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Existence of unpredictable solutions and chaos

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Abstract: Recently we introduced the concept of Poincaré chaos. In the present paper, by means of the Bebutov dynamical system, an unpredictable solution is considered as a generator of the chaos in a quasilinear system. The results can be easily extended to different types of differential equations. An example of an unpredictable function is provided. A proper irregular behavior in coupled Duffing equations is observed through simulations.

Key words: Poincaré chaos, unpredictable function, Poisson stability, Bebutov dynamical system, quasilinear differential equation, chaos control

1. Introduction

The row of periodic, quasiperiodic, almost periodic, recurrent, and Poisson stable motions was successively developed in the theory of dynamical systems. Then chaotic dynamics started to be considered, which is not a single motion phenomenon, since a prescribed set of motions is required for a definition [16, 22, 31]. Our manuscript serves for proceeding the row and involving chaos as a purely functional object in nonlinear dynamics. In our previous paper [13], we introduced unpredictable motions based on Poisson stability. This time, we introduce the concept of an unpredictable function as an unpredictable point in Bebutov dynamics [29].

It was proved in [13] that an unpredictable point gives rise to the existence of chaos in the quasiminimal set. Thus, if one shows the existence of an unpredictable solution of an equation, then the chaos exists. The present study as well as our previous results concerning replication of chaos [12] support the opinion of Holmes [20] that the theory of chaos has to be a part of the theory of differential equations. Since the main body of the results on chaotic motions has been formulated in terms of differential and difference equations, we may suggest that all these achievements have to be embedded and developed in the theory of dynamical systems or, more specifically, in the theory of differential equations or hybrid systems.

The rest of the paper is organized as follows. In the next section, we give the auxiliary results from the previous paper [13]. Section 3 is concerned with Bebutov dynamics and the description of unpredictable functions. The existence of unpredictable solutions in a quasilinear system is considered in Section 4. Section 5 is devoted to examples. Finally, some concluding remarks are given in Section 6.

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2. Preliminaries

Throughout the paper, we will denote by $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$, and $\mathbb{Z}$ the sets of real numbers, nonnegative real numbers, natural numbers, and integers, respectively. Moreover, we will make use of the usual Euclidean norm for vectors and the norm induced by the Euclidean norm for square matrices [21].

Let (X, d) be a metric space. A mapping $\pi : \mathbb{R}_+ \times X \to X$ is a semiflow on X [29] if:

(i) $\pi(0, p) = p$ for all $p \in X$;

(ii) $\pi(t, p)$ is continuous in the pair of variables t and p;

(iii) $\pi(t_1, \pi(t_2, p)) = \pi(t_1 + t_2, p)$ for all $t_1,\ t_2 \in \mathbb{R}_+$ and $p \in X$.

Suppose that π is a semiflow on X. A point $p \in X$ is stable P^+ (positively Poisson stable) if there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that $\pi(t_n, p) \to p$ as $n \to \infty$ [24]. For a fixed $p \in X$, let us denote by Θ_p the closure of the trajectory $\mathcal{T}(p) = \{\pi(t, p) : t \in \mathbb{R}_+\}$, i.e. $\Theta_p = \overline{\mathcal{T}(p)}$.

It was demonstrated by Hilmy [19] that if the trajectory corresponding to a Poisson stable point p is contained in a compact subset of X and it is neither a rest point nor a cycle, then the quasiminimal set contains an uncountable set of motions everywhere dense and Poisson stable. The following theorem can be proved by adapting the technique given in [19, 24].

Theorem 2.1 *Suppose that $p \in X$ is stable P^+ and $\mathcal{T}(p)$ is contained in a compact subset of X. If Θ_p is neither a rest point nor a cycle, then it contains an uncountable set of motions everywhere dense and stable P^+.*

The results of our paper are correct if one considers stable P^- (negatively Poisson stable) points for a semiflow with negative time or both stable P^+ and stable P^- (Poisson stable) points for a flow. The definition of a quasiminimal set is given for a Poisson stable point in [24].

The descriptions of an unpredictable point and trajectory are as follows.

Definition 2.1 ([13]) *A point $p \in X$ and the trajectory through it are unpredictable if there exist a positive number ϵ_0 (the unpredictability constant) and sequences $\{t_n\}$ and $\{\tau_n\}$, both of which diverge to infinity, such that $\lim_{n\to\infty} \pi(t_n, p) = p$ and $d(\pi(t_n + \tau_n, p), \pi(\tau_n, p)) \geq \epsilon_0$ for each $n \in \mathbb{N}$.*

An important point to discuss is the sensitivity or unpredictability. In famous research studies [16, 22, 23, 26, 30], sensitivity was considered as a property of a system on a certain set of initial data since it compares the behavior of at least a couple of solutions. Definition 2.1 allows us to formulate unpredictability for a single trajectory. Indicating an unpredictable point p, one can make an error by taking a point $\pi(t_n, p)$. Then $d(\pi(\tau_n, \pi(t_n, p)), \pi(\tau_n, p)) \geq \epsilon_0$, and this is unpredictability for the motion. Thus, we speak about the unpredictability of a single trajectory, whereas the former definitions considered the property in a set of motions.

