}_{k-1},\ldots,\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}}\ldots\lambda_{\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}} =$$

$$\lambda_{\mathbf{a}_k,\ldots,\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}}a^{(0)}_{\mathbf{a}_{k-1},\ldots,\mathbf{a}_1,\mathbf{0},\ldots,\mathbf{0}} = 1.$$

The lemma is proved. □

Lemma 4.2. *Let us consider N-valid tree T and tree $\tilde{T}$ and digraph Γ constructed using it. Let the values of $m_0(\chi)$ be defined as specified in equalities (8). Let also $(A^{(n)})_{n=0}^{\infty}$ be a sequence of arrays defined by equalities*

(13) and (14). Then the array $A^{(n)}$ has its elements corresponding to the vertices of level $l \leq N+n$ in tree $\tilde{T}$ equal to 1.

Proof We will prove the lemma by induction. The validity of the base for $n=0$ follows from the previous lemma. Now we prove that if in $A^{(n-1)}$ elements corresponding to vertices of level less than or equal to $N+n-1$ are equal to one, then in $A^{(n)}$ elements corresponding to vertices of level less than or equal to $N+n$ are equal to one. Let $\mathbf{A}_N = (\mathbf{a}_N, \mathbf{a}_{N-1}, \ldots, \mathbf{a}_1)$ be a vertex of level $l \leq N+n$ in $\tilde{T}$. In graph Γ it is connected to all vertices of lower level, which we denote as $(\mathbf{a}_{N-1}, \ldots, \mathbf{a}_1, \tilde{\mathbf{a}}_0)$; moreover, $\sum\limits_{\tilde{\mathbf{a}}_0} \lambda_{\mathbf{a}_N, \mathbf{a}_{N-1}, \ldots, \mathbf{a}_1, \tilde{\mathbf{a}}_0} = 1$ and $\lambda_{\mathbf{a}_N, \mathbf{a}_{N-1}, \ldots, \mathbf{a}_1, \mathbf{a}_0} = 0 \ \forall \mathbf{a}_0 \notin \{\tilde{\mathbf{a}}_0\}$.

Also, it should be mentioned that since vertices $(\mathbf{a}_{N-1}, \ldots, \mathbf{a}_1, \tilde{\mathbf{a}}_0)$ of $\tilde{T}$ have their level not higher than $l-1 \leq N+n-1$, then, by the induction hypothesis

$a^{(n-1)}_{\mathbf{a}_{N-1}, \ldots, \mathbf{a}_1, \tilde{\mathbf{a}}_0} = 1, \ \forall \tilde{\mathbf{a}}_0 \in \{\tilde{\mathbf{a}}_0\}$. Then

$$a^{(n)}_{\mathbf{a}_N, \mathbf{a}_{N-1} \ldots, \mathbf{a}_1} = \sum_{\mathbf{a}_0 \in GF(p^s)} \lambda_{\mathbf{a}_N, \mathbf{a}_{N-1}, \ldots, \mathbf{a}_1, \mathbf{a}_0} a^{(n-1)}_{\mathbf{a}_{N-1}, \ldots, \mathbf{a}_1, \mathbf{a}_0} =$$

$$\sum_{\tilde{\mathbf{a}}_0 \in \{\tilde{\mathbf{a}}_0\}} \lambda_{\mathbf{a}_N, \mathbf{a}_{N-1}, \ldots, \mathbf{a}_1, \tilde{\mathbf{a}}_0} a^{(n-1)}_{\mathbf{a}_{N-1}, \ldots, \mathbf{a}_1, \tilde{\mathbf{a}}_0} = \sum_{\tilde{\mathbf{a}}_0 \in \{\tilde{\mathbf{a}}_0\}} \lambda_{\mathbf{a}_N, \mathbf{a}_{N-1}, \ldots, \mathbf{a}_1, \tilde{\mathbf{a}}_0} = 1,$$

which proves the lemma.

These lemmas directly imply the following theorem. □

Theorem 4.3. *Let the tree $\tilde{T}$ and digraph Γ be constructed using N-valid tree T. Let the values of $m_0(\chi)$ be defined as specified by equalities (11). Let $\tilde{H} = height(\tilde{T})$. Then the equality*

$$\hat{\varphi}(\chi) = \prod_{k=0}^{\infty} m_0(\chi \mathcal{A}^{-k}) \in \mathfrak{D}_{-N}({F_M^{(s)}}^{\perp})$$

defines an orthogonal scaling function $\varphi(x) \in \mathfrak{D}_M(F_{-N}^{(s)})$, and $M \leq \tilde{H} - N$.

Remark. We would like to remind [5] that a discrete dynamical system consists of a nonempty set X and a map $f: X \to X$. For $n \in \mathbb{N}$, the nth iterate of f is the n-fold composition $f^n = f \circ \cdots \circ f$, and f^0 is considered an identity map. A point $x \in X$ is called a *fixed point* if $f(x) = x$. Starting at the initial conditions x_0 at the 0th iteration, we can apply the function n times to determine the state $x_n = f^n(x_0)$. The sequence $(x_n)_{n=0}^{+\infty}$ is called a *trajectory.*

Let us denote the collection of functions $f_N : \{0, 1, ..., p-1\}^N \to [0,1]$ as Φ_N and choose a function $\Lambda \in \Phi_{N+1}$. Function Λ may be viewed as $(N+1)$-dimensional array $\Lambda = (\lambda_{i_1, i_2, \ldots, i_N, i_{N+1}})$. Then the equalities (14) define discrete dynamic system $\Lambda : \Phi_N \to \Phi_N$, and the equality (13) defines the initial state. Theorem 4.3 specifies a class of discrete dynamical systems Λ with initial state $A^{(0)}$, which have a fixed point in their trajectory with initial point (13).

Theorem 4.3 for $s=1$, $N=1$ was proved by Kruss, for $s=1$, $N \in \mathbb{N}$ – by Berdnikov, and for any $s, N \in \mathbb{N}$ – by Kruss. The idea to consider the local field of positive characteristic as the vector space was proposed by Lukomskii.

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