It was proved in [13] that if $p \in X$ is an unpredictable point, then $\mathcal{T}(p)$ is neither a rest point nor a cycle, and that if a point $p \in X$ is unpredictable, then every point of the trajectory $\mathcal{T}(p)$ is also unpredictable. It is worth noting that the unpredictability constant ϵ_0 is common for each point on an unpredictable trajectory.

The dynamics on a set $S \subseteq X$ is sensitive [16, 23] if there exists a positive number ϵ_0 such that for each $u \in S$ and each positive number δ there exist a point $u_\delta \in S$ and a positive number τ_δ such that $d(u_\delta, u) < \delta$ and $d(\pi(\tau_\delta, u_\delta), \pi(\tau_\delta, u)) \geq \epsilon_0$.

A result concerning sensitivity in a quasiminimal set is given in the next theorem.

Theorem 2.2 ([13]) *The dynamics on Θ_p is sensitive if $p \in X$ is an unpredictable point.*

Theorem 2.2 mentions the presence of sensitivity in the set Θ_p if p is an unpredictable point in X. According to Theorem 2.1, if the trajectory $\mathcal{T}(p)$ of an unpredictable point $p \in X$ is contained in a compact subset of X, then Θ_p contains an uncountable set of everywhere dense stable P^+ motions. Additionally, since $\mathcal{T}(p)$ is dense in Θ_p, the transitivity is also valid in the dynamics. We named this type of chaos Poincaré chaos in our previous paper [13].

3. Unpredictable functions and chaos

This section is devoted to the description of unpredictable functions and their connection with chaos. For that purpose, the results provided in [29] will be utilized.

Let us denote by $C(\mathbb{R})$ the set of continuous functions defined on $\mathbb{R}$ with values in $\mathbb{R}^m$, and assume that $C(\mathbb{R})$ has the topology of uniform convergence on compact sets, i.e. a sequence $\{h_k\}$ in $C(\mathbb{R})$ is said to converge to a limit h if for every compact set $\mathcal{U} \subset \mathbb{R}$ the sequence of restrictions $\{h_k|_{\mathcal{U}}\}$ converges to $\{h|_{\mathcal{U}}\}$ uniformly.

One can define a metric d on $C(\mathbb{R})$ as [29]

$$d(h_1, h_2) = \sum_{k=1}^{\infty} 2^{-k} \rho_k(h_1, h_2), \tag{1}$$

where $h_1,\ h_2$ belong to $C(\mathbb{R})$ and

$$\rho_k(h_1, h_2) = \min\left\{1, \sup_{s \in [-k,k]} \|h_1(s) - h_2(s)\|\right\},\ k \in \mathbb{N}.$$

Let us define the mapping $\pi : \mathbb{R}_+ \times C(\mathbb{R}) \to C(\mathbb{R})$ by $\pi(t, h) = h_t$, where $h_t(s) = h(t+s)$. The mapping π defines a semiflow on $C(\mathbb{R})$ and it is called the Bebutov dynamical system [29].

We describe an unpredictable function as follows.

Definition 3.1 *An unpredictable function is an unpredictable point of the Bebutov dynamical system.*

According to Theorem $III.3$ [29], a motion $\pi(t, h)$ lies in a compact set if h is a bounded and uniformly continuous function. Assuming this, by means of Theorem 2.2, we obtain that an unpredictable function h determines chaos if it is bounded and uniformly continuous. On the basis of this result, one can say that if a differential equation admits an unpredictable solution that is uniformly continuous and bounded, then chaos is present in the set of solutions. In the next section, we will prove the existence of an unpredictable solution whose quasiminimal set is a chaotic attractor.

4. Unpredictable solutions of quasilinear systems

Consider the following quasilinear system,

$$x' = Ax + f(x) + g(t), \tag{2}$$

where the $m \times m$ constant matrix A has eigenvalues all with negative real parts, the function $f : \mathbb{R}^m \to \mathbb{R}^m$ is continuous, and $g : \mathbb{R} \to \mathbb{R}^m$ is a uniformly continuous and bounded function.

Since the eigenvalues of the matrix A have negative real parts, there exist positive numbers K and ω such that $\|e^{At}\| \leq Ke^{-\omega t},\ t \geq 0$ [18].

The following conditions are required:

(C1) There exists a positive number M_f such that $\sup_{x\in\mathbb{R}^m} \|f(x)\| \leq M_f$;

(C2) There exists a positive number L_f such that $\|f(x_1) - f(x_2)\| \leq L_f \|x_1 - x_2\|$ for all $x_1,\ x_2 \in \mathbb{R}^m$;

(C3) $KL_f - \omega < 0$.

The main result of the present study is mentioned in the next theorem.

Theorem 4.1 *Suppose that the conditions* $(C1) - (C3)$ *are valid. If the function* $g(t)$ *is unpredictable, then system (2) possesses a unique uniformly exponentially stable unpredictable solution, which is uniformly continuous and bounded on* $\mathbb{R}$.

Proof Using the technique for quasilinear equations [18], one can confirm under the conditions $(C1) - (C3)$ that system (2) possesses a unique bounded on $\mathbb{R}$ solution $\phi(t)$ that satisfies the relation

$$\phi(t) = \int_{-\infty}^{t} e^{A(t-u)}[f(\phi(u)) + g(u)]du. \tag{3}$$

Moreover, $\sup_{t\in\mathbb{R}} \|\phi(t)\| \leq M_\phi$, where $M_\phi = \dfrac{K(M_f + M_g)}{\omega}$ and $M_g = \sup_{t\in\mathbb{R}} \|g(t)\|$. The solution $\phi(t)$ is uniformly continuous on $\mathbb{R}$ since $\sup_{t\in\mathbb{R}} \|\phi'(t)\| \leq \|A\| M_\phi + M_f + M_g$.

Suppose that $x(t)$ is a solution of (2) such that $x(t_0) = x_0$ for some $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^m$. It can be verified that

$$\|x(t) - \phi(t)\| \leq K \|x_0 - \phi(t_0)\| e^{(KL_f - \omega)(t-t_0)},\ t \geq t_0,$$

and, therefore, $\phi(t)$ is uniformly exponentially stable.

Since the function $g(t)$ is unpredictable, there exist a positive number $\epsilon_0 \leq 1$ and sequences $\{t_n\},\ \{\tau_n\}$, both of which diverge to infinity, such that $d(g_{t_n}, g) \to 0$ as $n \to \infty$ and $d(g_{t_n+\tau_n}, g_{\tau_n}) \geq \epsilon_0$ for all $n \in \mathbb{N}$, where the distance function d is given by (1).

First of all, we shall show that $d(\phi_{t_n}, \phi) \to 0$ as $n \to \infty$. Fix an arbitrary small positive number $\epsilon < 1$, and suppose that α is a positive number satisfying $\alpha \leq \dfrac{\omega - KL_f}{2\omega + K - 2KL_f}$. Let k_0 be a sufficiently large natural number such that

$$k_0 \geq \max\left\{\frac{\ln(1/\alpha\epsilon)}{\ln 2}, \frac{1}{\omega - KL_f} \ln\left(\frac{2K(M_f + M_g)}{\omega\alpha\epsilon}\right)\right\}. \tag{4}$$

There exists a natural number n_0 such that if $n \geq n_0$, then $d(g_{t_n}, g) < 2^{-2k_0}\alpha\epsilon$. Therefore, for $n \geq n_0$, the inequality $\rho_{2k_0}(g_{t_n}, g) < \alpha\epsilon$ is valid. Since $\alpha\epsilon < 1$, we have that $\|g(t_n+s) - g(s)\| < \alpha\epsilon$ for $s \in [-2k_0, 2k_0]$.

Making use of the relation (3), one can obtain that

$$\phi(t_n+s) - \phi(s) = \int_{-\infty}^{s} e^{A(s-u)}[f(\phi(t_n+u)) - f(\phi(u)) + g(t_n+u) - g(u)]du.$$

Thus, if s belongs to the interval $[-2k_0, 2k_0]$, then it can be verified that

$$\begin{aligned} \|\phi(t_n+s) - \phi(s)\| &\leq \frac{2K(M_f+M_g)}{\omega} e^{-\omega(s+2k_0)} + \frac{K\alpha\epsilon}{\omega}\left(1 - e^{-\omega(s+2k_0)}\right) \\ &\quad + KL_f \int_{-2k_0}^{s} e^{-\omega(s-u)} \|\phi(t_n+u) - \phi(u)\| \, du. \end{aligned} \tag{5}$$

Now, let us define the functions

$$\psi_n(s) = e^{\omega s} \|\phi(t_n+s) - \phi(s)\|, \; n \geq n_0.$$

Inequality (5) implies that

$$\psi_n(s) \leq \frac{K\alpha\epsilon}{\omega} e^{\omega s} + \left(\frac{2K(M_f+M_g) - K\alpha\epsilon}{\omega}\right) e^{-2\omega k_0} + KL_f \int_{-2k_0}^{s} \psi_n(u) du.$$

Applying Gronwall's lemma [14], one can confirm that

$$\psi_n(s) \leq \frac{K\alpha\epsilon}{\omega - KL_f} e^{\omega s}\left(1 - e^{(KL_f-\omega)(s+2k_0)}\right) + \frac{2K(M_f+M_g)}{\omega} e^{KL_f s} e^{2(KL_f-\omega)k_0}.$$

Hence, the inequality

$$\|\phi(t_n+s) - \phi(s)\| < \frac{K\alpha\epsilon}{\omega - KL_f} + \frac{2K(M_f+M_g)}{\omega} e^{(KL_f-\omega)(s+2k_0)}$$

is valid. Since the number k_0 satisfies (4), we have $e^{(KL_f-\omega)k_0} \leq \dfrac{\omega\alpha\epsilon}{2K(M_f+M_g)}$ so that

$$\|\phi(t_n+s) - \phi(s)\| < \left(1 + \frac{K}{\omega - KL_f}\right)\alpha\epsilon$$

for $s \in [-k_0, k_0]$. Therefore, the inequality

$$\sup_{s\in[-k,k]} \|\phi(t_n+s) - \phi(s)\| < \left(1 + \frac{K}{\omega - KL_f}\right)\alpha\epsilon$$

holds for each integer k with $1 \leq k \leq k_0$. It is clear that $\left(1 + \dfrac{K}{\omega - KL_f}\right)\alpha\epsilon < 1$. Thus,

$$\rho_k(\phi_{t_n}, \phi) < \left(1 + \frac{K}{\omega - KL_f}\right)\alpha\epsilon, \; 1 \leq k \leq k_0.$$

For $n \geq n_0$, it can be obtained by using (4) one more time that

$$\begin{aligned} d(\phi_{t_n}, \phi) &= \sum_{k=1}^{\infty} 2^{-k} \rho_k(\phi_{t_n}, \phi) \\ &< \left(1 + \frac{K}{\omega - KL_f}\right) \alpha\epsilon \sum_{k=1}^{k_0} 2^{-k} + \sum_{k=k_0+1}^{\infty} 2^{-k} \\ &< \left(2 + \frac{K}{\omega - KL_f}\right) \alpha\epsilon \\ &\leq \epsilon. \end{aligned}$$

Hence, $d(\phi_{t_n}, \phi) \to 0$ as $n \to \infty$.

Next, we will verify the presence of a positive number $\overline{\epsilon}_0$ and a sequence $\{\overline{\tau}_n\}$, $\overline{\tau}_n \to \infty$ as $n \to \infty$, such that $d(\phi_{t_n+\overline{\tau}_n}, \phi_{\overline{\tau}_n}) \geq \overline{\epsilon}_0$ for all $n \in \mathbb{N}$.

Let N be a natural number such that $\sum_{k=N+1}^{\infty} 2^{-k} \leq \frac{\epsilon_0}{2}$. One can confirm that

$$\sum_{k=1}^{N} 2^{-k} \rho_k(g_{t_n+\tau_n}, g_{\tau_n}) \geq \frac{\epsilon_0}{2}.$$

In this case, for each $n \in \mathbb{N}$, there exist integers k_0^n between 1 and N such that

$$\rho_{k_0^n}(g_{t_n+\tau_n}, g_{\tau_n}) \geq \frac{2^{k_0^n}\epsilon_0}{2N} \geq \frac{\epsilon_0}{N}.$$

Therefore, it can be verified that

$$\sup_{s\in[-k_0^n, k_0^n]} \|g(t_n + \tau_n + s) - g(\tau_n + s)\| \geq \frac{\epsilon_0}{N}, \ n \in \mathbb{N}.$$

The last inequality implies the existence of numbers $\eta_n \in [-k_0^n, k_0^n]$ satisfying

$$\|g(t_n + \tau_n + \eta_n) - g(\tau_n + \eta_n)\| \geq \frac{\epsilon_0}{N}, \ n \in \mathbb{N}. \tag{6}$$

Suppose that $g(s) = (g_1(s), g_2(s), \ldots, g_m(s))$, where each g_i, $1 \leq i \leq m$, is a real-valued function. In accordance with (6), for each $n \in \mathbb{N}$, there is an integer j_n, $1 \leq j_n \leq m$, with

$$|g_{j_n}(t_n + \overline{\tau}_n) - g_{j_n}(\overline{\tau}_n)| \geq \frac{\epsilon_0}{Nm},$$

where $\overline{\tau}_n = \tau_n + \eta_n$, $n \in \mathbb{N}$. Since the function g is uniformly continuous, there exists a positive number $\Delta \leq 1$, which does not depend on the sequences $\{t_n\}$ and $\{\tau_n\}$, such that both of the inequalities

$$\|g(t_n + \overline{\tau}_n) - g(t_n + \overline{\tau}_n + s)\| \leq \frac{\epsilon_0}{4Nm}$$

and

$$\|g(\overline{\tau}_n) - g(\overline{\tau}_n + s)\| \leq \frac{\epsilon_0}{4Nm}$$

are valid for $s \in [-\Delta, \Delta]$. Thus, we have for $s \in [-\Delta, \Delta]$ that

$$\begin{aligned} |g_{j_n}(t_n + \overline{\tau}_n + s) - g_{j_n}(\overline{\tau}_n + s)| &\geq |g_{j_n}(t_n + \overline{\tau}_n) - g_{j_n}(\overline{\tau}_n)| \\ &- |g_{j_n}(t_n + \overline{\tau}_n) - g_{j_n}(t_n + \overline{\tau}_n + s)| \\ &- |g_{j_n}(\overline{\tau}_n) - g_{j_n}(\overline{\tau}_n + s)| \\ &\geq \frac{\epsilon_0}{2Nm}. \end{aligned} \tag{7}$$

For each $n \in \mathbb{N}$, one can find numbers $s_1^n, s_2^n, \ldots, s_m^n \in [-\Delta, \Delta]$ such that

$$\Big\| \int_{-\Delta}^{\Delta} [g(t_n + \overline{\tau}_n + u) - g(\overline{\tau}_n + u)]\, du \Big\| = 2\Delta \Big(\sum_{i=1}^{m} [g_i(t_n + \overline{\tau}_n + s_i^n) - g_i(\overline{\tau}_n + s_i^n)]^2 \Big)^{1/2}. \tag{8}$$

Hence, it can be deduced by means of (7) and (8) that

$$\begin{aligned} \Big\| \int_{-\Delta}^{\Delta} [g(t_n + \overline{\tau}_n + u) - g(\overline{\tau}_n + u)]\, du \Big\| &\geq 2\Delta \left| g_{j_n}(t_n + \overline{\tau}_n + s_{j_n}^n) - g_{j_n}(\overline{\tau}_n + s_{j_n}^n) \right| \\ &\geq \frac{\Delta \epsilon_0}{Nm}. \end{aligned}$$

Now, using the equation

$$\begin{aligned} \phi(t_n + \overline{\tau}_n + s) - \phi(\overline{\tau}_n + s) &= \phi(t_n + \overline{\tau}_n - \Delta) - \phi(\overline{\tau}_n - \Delta) \\ &+ \int_{-\Delta}^{s} A[\phi(t_n + \overline{\tau}_n + u) - \phi(\overline{\tau}_n + u)]du \\ &+ \int_{-\Delta}^{s} [f(\phi(t_n + \overline{\tau}_n + u)) - f(\phi(\overline{\tau}_n + u))]du \\ &+ \int_{-\Delta}^{s} [g(t_n + \overline{\tau}_n + u) - g(\overline{\tau}_n + u)]du, \end{aligned}$$

we attain that

$$\begin{aligned} \|\phi(t_n + \overline{\tau}_n + \Delta) - \phi(\overline{\tau}_n + \Delta)\| &\geq \Big\| \int_{-\Delta}^{\Delta} [g(t_n + \overline{\tau}_n + u) - g(\overline{\tau}_n + u)]du \Big\| \\ &- \|\phi(t_n + \overline{\tau}_n - \Delta) - \phi(\overline{\tau}_n - \Delta)\| \\ &- \int_{-\Delta}^{\Delta} (\|A\| + L_f) \|\phi(t_n + \overline{\tau}_n + u) - \phi(\overline{\tau}_n + u)\|\, du. \end{aligned}$$

The last inequality implies that

$$\sup_{s \in [-\Delta, \Delta]} \|\phi(t_n + \overline{\tau}_n + s) - \phi(\overline{\tau}_n + s)\| \geq \overline{\epsilon}_0, \ n \in \mathbb{N},$$

where $\bar{\epsilon}_0 = \dfrac{\Delta\epsilon_0}{2Nm[1+\Delta(\|A\|+L_f)]}$. Therefore, we have $d(\phi_{t_n+\bar{\tau}_n}, \phi_{\bar{\tau}_n}) \geq \bar{\epsilon}_0$ for each $n \in \mathbb{N}$.

The theorem is proved. □

In the definition of Devaney chaos, periodic motions constitute a dense subset. However, in our case, instead of periodic motions, Poisson stable motions take place in the dynamics. More precisely, we say that the dynamics on the quasiminimal set of functions on $\mathbb{R}$ is chaotic if the dynamics on it is sensitive and transitive, and there exists a continuum of Poisson stable trajectories dense in the quasiminimal set. Nevertheless, in the framework of chaos there may be infinitely many periodic motions. For instance, the symbolic dynamics of biinfinite sequences possesses both an uncountable set of nonperiodic Poisson stable motions and infinitely many cycles [13, 31].

5. Examples

5.1. Example 1

In this subsection, we will construct an unpredictable function.

Consider the function $z(t) = (z_1(t), z_2(t))$ defined as $z_1(t) = p_i,\ z_2(t) = q_i$ for $t \in [i, i+1),\ i \in \mathbb{Z}$, such that (p_i, q_i) is an unpredictable trajectory [13] of the Hénon map

$$\begin{aligned} p_{i+1} &= \alpha_0 - \beta_0 q_i - p_i^2 \\ q_{i+1} &= p_i, \end{aligned} \tag{9}$$

where $\beta_0 \neq 0$ and $\alpha_0 \geq (5+2\sqrt{5})(1+|\beta_0|)^2/4$. The unpredictable trajectory belongs to a Cantor set such that there exists a positive number R satisfying $\|(p_i, q_i)\| \leq R$ for each $i \in \mathbb{Z}$ [15, 27].

Define the continuous on $\mathbb{R}$ function $\psi(t)$ such that

$$\psi(t) = e^{-\gamma(t-i)}\psi(i) + \int_i^t e^{-\gamma(t-u)} z(u)du,\ t \in [i, i+1],\ i \in \mathbb{Z},$$

where γ is a positive number and $\psi(0) = \displaystyle\int_{-\infty}^0 e^{\gamma u} z(u)du.$

Let us show that $\psi(t)$ is an unpredictable function. Fix an arbitrary small positive number $\epsilon < 1$, and let β be a positive number such that $\beta \leq \dfrac{\gamma}{2\gamma+1}$. Suppose that r_0 is a sufficiently large natural number such that $r_0 \geq \max\left\{\dfrac{\ln(1/\beta\epsilon)}{\ln 2}, \dfrac{1}{\gamma}\ln\left(\dfrac{2R}{\gamma\beta\epsilon}\right)\right\}$. Since (p_i, q_i) is an unpredictable trajectory of (9), there exist a positive number ϵ_0 and sequences $\{i_n\},\ \{j_n\}$, both of which diverge to infinity, such that $\|(p_{i+i_n}, q_{i+i_n}) - (p_i, q_i)\| < \beta\epsilon$, $n \in \mathbb{N},\ i = -2r_0, -2r_0+1, \ldots, r_0-1$, and $\|(p_{i_n+j_n}, q_{i_n+j_n}) - (p_{j_n}, q_{j_n})\| \geq \epsilon_0,\ n \in \mathbb{N}$.

If $s \in [-r_0, r_0]$, then one can confirm that $\|\psi(i_n+s) - \psi(s)\| < \left(1+\dfrac{1}{\gamma}\right)\beta\epsilon$ for each $n \in \mathbb{N}$. Therefore, $\pi(i_n, \psi) \to \psi$ as $n \to \infty$ so that $\psi(t)$ is a positively Poisson stable point of the Bebutov dynamical system. On the other hand, we have $\sup\limits_{s\in[0,1]} \|\psi(i_n+j_n+s) - \psi(j_n+s)\| \geq \dfrac{(1-e^{-\gamma})\epsilon_0}{\gamma(1+e^{-\gamma})}$, and this proves that $\psi(t)$ is an unpredictable function.

5.2. Example 2

In this part of the paper, we will show how an unpredictable point may cause irregular dynamics. For that purpose, we will make use of coupled Duffing equations such that the first one is forced with a relay function and the second one is perturbed with the solutions of the former.

Let us consider the following forced Duffing equation,

$$x'' + 0.68x' + 1.6x + 0.008x^3 = \nu(t, \zeta, \lambda), \tag{10}$$

where the forcing term $\nu(t, \zeta, \lambda)$ is a relay function defined as

$$\nu(t, \zeta, \lambda) = \begin{cases} 1.2, & \text{if} \quad \zeta_{2j}(\lambda) < t \leq \zeta_{2j+1}(\lambda), \ j \in \mathbb{Z}, \\ 0.4, & \text{if} \quad \zeta_{2j-1}(\lambda) < t \leq \zeta_{2j}(\lambda), \ j \in \mathbb{Z}. \end{cases} \tag{11}$$

In (11), the sequence $\zeta = \{\zeta_j\}_{j \in \mathbb{Z}}$ of switching moments is defined through the equation $\zeta_j = j + \kappa_j, \ j \in \mathbb{Z}$, where the sequence $\{\kappa_j\}_{j \in \mathbb{Z}}$ is a solution of the logistic map

$$\kappa_{j+1} = \lambda \kappa_j (1 - \kappa_j). \tag{12}$$

By means of the variables $x_1 = x$ and $x_2 = x'$, equation (10) can be written as a system in the following form:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -1.6x_1 - 0.68x_2 - 0.008x_1^3 + \nu(t, \zeta, \lambda). \end{aligned} \tag{13}$$

We suppose that the parameter λ in (12) is greater than 4 such that the map possesses an invariant Cantor set $\Lambda \subset [0, 1]$ [27]. It was demonstrated in [13] that for such values of the parameter the map (12) possesses an unpredictable point in Λ. Let us consider system (13) with $\zeta_0 \in \Lambda$. For each natural number p, system (13) admits an unstable periodic solution with period $2p$ if p is odd and an unstable periodic solution with period p if p is even [5]. The reader is referred to [1–12] for more information about the dynamics of relay systems.

In order to illustrate the irregular dynamics of (13), we make use of the value $\lambda = 4.007$ in the system and depict the solution corresponding to the initial data $x_1(0.41) = 0.6, \ x_2(0.41) = 0.5$ and $\zeta_0 = 0.41$ in Figure 1. The simulation results seen in Figure 1 confirm the presence of irregular behavior in the dynamics of (13). Due to the instability, simulations of the system cannot be provided for large intervals of time.

Next, we take into account another Duffing equation,

$$y'' + 0.95y' + 1.8y + 0.005y^3 = 0. \tag{14}$$

Using the variables $y_1 = y$ and $y_2 = y'$, equation (14) can be reduced to the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -1.8y_1 - 0.95y_2 - 0.005y_1^3. \end{aligned} \tag{15}$$

We perturb (15) with the solutions of (13) and set up the system

$$\begin{aligned} z_1' &= z_2 + x_1(t) \\ z_2' &= -1.8z_1 - 0.95z_2 - 0.005z_1^3 + x_2(t). \end{aligned} \tag{16}$$

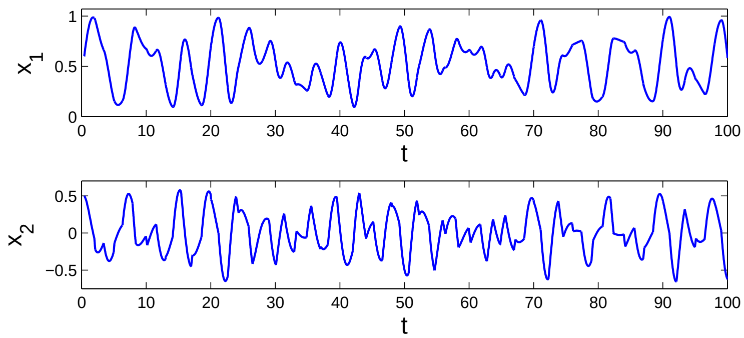

Figure 1. The solution of (13) with $x_1(0.41) = 0.6$, $x_2(0.41) = 0.5$, and $\zeta_0 = 0.41$. The value $\lambda = 4.007$ is used in the simulation. The figure reveals the presence of chaos in the dynamics of (13).

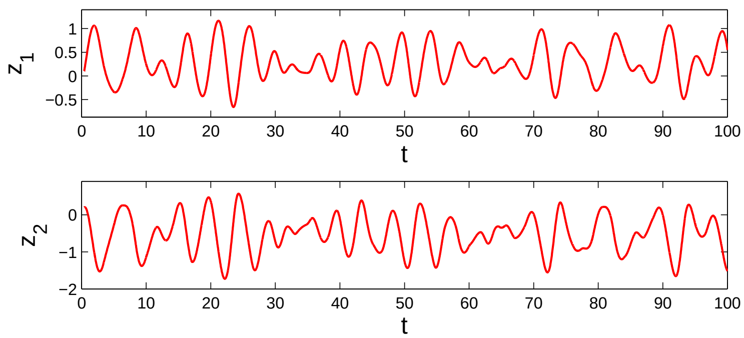

Figure 2. Chaotic behavior in the dynamics of system (16). The solution $(x_1(t), x_2(t))$ represented in Figure 1 is utilized as the perturbation in (16). The irregularity is observable in both z_1 and z_2 coordinates.

System (16) is in the form of (2), where $A = \begin{pmatrix} 0 & 1 \\ -1.8 & -0.95 \end{pmatrix}$, $f(z_1, z_2) = (0, -0.005z_1^3)$, and $g(t) = (x_1(t), x_2(t))$. Both eigenvalues of A have real parts -0.475, and the coefficient of the nonlinear term is chosen sufficiently small in absolute value so that the conditions $(C1)-(C3)$ are valid for (16).

Figure 2 shows the solution of (16) with $z_1(0.41) = 0.1$ and $z_2(0.41) = 0.2$. For the simulation, the solution $(x_1(t), x_2(t))$, which is represented in Figure 1, is used. One can observe in Figure 2 that the represented solution behaves irregularly.

Next, we will demonstrate the presence of periodic motions in system (16) by means of the Ott–Grebogi–Yorke (OGY) control technique [25]. Since the logistic map (12) is the main source of the chaotic behavior in the coupled system (13) + (16), we will apply the OGY method to the map. Let us explain briefly the method for the logistic map [28]. Suppose that the parameter λ in (12) is allowed to vary in the range of $[4.007 - \varepsilon, 4.007 + \varepsilon]$, where ε is a given small positive number. Consider an arbitrary solution $\{\kappa_j\}$, $\kappa_0 \in \Lambda$, of the map, and denote by $\kappa^{(i)}$, $i = 1, 2, \ldots, p$, the target p-periodic orbit to be stabilized. In the OGY control method [28], at each iteration step j after the control mechanism is switched on, we consider the logistic map

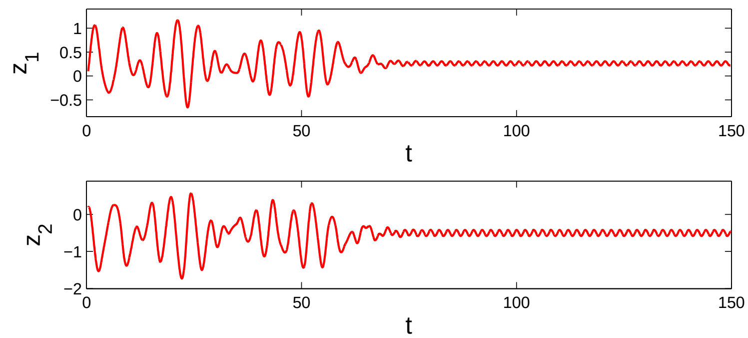

Figure 3. The stabilization of the 2-periodic solution of (16) corresponding to the fixed point 3.007/4.007 of the logistic map (12). The value $\varepsilon = 0.095$ is used and the control is switched on at $t = \zeta_{20}$.

with the parameter value $\lambda = \bar{\lambda}_j$, where

$$\bar{\lambda}_j = 4.007 \left(1 + \frac{(2\kappa^{(i)} - 1)(\kappa_j - \kappa^{(i)})}{\kappa^{(i)}(1 - \kappa^{(i)})} \right), \tag{17}$$

provided that the number on the right-hand side of the formula (17) belongs to the interval $[4.007-\varepsilon, 4.007+\varepsilon]$. In other words, formula (17) is valid if the trajectory $\{\kappa_j\}$ is sufficiently close to the target periodic orbit. Otherwise, we take $\bar{\lambda}_j = 4.007$, so that the system evolves at its original parameter value, and wait until the trajectory $\{\kappa_j\}$ enters a sufficiently small neighborhood of the periodic orbit $\kappa^{(i)}$, $i = 1, 2, \ldots, p$, such that the inequality $-\varepsilon \leq 4.007 \frac{(2\kappa^{(i)} - 1)(\kappa_j - \kappa^{(i)})}{\kappa^{(i)}(1 - \kappa^{(i)})} \leq \varepsilon$ holds. If this is the case, the control of chaos is not achieved immediately after switching on the control mechanism. Instead, there is a transition time before the desired periodic orbit is stabilized. The transition time increases if the number ε decreases [17].

Figure 3 shows the stabilization of an unstable 2-periodic solution of (16). Here, the OGY control method is used around the fixed point 3.007/4.007 of the logistic map (12), and the simulation is performed for the initial data $x_1(0.41) = 0.6$, $x_2(0.41) = 0.5$, $z_1(0.41) = 0.1$, $z_2(0.41) = 0.2$, $\zeta_0 = 0.41$. The control is switched on at $t = \zeta_{20}$ and the value $\varepsilon = 0.095$ is utilized. One can confirm that even if the control is switched on at $t = \zeta_{20}$ there is a transition time before the stabilization such that the control becomes dominant approximately at $t = 76$. Figure 3 manifests that the OGY control technique is appropriate for the stabilization of the unstable periodic motions of system (16).

On the other hand, Figure 4 shows the simulation result for (16) when the OGY method is applied around the 2-periodic orbit $\kappa^{(1)} \approx 0.34459$, $\kappa^{(2)} \approx 0.90497$ of (12). The represented solution corresponds again to the initial data $x_1(0.41) = 0.6$, $x_2(0.41) = 0.5$, $z_1(0.41) = 0.1$, $z_2(0.41) = 0.2$, $\zeta_0 = 0.41$. The value $\varepsilon = 0.072$ is used and the control is switched on at $t = \zeta_{25}$. The presence of a transition time before the stabilization is observable in Figure 4 such that the control becomes dominant approximately at $t = 46$. One can observe that the stabilized 2-periodic solutions seen in Figure 3 and Figure 4 are different, and this reveals the presence of periodic motions.

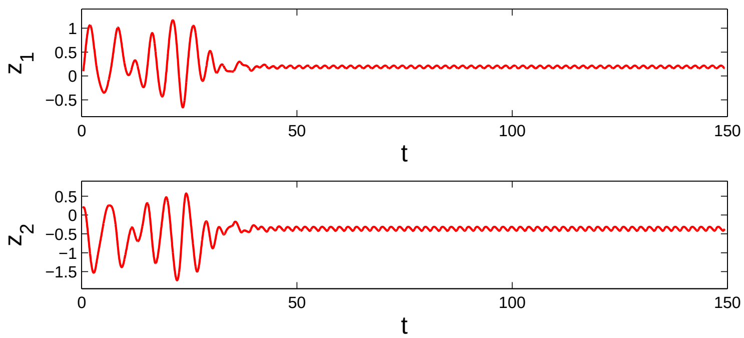

Figure 4. The stabilization of the 2-periodic solution of (16) corresponding to the 2-periodic orbit $\kappa^{(1)} \approx 0.34459$, $\kappa^{(2)} \approx 0.90497$ of (12). The value $\varepsilon = 0.072$ is used and the control is switched on at $t = \zeta_{25}$.

6. Conclusions

The unpredictable function has been defined as an unpredictable point of the Bebutov dynamics, and chaos in the quasiminimal set of the function is verified. This is the first time in the literature that the existence of an unpredictable solution for a quasilinear ordinary differential equation is proved.

The concept of unpredictable solutions can be useful for finding more delicate features of chaos in systems with complicated dynamics. Studies based on unpredictable functions may pave the way for the functional analysis of chaos to involve the operator theory results. Hopefully, our approach will give a basis for a deeper comprehension and possibility to unite different appearances of chaos. In this framework, the results can be developed for partial differential equations, integrodifferential equations, functional differential equations, and evolution systems.

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